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Vibrations and buckling of a cylindrical elastic thin shell, joined with plates of different shapes

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Contents

Introduction

Thin-walled shells and structures made from them are widely used in modern industry and mechanical engineering. Significant development of shell theory occurred in the mid-20th century when solving many applied problems. Thin shells are used in the oil and gas industry to store and transport liquids and gases, in the aerospace industry for the manufacture of rocket bodies and space modules, in construction of water towers and pipes, as well as in the automotive industry for design bodies, fuel tanks, gas cylinders and other components. Structures consisting of thin-walled shells usually withstand high dynamic loads, causing vibrations and deformations.

The general theory of plates and shells was advanced through the research of S.P. Timoshenko [1, 2, 3, 4], V.Z. Vlasov [5, 6, 7], A.I. Lurie [8], W.T. Koiter [9, 10], A.L. Goldenveizer [11, 12, 13], A.P. Filin [14], E. Reissner [15], L.G. Donnell [16], V.V. Novozhilov [17, 18], Kh.M. Mushtari [19], J. Arbocz [20, 21], and others, contributing significantly to the foundational principles of modern shell theory.

Vibrations of thin-walled structures are the focus of works by V.V. Bolotin [22, 23, 24], A.L. Goldenveizer [11], V.B. Lidsky [11], P.E. Tovstik [11, 25, 26, 27], V. Zödl [28], I.A. Birger [29]. These publications discuss the fundamental theoretical and practical aspects of the dynamics of shells and plates, encompassing various analysis methods, numerical modeling, and experimental research.

Research on the structure of the frequency spectrum of free vibrations of structures is of particular importance, as this information is critically necessary for the design of structures containing thin-walled elements to avoid dangerous resonances. Numerous studies dedicated to this issue are compiled in the reference [24]. In [11], A.L. Goldenveizer, V.B. Lidsky, and P.E. Tovstik investigate the properties of the eigenfrequency spectrum of vibrations of thin rotational shells.

The equations describing the vibrations of the structures discussed in the dissertation are complex, making exact analytical solutions challenging to obtain. The presence of a small thin-walled parameter allows for the use of asymptotic methods to derive asymptotic formulas for eigenfrequencies of vibrations. This work pays particular attention to the influence of various methods of reinforcing a cylindrical shell on the natural frequencies of the structure. The dissertation uses the asymptotic method developed in the works of S.B. Filippov [30, 31], the Rayleigh-Ritz method, and the finite element method to study the natural vibrations of a cylindrical shell. Comparative analysis showed good agreement of results, indicating the reliability and effectiveness of the approach used, especially in the context of analyzing various reinforcement configurations of cylindrical shells.

Structures consisting of thin-walled elements, under certain conditions, can become unstable, moving from the initial stress-strain state to the adjacent equilibrium state. This dictates the need to solve the problems of buckling of such structures. The stability of thin shells was considered in the papers by P.E. Tovstik [25], E.I. Grigolyuk [32], A.S. Volmir [33], V.Z. Vlasov [5], L.G. Donnell [16], R. Southwell [34], P.F. Papkovich [35], W. Koiter [9, 10], E. Reissner [15], N.A. Alfutov [36], V.V. Bolotin [22], A.N. Dinnik [37], A.R. Rzhanitsyn [38], R. Lorenz [39], P.M. Ogibalov [40], I.I. Vorovich [41], V.I. Feodosiev [42], W. Flügge [43], and A.V. Pogorelov [44].

Circular cylindrical shells, often reinforced with stiffness ribs, are commonly used in practice. Depending on the application, these can be longitudinal ribs — stringers, or transverse circular ribs — frames. Reinforced shells withstand higher critical pressure compared to similar smooth shells of the same mass.

Along with reinforced shells, structures incorporating several conjugated shells or shells conjugated with plates are often used in practice. The works [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 63] consider problems of vibrations of a cylindrical shell conjugated with spherical and conical shells. The issues of stability loss in such structures are investigated by P.E. Tovstik [25] and S.B. Filippov [30, 31, 55, 56, 57, 58, 59, 60, 61, 62].

In addition to analytical and asymptotic research methods, numerical methods are actively used for investigating thin-walled structures. With the advancement of computing technology, research on complex composite structures has become possible. The works [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] present research on reinforced cylindrical shells using the finite element method.

This dissertation is dedicated to improving the efficiency of structures by optimizing their geometric parameters. The goal is to explore the possibility of reducing the mass of structures without decreasing the critical pressure and fundamental frequency. Methods for efficiently calculating optimal parameters of a reinforced cylindrical shell are proposed. An analysis of the accuracy of the proposed models and research methods is conducted.

The dissertation examines some specific problems of free vibrations and stability of reinforced shells, which have practical applications. For analyzing such problems, the classical (linear) system of shell theory equations based on Kirchhoff-Love hypotheses is used. The Kirchhoff-Love hypotheses assume that lines perpendicular to the mid-surface of the shell remain undeformed and orthogonal to the deformed surface, and the thickness of the plate does not change during deformation. These hypotheses simplify the mathematical model of the shell, which is convenient for deriving analytical approximate formulas. Using the classical system of shell theory equations with Kirchhoff-Love hypotheses provides sufficient accuracy for solving the problems set in this work and allows for deriving approximate formulas that can be used in practical calculations when designing thin-walled structures.

Structure of the Dissertation

In the introduction, the relevance of research dedicated to the vibrations and stability loss of thin-walled elastic cylindrical shells is substantiated. An analysis of existing literature related to the dissertation's theme is presented.

The first chapter of the dissertation addresses the problem of natural vibrations of a circular reinforced cylindrical shell. This study is of significant practical importance for industry, as the fundamental frequency of natural vibrations of structures is a key indicator of their reliability and safety. To increase the fundamental frequency of vibrations, transverse stiffening ribs — frames — are installed on the shell. Most works consider cases of shell reinforcement with identical frames. The dissertation investigates the vibrations and stability of a shell reinforced with ribs of varying heights. The classical system of shell theory equations, based on Kirchhoff-Love hypotheses, is used for modeling the structure. The equation system for the case of shell reinforcement with stiffness ribs, whose heights change along the shell's generatrix, is examined using the asymptotic method. A comparison of the fundamental vibration frequencies of these structures, found using the Rayleigh-Ritz method and the finite element method, is conducted. The effect of the distribution law of frame stiffnesses on the fundamental frequencies and vibration modes of the structure is investigated. The study identifies two types of structure vibrations and derives formulas for calculating approximate values of the fundamental frequencies of the structure for various edge support conditions of the shell. A single-parameter optimization problem is solved, with the relative height of the frames chosen as the parameter. For a structure of fixed mass, the parameter value at which the structure has the highest fundamental frequency is found. As the height of the frames increases, they transform into annular plates, necessitating solving the problem of natural vibrations of a plate. In this case, optimization is achieved for such a set of parameters where the fundamental frequency of the reinforced cylinder coincides with the fundamental frequency of vibrations of the plate with the largest radius.

The second chapter of the dissertation examines cylindrical shells conjugated with spherical end segments. The frequency spectrum of natural vibrations of a structure consisting of a cylindrical shell, with one end rigidly fixed and the other end conjugated with the edge of a spherical segment, is investigated. As the curvature of the spherical segment increases indefinitely, it degenerates into a circular plate. Numerical research on vibrations in the finite element package *Comsol* revealed that the vibrations could be divided into three groups. The frequencies and modes of the first group ("shell-like") are close to the frequencies and modes of vibrations of a cylindrical shell with one end rigidly fixed and the other hinge-supported. The second group's frequencies and modes ("cap-like in the degenerate case - "plate-like") are close to the frequencies and modes of natural vibrations of the spherical segment with a rigidly fixed edge. The third group ("beam-like") has frequencies and modes of vibrations barely differing from those of a beam, with one end clamped and a mass concentrated on the other. Asymptotic and numerical solutions show that for a cylindrical shell, the end cap plays the role of an elastic fixture.

The third chapter of the dissertation is devoted to investigating the stability loss of a structure consisting of a cylindrical shell conjugated with circular frames of different stiffness under normal external pressure. An approximate analytical solution to the problem of stability loss of the structure is obtained. A comparison of solutions for several sets of parameters of the height distribution function of the frames along the shell's generatrix is conducted. Two optimization problems are solved. In the first problem, the parameters of a fixed-mass structure with the highest critical pressure are found. In the second problem, for a given critical pressure, the geometric parameters of the structure with the lowest mass are determined.

At the end of each chapter, research results are presented, and brief conclusions are made.

In the dissertation's conclusion, general conclusions and provisions are described.

General Characteristics of the Dissertation

Relevance of the topic is dictated by the need to develop methods for investigating and optimizing structures composed of cylindrical shells reinforced with ring plates and cylindrical shells conjugated with spherical end caps. In the context of active urbanization, the optimization of pipeline and tank structures at the design stage is critically important, ensuring their resilience and durability under all dynamic loads and operating conditions. Effective modeling and analysis of such complex systems contribute to the creation of safe and efficient engineering solutions.

The aim of the work includes:

— Developing an efficient method for analytical calculation of the fundamental frequency of natural vibrations of a cylindrical shell reinforced with frames of varying stiffness.

— Assessing the influence of curvature and thickness of the end cap on the fundamental frequency of natural vibrations of a structure consisting of a cylindrical shell conjugated with an end cap.

— Developing a method for determining the fundamental frequencies of composite structures for different vibration modes of the structure.

— Modeling various conditions of shell conjugation with plates.

— Solving the problem of stability loss of a cylindrical shell reinforced with frames of varying stiffness.

Reliability of the results is determined by the use of equations from the technical theory of shells and semi-momentless theory; application of tested numerical and analytical methods for solving systems of differential equations; comparison of results obtained through analytical, asymptotic, and numerical analysis; systematic assessments of the error and order of accuracy of approximation formulas and results obtained using the finite element method; and alignment of simulation results with other authors' findings.

Scientific value of the dissertation consists of the following:

— An algorithm for investigating vibrations and stability loss of a cylindrical shell reinforced with frames of varying stiffness is developed, where the distribution function of frame stiffnesses along the shell's generatrix can be arbitrary.

— A method for estimating fundamental frequencies of described structures by dividing the vibration spectrum into types is proposed and tested. The change in vibration modes when frequencies of different types coincide is investigated.

— Various approaches to optimization problems of structures are proposed: minimizing the mass of the structure at a fixed critical pressure and maximizing the fundamental frequency of the structure at a fixed mass. Examples of their solutions are provided.

Practical value of the dissertation consists of the following:

— Using approximate formulas for estimating the fundamental frequency of a cylindrical shell reinforced with ring plates significantly accelerates the design process.

— Recommendations for the design of medium-length reinforced cylindrical shells are provided, which can reduce their mass without losing strength.

— A method for calculating the geometric parameters of a reinforced cylindrical shell with a given critical pressure is developed.

Publications. The main results presented in the dissertation are published in the following journals:

— "Vestnik of St. Petersburg University. Mathematics. Mechanics. Astronomy"[77, 78, 79];

— "Vestnik of St.Petersburg University. Mathematics"[80, 81, 82];

— In the book series "Advanced Structured Materials"[83, 84];

— In the book series "AIP Conference Proceedings"[85];

— Proceedings of the seminar "Computer Methods in Continuum Mechanics"[86, 87].

Among these, 3 are in journals included in the list of peer-reviewed scientific journals recommended by the Higher Attestation Commission, 6 in peer-reviewed publications indexed in the international citation database Scopus, and 2 in collections indexed in RSCI.

Validation of results. The research findings were reported at the following international and all-Russian conferences:

— "All-Russian Congress on Theoretical and Applied Mechanics"[88];

— "Current Problems in Mechanics (APM)"[89];

— "29th Nordic Seminar on Computational Mechanics"[90];

— VIII and IX Polyakhov's Readings [91, 92];

— Section of Theoretical Mechanics at the Gorky House of Scientists (St. Petersburg) [93].

Author's contribution to the preparation of publications. In joint publications, the scientific supervisor S.B. Filippov contributed to the problem setting and discussion of results. In the works [78, 79, 81, 82, 84], A.L. Smirnov contributed to the results of the asymptotic analysis of the problem of vibrations of a cylindrical shell with a cap.

Structure and volume of the dissertation. The dissertation consists of an introduction, three chapters, and a bibliography. The volume of the dissertation work is 111 pages, including 25 figures and 20 tables. The bibliography contains 93 items.

The work was carried out with the financial support of the Russian Foundation for Basic Research (project 19-01-00208) and the Russian Science Foundation (project 3-21-00111).

Main scientific results:

— The Rayleigh-Ritz method is applied for the first time to find the fundamental frequency of a cylindrical shell conjugated with frames of varying stiffness. Different cases of frame stiffness distribution along the shell's generatrix are considered [85, 86, 87].

— The feasibility of using the ring plate model to study the vibration of transverse frames is investigated [85, 86, 87].

— The influence of a cylindrical shell on conjugated plates in solving vibration and stability problems is examined [77, 78, 79, 80, 81, 82].

— Problems of vibrations and stability loss for cylindrical shells reinforced with frames of varying stiffness and different edge conditions are solved [77, 80].

— The effect of curvature and thickness of the end plate on the fundamental frequency of the structure is studied [78, 79, 81, 82, 84].

Provisions for Defense:

— Equations describing the natural vibrations of thin cylindrical shells reinforced with frames of varying stiffness are obtained. Approximate formulas for the fundamental frequencies of the structure's vibrations are derived. As an example, an optimization problem is considered involving the maximization of the fundamental vibration frequency of a structure of a given mass.

— Approximate formulas for the lowest natural frequencies of a cylindrical shell conjugated with a spherical segment or a circular plate are obtained. The influence of the curvature and thickness of the spherical segment on the fundamental frequency of the structure is investigated.

— Asymptotic formulas for the critical external normal pressure of a cylindrical shell conjugated with ring plates are obtained. Examples of solved problems include maximizing the critical pressure for a structure of a given mass and minimizing the mass of the structure for a fixed value of critical pressure.

1 Natural vibrations of a thin cylindrical shell stiffened with rings of varying stiffness

This chapter investigates the lowest frequencies and modes of vibrations of a structure consisting of a thin-walled elastic cylindrical shell stiffened with rings of varying stiffness. An example of such a shell is shown in Figure 1.1.

Two types of structure vibrations are identified. The modes of the first type have a large number of waves in the circumferential direction and are similar to the modes of natural vibrations of an unstiffened cylindrical shell. The modes and frequencies of the second type are close to the modes and frequencies of vibrations of a ring plate. The influence of changing the distribution law of ring stiffnesses along the generatrix on the lowest frequency of the shell is investigated using numerical and asymptotic methods. Formulas for calculating approximate values of the fundamental frequencies of the structures are obtained for cases of simple support and rigid fixing of the shell edges.

An optimization problem is solved to find the values of the coefficients of the ring height distribution function for a structure of fixed mass, where the value of the fundamental frequency reaches its highest.

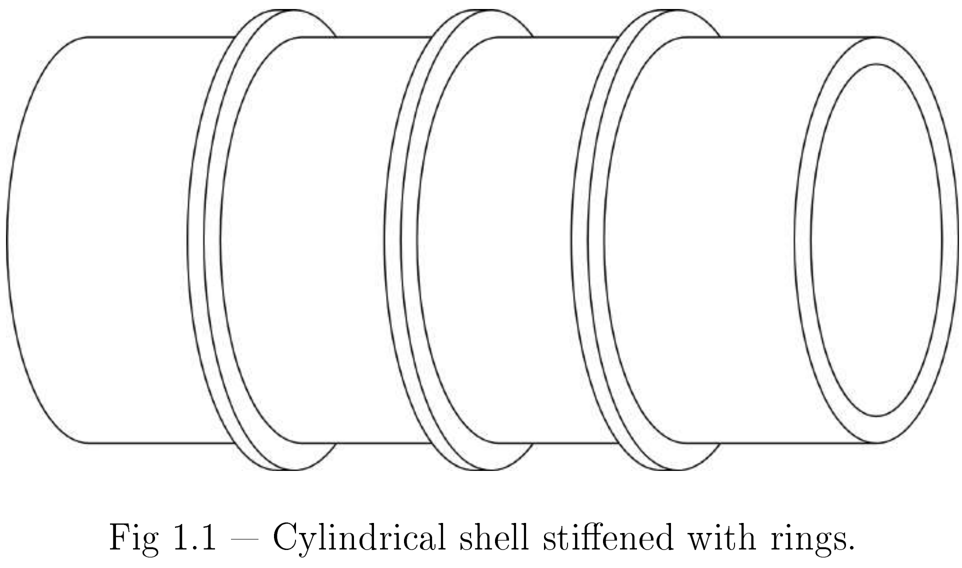


Fig 1.1 — Cylindrical shell stiffened with rings.

1.1 Natural vibrations of a stiffened cylindrical shell

1.1.1 Problem statement

The problem of vibrations of a thin-walled elastic cylindrical shell, which is stiffened with n_s transverse stiffness ribs (rings) with zero eccentricity to increase the first natural frequency, is considered. After separating the variables, the dimensionless system of equations describing small free vibrations of the cylindrical shell takes the form [11, 14]:

$$\mu^4 \cdot \Delta^2 w - \sigma \Delta_k \Phi - \lambda w = 0, \quad \Delta^2 \Phi + \Delta_k w = 0, \tag{1.1}$$

where

$$\Delta w = \frac{d^2 w}{ds^2} - m^2 w, \quad \Delta_k w = \frac{d^2 w}{ds^2}, \quad \sigma = 1 - \nu^2.$$

Here, s is the coordinate directed along the generatrix, Φ is the force function, w is the projection of displacement in the normal direction, m is the number of waves along the parallel, ν is the Poisson's ratio, $\mu^4 = h^2/12$ is a small parameter, h is the shell thickness, $\lambda = \sigma\rho\omega^2 R^2 E^{-1}$, ρ is the material density, E is Young's modulus, ω is the frequency of natural vibrations. The radius R of the cylinder base is chosen as the unit of length.

We will limit ourselves to determining the lowest frequencies of vibrations. Assume that the boundary conditions do not allow bending of the shell's mid-surface. Expressing the solution of system (1.1) as the sum of the main semi-momentless state and a simple edge effect near the shell edges, the lowest frequencies correspond to $\lambda \sim \mu^2$, $m \sim \mu^{-1/2}$ [11]. Excluding the force function Φ from the system and considering that $\Delta \sim m^2$, we obtain in the first approximation the equation

$$w_0^{IV} - \alpha^4 w_0 = 0, \quad \alpha^4 = \frac{m^4 \lambda_0 - \mu^4 m^8}{\sigma}, \tag{1.2}$$

where w_0 describes the semi-momentless state, λ_0 is the approximate value of λ, $w' = dw/ds$ (see [11, 14]). In the following, only the approximate solution is considered, and instead of w_0 and λ_0, the notations w and λ are used, respectively.

For simple support of the shell edges, the boundary conditions for equation (1.2) are:

$$w(0) = w''(0) = w(l) = w''(l) = 0, \tag{1.3}$$

and for rigid fixing:

$$w(0) = w'(0) = w(l) = w'(l) = 0, \tag{1.4}$$

where l is the dimensionless length of the shell.

If the shell is stiffened along the parallels with coordinates $s = s_i, i = 1, 2, \ldots, n-1$ with circular rods (rings), then $w = w^{(i)}$ for $s \in [s_{i-1}, s_i], i = 1, 2, \ldots, n$, with $s_0 = 0$, $s_n = l$:

$$w^{(i)^{IV}} - \alpha^4 w^{(i)} = 0, \ i = 1,2,\ldots,n. \tag{1.5}$$

Assuming that the characteristic size of the cross-section of the ring $a_i \ll \mu$, the following conjugation conditions are satisfied on the parallels stiffened with rings, which can have different heights and stiffnesses [30]:

$$\begin{gathered} w^{(i)} = w^{(i+1)}, \quad w^{(i)'} = w^{(i+1)'}, \\ w^{(i)''} = w^{(i+1)''}, \quad w^{(i)'''} - w^{(i+1)'''} = -c_i w^{(i+1)}, \\ s = s_i, \quad i = 1, 2, \ldots, n-1, \end{gathered} \tag{1.6}$$

where

$$c_i = \frac{m^8 \mu^4 l \eta_i}{\sigma n}, \quad \eta_i = \frac{12 \sigma n E_c J_i}{h^3 E l}.$$

Here, E_c is Young's modulus of the material of the rings, η_i is the dimensionless stiffness of the i-th ring, proportional to the ratio of bending stiffnesses of the ring and the shell, introduced in [36], J_i is the moment of inertia of the cross-section of the i-th ring relative to the generatrix of the cylinder.

The approximate value of the frequency parameter of the stiffened shell is determined by the formula:

$$\lambda = \frac{\sigma\alpha^4(\mathbf{c})}{m^4} + \mu^4 m^4, \quad \mathbf{c} = \{c_i\}_{i=1}^n,$$

where $\alpha(\mathbf{c})$ is the eigenvalue of the boundary problem (1.5), (1.6) with boundary conditions (1.3) or (1.4).

1.1.2 Moment of inertia of the ring

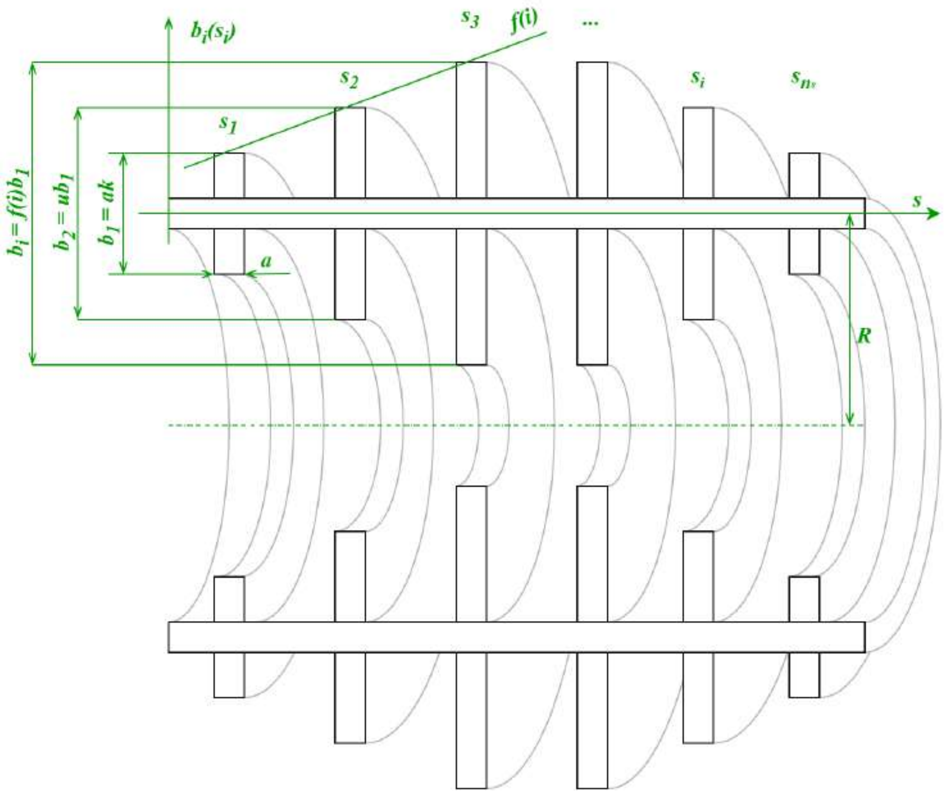


Fig 1.2 — Axial section of the shell stiffened with rings.

Figure 1.2 shows a shell with rings in a cut along the shell's generatrix. It is assumed that the shell and rings are made of the same material. It is assumed that all rings have the same width a, and the height of the first ring is $b = ka$. Let's introduce $f(i)$ — a function describing the distribution of ring heights along the generatrix of the cylinder: $b_i = bf(i) = kaf(i)$.

The eccentricity of a ring is the distance between the center of gravity of the ring's cross-section and the mid-surface of the shell. For a ring with zero eccentricity, the moment of inertia of the i-th ring is calculated using the formula:

$$J_i = \frac{ab_i^3}{12} = \frac{a^4k^3}{12}f^3(i) = Jf^3(i), \quad J = \frac{a^4k^3}{12}.$$

Then the dimensionless stiffness of the ring can be written as:

$$c_i = \frac{m^8\mu^4 l\eta_i}{\sigma n} = \frac{m^8\mu^4 l\eta}{\sigma n} \cdot f^3(i) = c \cdot f^3(i), \tag{1.7}$$

$$\eta_i = \frac{12\sigma n E_c J_i}{h^3 E l} = \frac{12\sigma n J}{h^3 l} \cdot f^3(i) = \eta \cdot f^3(i),$$

where

$$c = \frac{m^8\mu^4 l\eta}{\sigma n}, \quad \eta = \frac{12\sigma n J}{h^3 l}. \tag{1.8}$$

Rings with non-zero eccentricity, located either inside or outside the shell, are more commonly used in practice. When the eccentricity of the ring is equal to $b_i/2$, to get an upper estimate of the first frequency of the shell, the moment of inertia $J = ab^3/3$ can be taken.

The function of the structure profile $f(i)$ can have an arbitrary form, but it is practical to stiffen the shell with rings whose heights are symmetrical about the middle. Specifically, for a linear distribution of ring heights as shown in Figure 1.3 (a), the function $f(i)$ is

$$f_{lin}(i) = (\kappa(i) - 1)(u - 1) + 1, \quad u = \frac{b_2}{b_1}. \tag{1.9}$$

For the case of ring heights distributed parabolically as shown in Figure 1.3 (b),

$$f_{parab}(i) = a_p\kappa^2(i) - na_p\kappa(i) + na_p - a_p + 1, \quad where \quad a_p = \frac{1-u}{n-3}. \tag{1.10}$$

For the case of exponential distribution of ring heights as shown in Figure 1.3 (c),

$$f_{exp}(i) = \frac{u-1}{e^2-e}e^{\kappa(i)} + \frac{e-u}{e-1}. \tag{1.11}$$

Consequently, for the case of reinforcing the shell with identical rings

$$f_0(i) = 1. \tag{1.12}$$

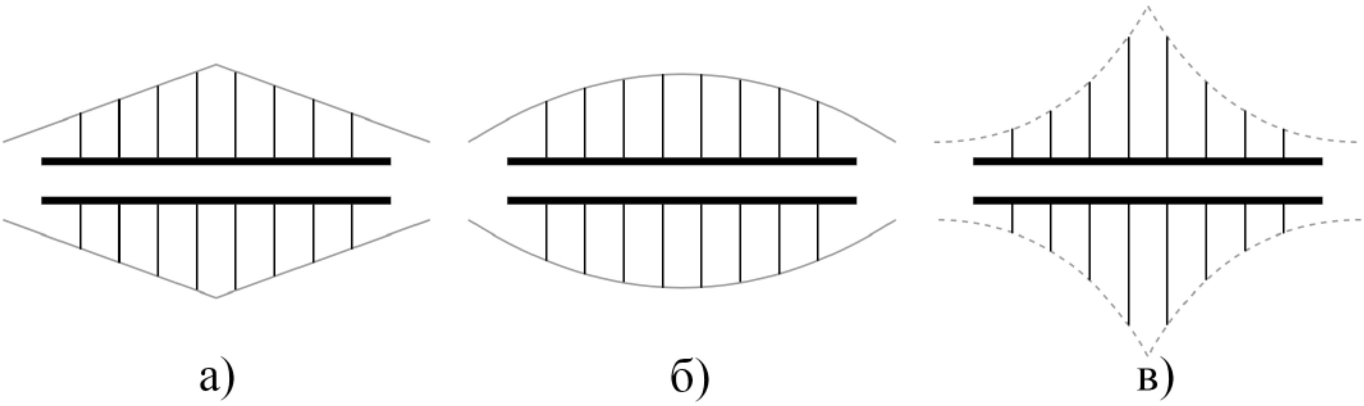


Fig 1.3 — Structure profiles for the case of
a). linear b). parabolic c). exponential
ring height distribution functions.

In formulas (1.9, 1.10, 1.11), the function

$$\kappa(i) = \frac{n}{2} - \left|\frac{n}{2} - i\right| = \begin{cases} i, & i < \frac{n}{2} \\ n-i, & i \geqslant \frac{n}{2} \end{cases}$$

ensures the symmetry of the structure profile functions, and the parameter $u = b_2/b_1$ characterizes the amplitude of the distribution function.

1.1.3 Optimal placement of rings

The boundary problems (1.3, 1.5, 1.6) and (1.4, 1.5, 1.6) are equivalent to the problems of determining the lowest frequencies of transverse vibrations of beams, respectively with simply supported and clamped ends (Figure 1.4), stiffened with springs of stiffness c_i at points $s = s_i$. The case of simply supported ends of the beam with uniformly spaced ($s_i = l/n \cdot i$) identical springs ($c_i = c$) is studied in the monograph [30].

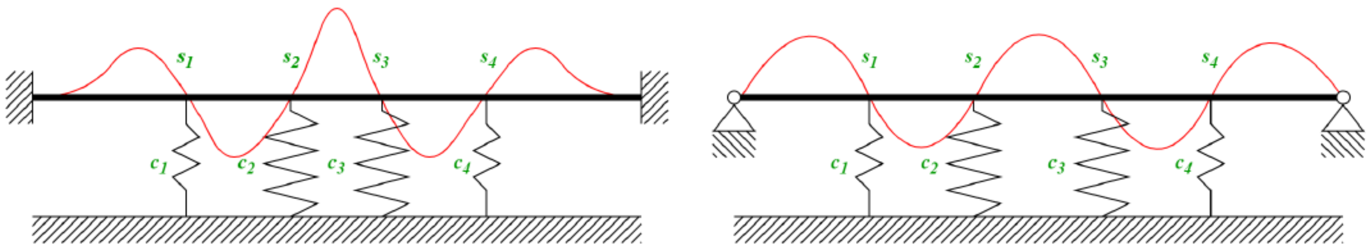


Fig 1.4 — Vibration modes of a beam supported with springs with

a). clamped, b). simply supported

ends.

The work [63] analyzed cases of non-uniform placement of springs. By iterating through options, the optimal locations of springs were determined, corresponding to the maximum value of the first eigenvalue $\alpha_1(c)$ of the boundary problems (1.3, 1.5, 1.6) and (1.4, 1.5, 1.6). It was found that as $c \to \infty$, the coordinates of the optimal placement points of the springs $s_i(c)$ tend towards certain limit values s_i^*. The optimal placement of springs at points s_i^*, corresponding to $c = \infty$, is referred to as the limit optimal placement.

In the article [63], for $n = 3, 5, 7$, it was shown that in the case of simply supported ends of the beam, the limit optimal placement of springs is their uniform placement, i.e., $s_i^* = l/n \cdot i$. The points $s_i^* = l/n \cdot i$ are the nodes of the vibration mode of the unstiffened beam

$$w_n(s) = \sin(\alpha_n s), \quad \alpha_n = \frac{n}{l} \cdot \pi. \tag{1.13}$$

The work [56] showed that in the case of rigid fixing, the nodes of the vibration mode of the unstiffened beam coincide with the points of the limit optimal placement of springs. The solution of equation (1.2) with boundary conditions (1.4) is

$$w_n(s) = [U(\alpha_n s) - \varkappa_n V(\alpha_n s)] \tag{1.14}$$

where

$$S(x) = \operatorname{ch} x + \cos x, \quad T(x) = \operatorname{sh} x + \sin x,$$

$$U(x) = \operatorname{ch} x - \cos x, \quad V(x) = \operatorname{sh} x - \sin x,$$

$$S(x) = T'(x) = U''(x) = V'''(x) = S''''(x)$$

are the Krylov function system,

$$\alpha_n = \frac{z_n}{l}, \quad \varkappa_n = \frac{\operatorname{ch} z_n - \cos z_n}{\operatorname{sh} z_n - \sin z_n},$$

and the values

$$z_n \simeq \pi(2n+1)/2, \quad n = 1,2,\ldots$$

are the roots of the equation $\operatorname{ch} z \cdot \cos z = 1$. In this case, the points of optimal placement of springs coincide with the roots of the equation $w_n(s) = 0$.

For simplicity, instead of $\alpha_0 = \frac{\pi}{2l}$, α is used, instead of $\varkappa_1 = U(\frac{3\pi}{2})/V(\frac{3\pi}{2})$, $\varkappa$ is used, and the first vibration mode of the unstiffened beam $w_1(s)$ is denoted $w(s)$.

If $c \gg 1$, then the optimal placement of rings differs little from the limit optimal placement.

1.1.4 Determining eigenvalues of vibrational frequencies of a beam supported with springs

Consider the boundary problem (1.5, 1.6) of vibrations of a beam stiffened with springs. An approximate solution to this problem in the case of simply supported beam ends and uniformly distributed identical springs is obtained in [30] using the method of averaging elastic characteristics. The formula for estimating the first eigenvalue α_1 is:

$$\alpha_1^4 = \frac{\pi^4}{l^4} + c\frac{n}{l}. \tag{1.15}$$

The averaging method is only applicable for the case of uniformly distributed identical springs. Let's consider the boundary problem of vibrations of a beam stiffened with springs of varying stiffness. Instead of the averaging method, the Rayleigh method will be used. The Rayleigh formula for a beam stiffened with springs can be written in the following dimensionless form:

$$\alpha_1^4 = \frac{I_1 + I_2}{I_0}, \tag{1.16}$$

Here I_0 is the kinetic energy of the beam (the work of the compressing force on the shell), I_1 is the energy of deformation of the beam (potential energy of bending of the shell), I_1 is the energy of deformation of the springs (deformation of the reinforcing rings of the shell).

$$I_1 = \int_0^l (w''(s))^2 ds, \quad I_2 = \sum_{i=1}^{n-1} c_i w^2(s_i), \quad I_0 = \int_0^l w^2(s) ds.$$

For I_1, the following transformations can be made:

$$I_1 = \int_0^l (w''(s))^2 ds = (w''(s)w'(s))\Big|_0^l - \int_0^l w'(s)w'''(s) ds =$$

$$= (w''(s)w'(s) - w'''(s)w(s))\Big|_0^l + \int_0^l w(s)w''''(s) ds.$$

For both the simple support (1.3) and the rigid fixation (1.4) of the beam ends, the non-integral term becomes zero, therefore

$$I_1 = \alpha^4 \int_0^l w^2(s) ds = \alpha^4 I_0.$$

In the case of simple support (1.3) of the beam ends, substitute $w = w_1$ in (1.16), where the function w_1 is defined by formula (1.13). We get

$$I_0 = \int_0^l \left(\sin\left(\frac{\pi s}{l}\right)\right)^2 ds = \frac{1}{2}\int_0^l \left(1 - \cos\left(\frac{2\pi s}{l}\right)\right) ds = \frac{l}{2},$$

$$I_1 = \alpha^4 I_0 = \frac{\pi^4}{l^4} \cdot \frac{l}{2}.$$

With (1.7)

$$I_2 = \sum_{i=1}^{n-1} c_i w^2(s_i) = c \sum_{i=1}^{n-1} f^3(i) \cdot w^2(s_i).$$

Thus, for the case of simple support of the beam ends (1.3), the formula for estimating the first eigenvalue is:

$$\alpha_s^4 = \frac{I_1}{I_0} + \frac{I_2}{I_0} = \frac{\pi^4}{l^4} + c\frac{2T_s(n)}{l}, \quad T_s(n) = \sum_{i=1}^{n-1} f^3(i) \sin^2\left(\frac{\pi i}{n}\right). \tag{1.17}$$

For the case of reinforcing the beam with springs of the same stiffness (1.12), both the averaging method and the Rayleigh method give the same approximate formula (1.15) for the first eigenvalue:

$$T_s^0(n) = \sum_{i=1}^{n-1} \sin^2\left(\frac{\pi i}{n}\right) = \frac{n}{2}, \quad \alpha_s^4 = \frac{\pi^4}{l^4} + c\frac{n}{l}.$$

Similar calculations are carried out for the case of rigidly fixed ends (1.4) of the beam: substitute $w = w_1$ in (1.16), where the function w_1 is defined by formula (1.14). Then, the values of I_0, I_1 and I_2 are calculated by the following formulas:

$$I_0 = \int_0^l (U(\alpha s) - \varkappa \cdot V(\alpha s))^2 ds, \quad I_1 = \int_0^l (S(\alpha s) - \varkappa \cdot T(\alpha s))^2 ds,$$

$$I_2 = \sum_{i=1}^{n-1} c_i w^2(s_i) = c \sum_{i=1}^{n-1} f^3(i) \cdot w^2(s_i).$$

Therefore, the value of the approximate parameter for the first frequency of vibrations of the beam with clamped ends can be calculated using the formula

$$\alpha_c^4 = \frac{I_1}{I_0} + \frac{I_2}{I_0} = \left(\frac{3\pi}{2l}\right)^4 + c\frac{T_c(n)}{I_0}, \quad T_c(n) = \sum_{i=1}^{n-1} f^3(i) w^2(s_i). \tag{1.18}$$

1.1.5 Determining eigenvalues in the problem of vibrations of a stiffened shell

Consider the problem of determining the lowest value of the frequency parameter λ_1 for a cylindrical shell with simply supported edges (1.3). The shell is stiffened with rings of stiffness c_i along parallels with coordinates s_i, which are nodes of the vibration mode of the unstiffened simply supported shell (1.13).

Denote $\alpha_s(\eta, m)$ as the eigenvalue of the boundary problem for the case of simple support (1.3, 1.5, 1.6), corresponding to the lowest frequency parameter. Denote the corresponding frequency parameter value $\lambda_1(\eta)$ as $\lambda^s(\eta)$

$$\lambda^s(\eta) = \lambda_1(\eta) = \min_m \left[\frac{\sigma\alpha_s^4(\eta{,}m)}{m^4} + \mu^4 m^4\right]. \tag{1.19}$$

Considering (1.8) and (1.17),

$$\alpha_s^4 = \frac{\pi^4}{l^4} + c\frac{2T_s(n)}{l} = \frac{\pi^4}{l^4} + \frac{m^8\mu^4 l\eta}{\sigma n}\frac{2T_s(n)}{l} = \frac{\pi^4}{l^4} + \frac{2T_s(n)\mu^4\eta}{\sigma n}m^8.$$

Substituting the obtained value of α_s^4 into (1.19) gives the following expression:

$$\lambda^s(\eta) = \min_m \left[\frac{\sigma\pi^4}{l^4}m^{-4} + \frac{2T_s(n)\mu^4\eta}{n}m^4 + \mu^4 m^4\right]$$

In general, the minimizable function $\lambda(\eta{,}m)$ takes the form:

$$\lambda(\eta{,}m) = Xm^{-4} + Ym^4, \tag{1.20}$$

where

$$X = \frac{\sigma\pi^4}{l^4}, \quad Y = \left(1 + \frac{2T_s(n)}{n}\eta\right)\mu^4.$$

Minimizing the obtained function $\lambda(\eta, m)$ with respect to m:

$$\frac{d}{dm}\lambda(\eta{,}m) = -4Xm^{-5} + 4Ym^3, \quad -4Xm_0^{-5} + 4Ym_0^3 = 0,$$

then

$$m_0^4 = \sqrt{\frac{X}{Y}}, \quad \min_m \left[Xm^{-4} + Ym^4\right] = 2\sqrt{XY}.$$

Therefore, for the lowest frequency parameter value, we have

$$\lambda^s(\eta) = 2\sqrt{\frac{\sigma\pi^4}{l^4}\cdot\left(1+\frac{2T_s(n)}{n}\eta\right)\mu^4} = \frac{2\pi^2\mu^2\sqrt{\sigma}}{l^2}\sqrt{1+\frac{2T_s(n)}{n}\eta}$$

Since $\mu^4 = h^2/12$, the value of λ^s for the case of simply supported shell edges is written as:

$$\lambda^s(\eta) = \lambda^s(0)\sqrt{1+\frac{2T_s(n)}{n}\eta}, \quad where \quad \lambda^s(0) = \frac{\pi^2 h\sqrt{\sigma}}{l^2\sqrt{3}}. \tag{1.21}$$

With an increase in η, $\lambda^s(\eta)$ also increases, but formula (1.21) cannot be used for large values of η. There exists a vibration frequency value independent of η. Indeed, the eigenvalue α_n of the boundary problem (1.3), (1.5) is also an eigenvalue of the boundary problem (1.3), (1.5), (1.6), as the vibration mode of the unstiffened shell $w_n(s)$ satisfies the boundary conditions (1.6). For the parameter λ_n^s corresponding to this eigenvalue α_n, we have:

$$\lambda_n^s = \min_m\left[\frac{\sigma\alpha_n^4}{m^4}+\mu^4 m^4\right] = 2\sqrt{\sigma\alpha_n^4\mu^4} = 2\alpha_n^2\mu^2\sqrt{\sigma} =$$
$$= \frac{2\pi^2 n^2}{l^2}\frac{h}{\sqrt{12}}\sqrt{\sigma} = \frac{\pi^2 h\sqrt{\sigma}}{l^2\sqrt{3}}n^2 = \lambda^s(0)n^2.$$

This parameter does not depend on η.

The parameter η_s^*, which is the root of the equation

$$\lambda^s(0)\sqrt{1+\frac{2T_s(n)}{n}\eta} = \lambda_n^s$$

is called the *effective stiffness of the ring.* Increasing the stiffness of the ring η after reaching the value of η_s^* does not lead to an increase in the lowest frequency parameter. Therefore,

$$\frac{\lambda^s(\eta)}{\lambda^s(0)} = \begin{cases}\sqrt{1+\frac{2T_s(n)}{n}\eta}, & \eta \leqslant \eta_s^* \\ n^2, & \eta > \eta_s^*\end{cases}, \quad \eta_s^* = \frac{n(n^4-1)}{2T_s(n)}. \tag{1.22}$$

Consider a similar problem of determining the lowest value of the frequency parameter λ^c for a cylindrical shell with rigidly fixed (1.4) edges. In this case, the coordinates of the parallels where the shell is stiffened with rings of stiffness c_i are the nodes of the vibration mode of the unstiffened rigidly fixed shell (1.14).

Substituting (1.8) into the eigenvalue (1.18) of the boundary problem(1.4, 1.5, 1.6):

$$\alpha_c^4 = \left(\frac{3\pi}{2l}\right)^4 + c\frac{T_c(n)}{I_0} = \left(\frac{3\pi}{2l}\right)^4 + \frac{m^8\mu^4 l\eta}{\sigma n}\frac{T_c(n)}{I_0} = \\ = \left(\frac{3\pi}{2l}\right)^4 + \frac{lT_c(n)\mu^4\eta}{\sigma n I_0}m^8$$

Then the smallest eigenvalue for the frequency parameter of the rigidly fixed cylindrical shell (1.19) is

$$\lambda^c(\eta) = \min_m\left[\frac{\sigma}{m^4}\left(\left(\frac{3\pi}{2l}\right)^4 + \frac{lT_c(n)\mu^4\eta}{\sigma n I_0}m^8\right) + \mu^4 m^4\right] = \\ = \min_m\left[\frac{\sigma(3\pi)^4}{(2l)^4}m^{-4} + \frac{lT_c(n)\mu^4\eta}{nI_0}m^4 + \mu^4 m^4\right].$$

After minimizing this expression with respect to m, similarly to (1.20), we obtain

$$\lambda^c(\eta) = 2\sqrt{\frac{\sigma(3\pi)^4}{(2l)^4}\cdot\left(1 + \frac{lT_c(n)\eta}{nI_0}\right)\mu^4} = \frac{9\pi^2\mu^2\sqrt{\sigma}}{2l^2}\sqrt{1 + \frac{lT_c(n)\eta}{nI_0}}.$$

or

$$\lambda^c(\eta) = \lambda^c(0)\sqrt{1 + \frac{lT_c(n)\eta}{nI_0}}, \quad \text{where} \quad \lambda^c(0) = \frac{3\pi^2 h\sqrt{3\sigma}}{4l^2}. \tag{1.23}$$

As η increases, $\lambda^c(\eta)$ also increases. However, like in the case of simply supported shell edges, formula (1.23) cannot be used for large values of η because there exists a frequency parameter λ_n^c, which is independent of η, and for which the approximate formula is:

$$\lambda_n^c = \min_m\left[\frac{\sigma\alpha_n^4}{m^4} + \mu^4 m^4\right] = 2\sqrt{\sigma\alpha_n^4\mu^4} = \frac{(2n+1)^2\pi^2}{4l^2}\frac{h\sqrt{\sigma}}{\sqrt{3}} = \\ = \frac{3\pi^2 h\sqrt{3\sigma}}{4l^2}\frac{(2n+1)^2}{3^2} = \lambda^c(0)\cdot\left(\frac{2n+1}{3}\right)^2.$$

The value of the effective stiffness of the ring η_c^* in this case is the root of the equation

$$\lambda^c(0)\sqrt{1+\frac{lT_c(n)\eta_c^*}{nI_0}} = \lambda^c(0)\cdot\left(\frac{2n+1}{3}\right)^2$$

Increasing the stiffness of the ring η after reaching the value of η_c^* does not lead to an increase in the lowest frequency parameter. Therefore:

$$\frac{\lambda^c(\eta)}{\lambda^c(0)} = \begin{cases} \sqrt{1+\frac{lT_c(n)\eta}{nI_0}}, & \eta \leqslant \eta_c^* \\ \left(\frac{2n+1}{3}\right)^2, & \eta > \eta_c^* \end{cases}, \quad \eta_c^* = \frac{nI_0}{lT_c(n)}\left(\left(\frac{2n+1}{3}\right)^4 - 1\right). \tag{1.24}$$

1.2 Optimization of parameters of a stiffened cylindrical shell for maximum increase of the first frequency

Let's assume the mass of the stiffened shell is fixed. We consider the problem of determining the optimal distribution of mass between the rings and the shell (cladding) that corresponds to the highest value of the first frequency.

The lowest vibration frequency ω_0 of a cylindrical shell with simply supported edges, having a non-dimensional length l and thickness h_0, can be determined using the approximate formula, assuming $\lambda = \lambda_s(0)$ (1.21)

$$\omega_s^0 = \sqrt{\frac{E\lambda^s(0)}{\rho R^2\sigma}} = \sqrt{\frac{E}{\rho R^2\sigma}\frac{\pi^2 h_0\sqrt{\sigma}}{l^2\sqrt{3}}} = \sqrt{\frac{Eh_0\pi^2}{\rho R^2 l^2\sqrt{3\sigma}}}. \tag{1.25}$$

Assume that due to the reduction of the shell thickness to h, $n-1$ rings are installed on it. Assume that the rings have a rectangular cross-section, equal width a, and height $b_i (i = 1,\ldots,n-1)$. The mass of the stiffened shell, in this case,

$$M^s = M(h) + M^r, \tag{1.26}$$

where $M(h) = \rho 2\pi R\cdot Rh\cdot Rl$ — mass of the cladding, and the mass of the rings

$$M^r = \sum_{i=1}^{n-1}\rho 2\pi R\cdot aR\cdot b_i R = 2\pi R^3\rho\cdot a^2 k\cdot P(n), \quad P(n) = \sum_{i=1}^{n-1} f(i).$$

Let the mass of the stiffened shell be equal to the mass of the smooth shell $M_0 = M(h_0)$.

To determine the first frequency of vibration of the stiffened shell ω_s, we use formulas (1.21) and (1.25). We introduce a function of the ratio of the first frequency of vibration of the stiffened shell to the first frequency of vibration of the smooth shell. In the case of simply supported shell edges considering (1.22), we obtain the following expression

$$f_s = \frac{\omega_s}{\omega_s^0} = \begin{cases} \sqrt[4]{1 + \frac{2T_s(n)}{n}\eta} \cdot \sqrt{d}, & \eta \leqslant \eta_s^*, \\ n \cdot \sqrt{d}, & \eta_s^* < \eta, \end{cases}, \quad \eta_s^* = \frac{n(n^4-1)}{2T_s(n)}, \quad d = \frac{h}{h_0}.$$

Transforming the second term under the square root considering the value of η from (1.8)

$$\frac{2T_s(n)}{n}\eta = \frac{2T_s(n)}{n}\frac{12\sigma n}{h^3 l}J = \frac{2T_s(n)}{n}\frac{12\sigma n}{h^3 l}\frac{a^4 k^3}{12}\frac{h_0^3}{h_0^3} = \frac{2T_s(n)\sigma k^3}{l h_0^3}\cdot\frac{a^4}{d^3}.$$

Therefore

$$f_s = \begin{cases} n \cdot \sqrt{d}, & 0 < d \leqslant d_s \\ \sqrt[4]{1 + \frac{B_s a^4}{d^3}} \cdot \sqrt{d}, & d_s < d \leqslant 1 \end{cases}, \quad B_s = \frac{2\sigma k^3}{l h_0^3} \cdot T_s(n). \tag{1.27}$$

Consider the condition of mass equality of the stiffened shell and the smooth shell (1.26):

$$2\pi R^3 \rho \left(hl + a^2 k P(n)\right) = 2\pi R^3 \rho h_0 l,$$

$$hl + a^2 k P(n) = h_0 l.$$

From which

$$a^2 = \frac{1-d}{A}, \quad \text{where} \quad A = \frac{k}{h_0 l}P(n).$$

Using the expression for a^2, the function $f_s^2(d)$ can be represented as follows:

$$f_s(d) = \begin{cases} n \cdot \sqrt{d}, & 0 < d \leqslant d_s \\ \sqrt[4]{1 + \frac{B_s}{A^2} \cdot \frac{(1-d)^2}{d^3}} \cdot \sqrt{d}, & d_s < d \leqslant 1 \end{cases},$$

From the continuity condition of the function f_s at d_s, we have

$$n^4 - 1 = \frac{B_s}{A^2} \cdot \frac{(1-d_s)^2}{d_s^3},$$

from which

$$d_s^3 - \frac{B_s}{A^2(n^4-1)}(1-d_s)^2 = 0,$$

or

$$d_s^3 - \frac{2\sigma k l}{h_0} \cdot \frac{T_s(n)}{P^2(n)(n^4-1)} \cdot (d_s - 1)^2 = 0. \qquad (1.28)$$

The value of d_s is determined as the root of this equation from the interval $[0, 1]$, which corresponds to the maximum first frequency of vibration of the simply supported shell, stiffened with rings of various stiffness.

The non-dimensional width of the rings a_s and the value of the target function f_s^*, corresponding to d_s, are found by the formulas:

$$a_s = \sqrt{\frac{1-d_s}{A}}, \quad f_s^* = n \cdot \sqrt{d_s}.$$

In the problem of vibrations of a stiffened cylindrical shell with rigidly fixed edges, a similar mass optimization can be performed. The lowest vibration frequency ω_0 of a cylindrical shell with clamped edges, having a non-dimensional length l and thickness h_0, can be determined by the approximate formula (assuming $\lambda = \lambda_c(0)$ according to (1.23))

$$\omega_c^0 = \sqrt{\frac{E\lambda^c(0)}{\rho R^2 \sigma}} = \sqrt{\frac{E}{\rho R^2 \sigma} \frac{3\pi^2 h_0 \sqrt{3\sigma}}{4l^2}} = \sqrt{\frac{9Eh_0\pi^2}{4\rho R^2 l^2 \sqrt{3\sigma}}}. \qquad (1.29)$$

To determine the first frequency of vibration of the stiffened shell w_c, we can use formulas (1.23) and (1.29). Considering the relationship (1.24), the function of the ratio of the first frequency of vibration of the stiffened shell to the first frequency of vibration of the smooth shell f_c will have the following form:

$$f_c = \frac{\omega_c}{\omega_c^0} = \begin{cases} \sqrt[4]{1 + \frac{lT_c(n)\eta}{nI_0}} \cdot \sqrt{d}, & \eta \leqslant \eta_c^* \\ \frac{2n+1}{3} \cdot \sqrt{d}, & \eta > \eta_c^* \end{cases}, \quad \eta_c^* = \frac{nI_0}{lT_c(n)} \left(\left(\frac{2n+1}{3} \right)^4 - 1 \right).$$

Let's perform similar transformations of the second term under the square root:

$$\frac{lT_c(n)\eta}{nI_0} = \frac{lT_c(n)}{nI_0}\frac{12\sigma n}{h^3 l}J = \frac{lT_c(n)}{nI_0}\frac{12\sigma n}{h_0^3 d^3 l}\frac{a^4 k^3}{12} = \frac{T_c(n)\sigma k^3}{I_0 h_0^3}\frac{a^4}{d^3},$$

then

$$f_c = \begin{cases} \frac{2n+1}{3}\cdot\sqrt{d}, & 0 < d \leqslant d_c \\ \sqrt[4]{1+\frac{B_c a^4}{k^3}}\cdot\sqrt{d}, & d_c < d \leqslant 1 \end{cases}, \quad B_c = \frac{\sigma k^3}{I_0 h_0^3}\cdot T_c(n).$$

Considering the equality of mass of the shell, stiffened with rings, and the smooth shell

$$f_c = \begin{cases} \frac{2n+1}{3}\cdot\sqrt{d}, & 0 < d \leqslant d_c \\ \sqrt[4]{1+\frac{B_c}{A^2}\cdot\frac{(1-d)^2}{d^3}}\cdot\sqrt{d}, & d_c < d \leqslant 1 \end{cases}, \quad a^2 = \frac{1-d}{A}, \quad A = \frac{kP(n)}{h_0 l}.$$

Where d_c satisfies the following equation:

$$\left(\frac{2n+1}{3}\right)^4 = 1 + \frac{B_c}{A^2}\cdot\frac{(1-d_c)^2}{d_c^3},$$

from which

$$d_c^3 - \frac{B_c}{A^2\left(\left(\frac{2n+1}{3}\right)^4 - 1\right)}(1-d_c)^2 = 0,$$

or

$$d_c^3 - \frac{\sigma k l^2}{I_0 h_0}\frac{T_c(n)}{P^2(n)\left(\left(\frac{2n+1}{3}\right)^4 - 1\right)}(1-d_c)^2 = 0. \tag{1.30}$$

The non-dimensional width of the rings a_c and the target function value f_c^*, corresponding to d_c, can be found by the following formulas:

$$a_c = \sqrt{\frac{1-d_c}{A}}, \quad f_c^* = \frac{2n+1}{3}\cdot\sqrt{d_c}.$$

1.3 Analytical determination of the fundamental frequency of a stiffened shell

Using numerical and asymptotic methods, the spectrum of natural frequencies of a stiffened shell has been obtained. Two types of vibrations have been identified.

The modes of the first type of vibrations have a large number of waves in the circumferential direction, similar to the modes of an unstiffened cylindrical shell. The natural frequencies and modes of the second type are similar to the frequencies and modes of vibrations of a circular plate.

In this section, the "shell"frequencies are determined analytically and numerically. As an example, consider a copper cylindrical shell with a length of $l = 4$ and a thickness of $h_0 = 0.01$, with Young's modulus $E = 11 \cdot 10^{10}$ Pa, Poisson's ratio $\nu = 0.35$, and density $\rho = 8920$ kg/m^3. The approximate value of the fundamental frequency of vibration (ω_0) of the unstiffened shell is calculated using formula (1.25) in the case of simple edge support, and using formula (1.29) in the case of rigid fixation. The corresponding values ω_s^0 and ω_c^0 are as follows

$$\omega_s^0 = \sqrt{\frac{Eh_0}{R^2 l^2 \sqrt{3\sigma}}} \simeq 216.5 \ \frac{\text{rad}}{\text{s}}, \quad \omega_c^0 = \sqrt{\frac{9Eh_0}{4R^2 l^2 \sqrt{3\sigma}}} \simeq 324.8 \ \frac{\text{rad}}{\text{s}}.$$

In the case of reinforcing the shell with n_s rings of width a and heights $b_i = kaf(i)(i = 1 \ldots n_s)$, the function $f(i)$ determines the distribution of ring heights along the shell's generatrix, and thus the distribution of ring stiffness and the construction profile. For the case of linear distribution

$$f_{lin}(i) = (\kappa(i) - 1)(u - 1) + 1, \quad u = \frac{b_2}{b_1},$$

where

$$\kappa(i) = \frac{n}{2} - \left|\frac{n}{2} - i\right| = \begin{cases} i, & i < \frac{n}{2} \\ n - i, & i \geqslant \frac{n}{2} \end{cases}, \quad n = n_s + 1.$$

For the case of parabolic distribution

$$f_{parab}(i) = a_p \kappa^2(i) - n a_p \kappa(i) + n a_p - a_p + 1, \quad where \quad a_p = \frac{1 - u}{n - 3},$$

and for the case of exponential distribution

$$f_{exp}(i) = \frac{u - 1}{e^2 - e} e^{\kappa(i)} + \frac{e - u}{e - 1}.$$

For a shell stiffened with rings of equal height, $u = 1$.

The approximate value of the "shell"frequency of such a structure can be obtained using the formula $\omega_s = \omega_s^0 f_s$ in the case of simple edge support, and using the formula $\omega_c = \omega_c^0 f_c$ in the case of rigid edge fixation. The values of f_s and f_c for different construction profiles are given in Tables 1.1 and 1.2. Figures 1.5 and 1.6 show the dependencies of the functions f_s and f_c on the number of rings for various ratios of ring width and height.

Based on the obtained results, it can be concluded that using unequal rings to stiffen a cylindrical shell leads to a more significant increase in its first natural frequency compared to stiffening with equal rings. Among the three construction profiles considered, the profile with an exponential law of height distribution of rings provides the greatest increase in the first frequency. Furthermore, it can be noted that fixing the edges of the shell leads to higher fundamental frequencies than in the case of simple edge support. These results underscore the importance of considering both the type of construction profile and the boundary conditions when improving constructions with cylindrical shells.

1.4 Natural vibrations of an annular plate

As can be seen from the results of the previous section, the first frequency of shell vibrations increases with the height of the rings. In most studies on the vibrations of cylindrical shells stiffened with rings, the rod model of the ring is used, as the natural frequency of the ring's vibrations exceeds that of the shell when the height is small. However, as the height increases, the natural frequency of the ring's vibrations decreases, leading to a decrease in the frequency of vibrations of the entire structure. Therefore, when the ratio of the ring's height to its width $k = b/a$ is large, to obtain more accurate results, rings should be considered as annular plates instead of using the

Table 1.1 — Values of the function f_s for a simply supported shell stiffened with n_s rings.

		$f_{lin}(i)$			$f_{parab}(i)$			$f_{exp}(i)$		
	n_s \ u	1	2	3	1	2	3	1	2	3
$k = 1$	4	3,16	3,41	3,51	3,16	3,41	3,51	3,16	3,41	3,51
	5	3,36	3,66	3,90	3,36	3,66	3,82	3,36	3,66	4,13
	6	3,51	3,85	4,17	3,51	3,85	4,08	3,51	3,85	4,45
	7	3,64	4,00	4,47	3,64	4,00	4,30	3,64	4,00	5,32
	8	3,74	4,13	4,68	3,74	4,13	4,50	3,74	4,13	5,64
$k = 1,5$	4	3,61	3,66	3,64	3,61	3,66	3,64	3,61	3,66	3,64
	5	4,02	4,14	4,14	3,94	4,02	3,99	4,25	4,45	4,49
	6	4,31	4,45	4,44	4,22	4,32	4,29	4,60	4,82	4,84
	7	4,63	4,83	4,83	4,46	4,59	4,56	5,49	5,82	5,86
	8	4,86	5,07	5,07	4,68	4,83	4,79	5,83	6,17	6,20
$k = 2$	4	3,77	3,73	3,95	3,77	3,73	3,95	3,77	3,73	3,95
	5	4,31	4,27	4,55	4,16	4,10	4,40	4,65	4,63	4,89
	6	4,65	4,59	4,94	4,50	4,43	4,78	5,04	5,00	5,33
	7	5,06	5,00	5,40	4,79	4,71	5,11	6,08	6,04	6,40
	8	5,33	5,26	5,71	5,05	4,97	5,41	6,46	6,40	6,83

ring rod model. The optimal use of rings is when the first frequency of their vibrations coincides with the first frequency of shell vibrations.

Figure 1.7 shows a cylindrical shell stiffened in the middle with an annular plate. As before, the radius of the cylindrical shell is chosen as the unit of length $R = 1$.

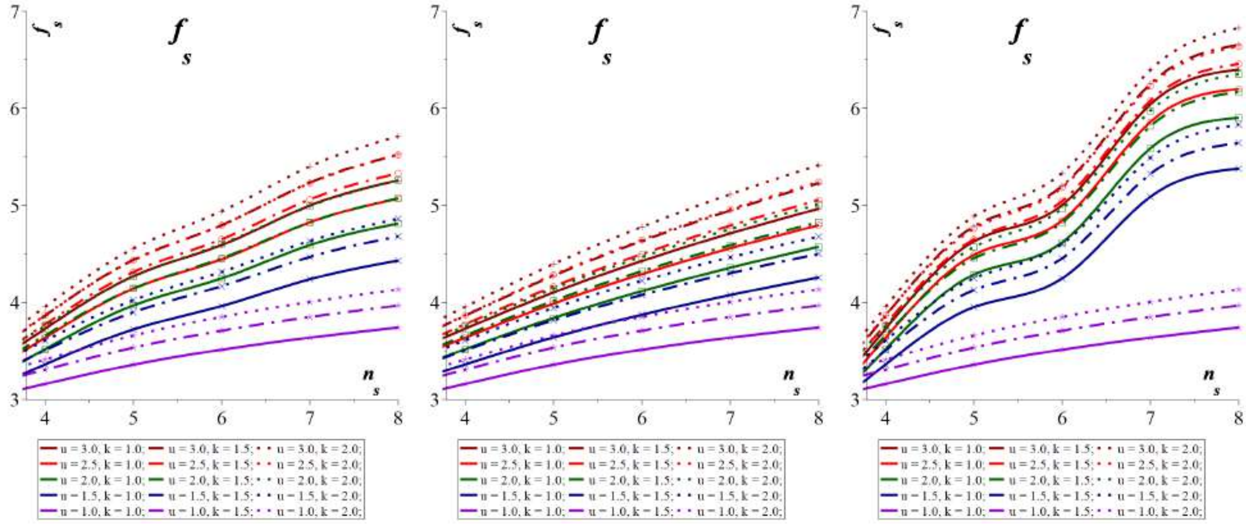


Fig 1.5 — Values of the function f_s for a) linear, b) parabolic, c) exponential construction profiles.

Numerical results show that for large values of k, the low-frequency vibrations are axisymmetric, and the lower frequencies of the stiffened shell are close to the frequencies corresponding to the bending vibrations of the plate.

The dimensionless equations describing axisymmetric vibrations of the cylindrical shell are as follows:

$$\mu^4 \frac{d^4 w}{ds^4} + \nu \frac{du}{ds} + w = \lambda w, \quad \frac{d^2 u}{ds^2} + \nu \frac{dw}{ds} = -\lambda u, \tag{1.31}$$

Here $s \in [0, l]$ is the longitudinal coordinate, (u, w) are the displacement components (u is axial, w is normal), $\mu^4 = h^2/12$ is a small parameter, h is the shell thickness, ν is the Poisson's ratio, and $\lambda = \rho \omega^2 R^2 E^{-1}$ is the frequency parameter.

Axisymmetric vibrations of an annular plate are described by the dimensionless equation:

$$\Delta^2 w_p = \gamma^4 \cdot w_p, \quad \Delta w_p = \frac{1}{s_p} \frac{d}{ds_p} \left(s_p \frac{dw_p}{s_p} \right), \quad \gamma^4 = \frac{12\lambda}{a^2}, \tag{1.32}$$

where $s_p \in [1, 1+b]$ is the radial coordinate, and w_p is the displacement along the normal.

Table 1.2 — Values of the function f_c for a rigidly fixed shell stiffened with n_s rings.

		$f_{lin}(i)$			$f_{parab}(i)$			$f_{exp}(i)$		
	n_s \ u	1	1,5	2	1	1,5	2	1	1,5	2
$k=1$	4	2,67	2,77	2,84	2,67	2,77	2,84	2,67	2,77	2,84
	5	2,86	2,99	3,08	2,86	2,99	3,08	2,86	2,99	3,08
	6	3,01	3,16	3,26	3,01	3,16	3,26	3,01	3,16	3,26
	7	3,13	3,30	3,42	3,13	3,30	3,42	3,13	3,30	3,42
	8	3,23	3,41	3,54	3,23	3,41	3,54	3,23	3,41	3,54
$k=1,5$	4	2,82	2,91	2,98	2,82	2,91	2,98	2,82	2,91	2,98
	5	3,14	3,26	3,35	3,08	3,20	3,29	3,31	3,42	3,50
	6	3,37	3,52	3,62	3,30	3,45	3,55	3,58	3,72	3,82
	7	3,63	3,80	3,91	3,50	3,66	3,78	4,25	4,40	4,50
	8	3,81	4,00	4,13	3,67	3,85	3,99	4,52	4,70	4,82
$k=2$	4	2,93	3,01	3,07	2,93	3,01	3,07	2,93	3,01	3,07
	5	3,32	3,43	3,50	3,22	3,34	3,42	3,52	3,62	3,69
	6	3,58	3,72	3,82	3,48	3,62	3,72	3,84	3,96	4,05
	7	3,89	4,05	4,17	3,71	3,88	3,99	4,56	4,70	4,78
	8	4,11	4,29	4,42	3,91	4,10	4,23	4,87	5,03	5,14

Assuming that the edge of the plate ($s_p = 1 + b$) is free, we write the boundary conditions as follows:

$$M_p = Q_p = 0 \quad \text{at} \quad s_p = 1 + b, \tag{1.33}$$

where the bending moment M_p and the shearing force Q_p are defined as:

$$M_p = \frac{d^2 w_p}{ds_p^2} + \frac{\nu}{s_p}\frac{dw_p}{ds_p}, \quad Q_p = \frac{d^3 w_p}{ds_p^3} + \frac{1}{s_p}\frac{d^2 w_p}{ds_p^2} - \frac{1}{s_p^2}\frac{dw_p}{ds_p}.$$

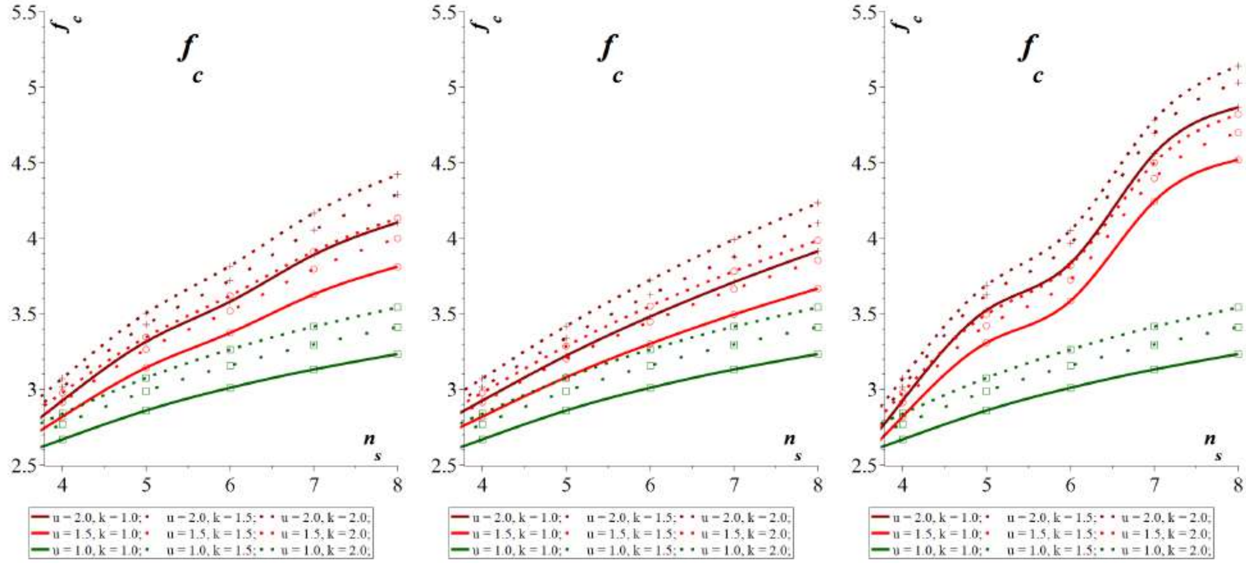


Fig 1.6 — Values of the function f_c for a) linear, b) parabolic, c) exponential construction profiles.

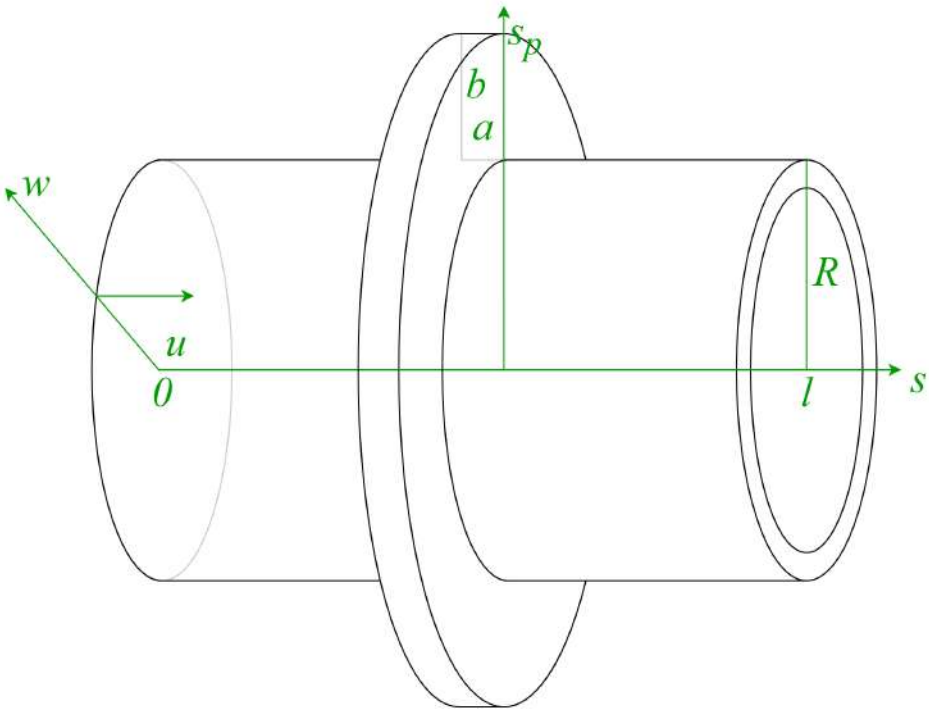


Fig 1.7 — Cylindrical shell conjugated with an annular plate.

In the study [57], it is shown that in the case when the width of the ring is close to the thickness of the shell $a \sim h$, in the first approximation, the conjugation boundary conditions (1.34) can be written as:

$$w_p(1) = 0, \quad \frac{dw_p}{ds_p}(1) = 0. \tag{1.34}$$

In general, the exact solution of the system (1.32), (1.33), (1.34) is represented through Bessel functions:

$$w_p = C_1 J_0(\gamma s_p) + C_2 Y_0(\gamma s_p) + C_3 I_0(\gamma s_p) + C_4 K_0(\gamma s_p), \tag{1.35}$$

where $\{C_i\}_{i=1}^4$ are arbitrary constants, J_0, Y_0 are Bessel functions of the first and second kind, I_0, K_0 are modified Bessel functions. In general, to find the parameter γ, it is necessary to substitute the solution (1.35) into the boundary conditions (1.33) and (1.34). The condition for the existence of a non-trivial solution is the zeroing of the main determinant of the system of linear equations. The resulting complex equation can only be solved numerically.

However, in the case of conjugation with a sufficiently narrow ($b \ll 1$) plate, a simple approximate formula for γ can be obtained.

After changing the variable $s_p = 1 + bx$ in equation (1.32) and boundary conditions (1.33), (1.34) and neglecting small terms, we obtain an eigenvalue problem:

$$\frac{d^4 w}{dx^4} - \beta^4 w = 0, \quad \text{where} \quad \beta = b\gamma, \tag{1.36}$$

$$w(0) = \frac{dw}{dx}(0) = \frac{d^2 w}{dx^2}(1) = \frac{d^3 w}{dx^3}(1) = 0. \tag{1.37}$$

The solution to equation (1.36) takes the form:

$$w = D_1 \sin(\beta x) + D_2 \cos(\beta x) + D_3 \operatorname{sh}(\beta x) + D_4 \operatorname{ch}(\beta x). \tag{1.38}$$

Substituting (1.38) into (1.37) results in a system of homogeneous equations for $\{D_i\}_{i=1}^4$, which has non-trivial solutions when the main determinant equals zero:

$$\operatorname{sh}(\beta)\cos(\beta) + 1 = 0.$$

The smallest root of this equation $\beta_0 \simeq 1.875$ corresponds to the first frequency. An approximate value of the parameter γ can be found using the formula $\gamma_0 = 1.875/b$.

For $a \ll b \ll 1$ and $a \sim h$, an approximate value of the frequency parameter λ_1, using (1.32) and (1.5),

$$\beta = b\gamma, \quad 12\lambda = \gamma^4 a^2,$$

can be obtained by the formula:

$$\lambda_1 \simeq 1.03\frac{a^2}{b^4}. \tag{1.39}$$

Next, we will use the obtained formula when solving the problem of choosing optimal coefficients for the distribution function.

1.5 Analytical and numerical evaluation of the fundamental frequency of the structure's vibrations

The lower part of the frequency spectrum of a cylindrical shell with annular stiffness ribs is formed by the eigenvalues (1.21) and (1.39) in the case of simply supported shell edges, and by the eigenvalues (1.23) and (1.39) in the case of rigidly fixed edges. Numerical results presented in Tables 1.1 and 1.2 show that as the parameter u increases, so does the first "shell"frequency of the structure. The corresponding vibration mode is shown in Figure 1.8.

However, as follows from the approximate formula (1.39), the first "plate"frequency decreases at the same time. The vibration mode corresponding to the first "plate"frequency is shown in Figure 1.9.

The optimal value of the parameter u is one where the first "shell"and "plate"frequencies coincide.

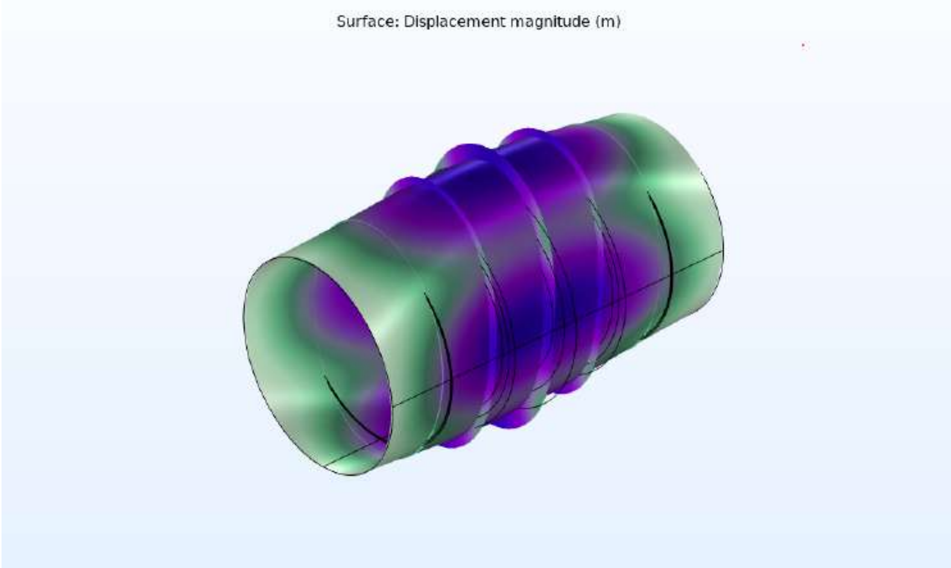


Fig 1.8 — "Shell"vibration mode of a cylindrical shell with thickness $h = 0.01$, length $l = 4$, stiffened with 5 rings for the linear distribution function case with $u = 7$.

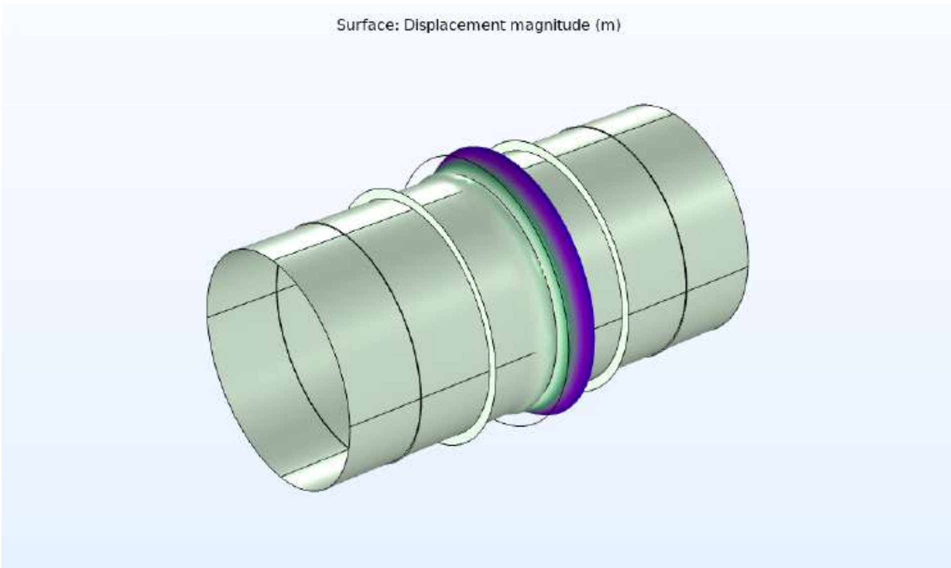


Fig 1.9 — "Plate"vibration mode of a cylindrical shell with thickness $h = 0.01$, length $l = 4$, stiffened with 5 rings for the linear distribution function case with $u = 8$.

Asymptotic formulas (1.22) and (1.24) for the eigenvalue of the first frequency of shell vibrations provide a good approximation for thin shells ($h \sim 0.01$) of medium length ($3 < l < 15$).

As the length of the shell increases, the frequency corresponding to the "beam"vibrations of the shell becomes minimal. That is, a long cylindrical shell should be modeled as a rod of constant cross-section, stiffened with springs. The vibration mode corresponding to the fundamental "beam"frequency is shown in Figure 1.10).

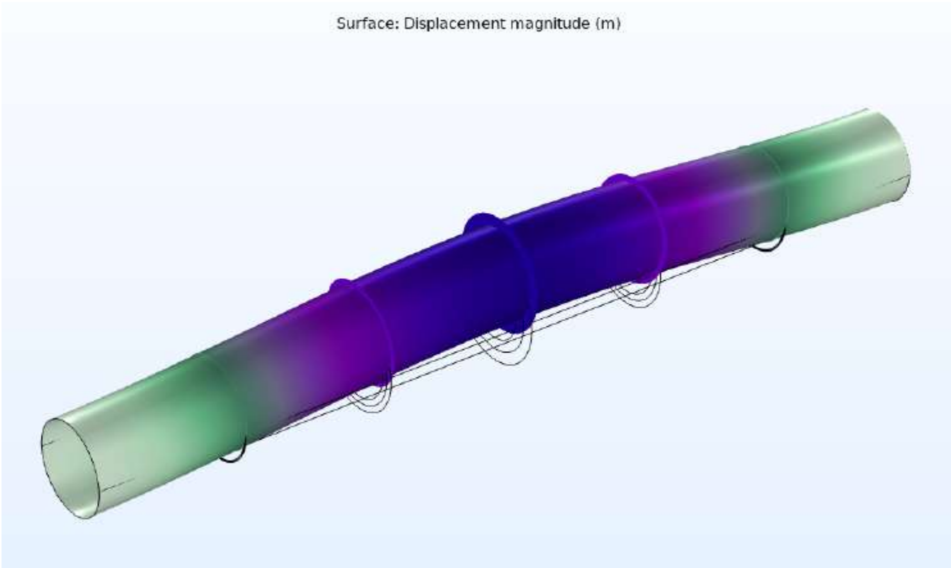


Fig 1.10 — "Beam"vibration mode of a cylindrical shell with thickness $h = 0.01$, length $l = 20$, stiffened with 5 rings for the linear distribution function case with $u = 7$.

For short lengths of the shell, formulas (1.21) and (1.23) lose their accuracy, as vibrations do not decay towards the ends of the intervals between installed rings.

With a high variability of the structure's profile (u), depending on the thickness and length of the shell, the vibration mode of the shell is localized in the vicinity of the first ring (Figure 1.11). In this case, the fundamental frequency of the structure will be the first frequency of vibrations of the shell segment, stiffened by the first ring with an elastic fixture at the location of the second ring. The condition of elastic fixture occupies an intermediate position between rigid fixture and simple support; in the future, it will be chosen the same as at the edge of the shell.

Below are the optimal values of the parameter u for some values of the construction's geometric parameters (k, n_s). For example, consider a copper cylindrical shell with a length of $l = 4$ and a thickness of $h_0 = 0.01$, with a Young's modulus of

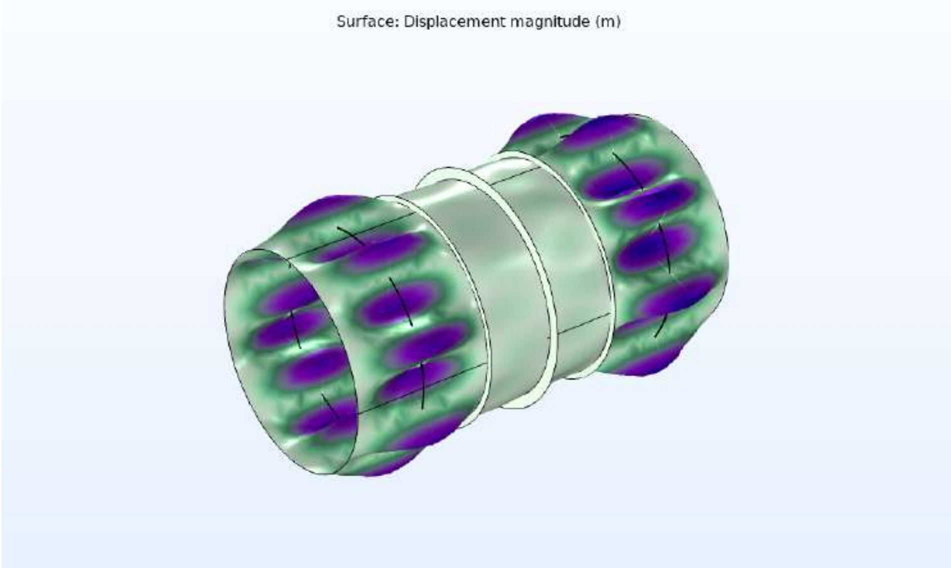


Fig 1.11 — Localization of "shell"vibration mode of a structure with high profile variability u.

$E = 11 \cdot 10^{10}$ Pa, Poisson's ratio $\nu = 0.35$, and density $\rho = 8920$ kg/m^3. Numerical results were obtained using the finite element method in Ansys and Comsol packages.

For different construction profiles, optimal values of the parameter u and their corresponding values of the function f_s (for the case of simply supported shell edges) and the function f_c (for the case of rigid fixation) are shown in Tables 1.3 and 1.4.

Table 1.5 for the optimal value of the parameter u shows the values of cyclic oscillation frequencies ω_{asymp} obtained by the asymptotic method described in the chapter, and the corresponding values ω_{fea} obtained by the finite-state method elements in the *Ansys* package for linear, parabolic and exponential profiles of the structure. Note that for $n_s = 3$ and $n_s = 4$ the profiles are the same.

The figure 1.12 for the case of shell stiffening with three rings shows the vibration shape of the optimal structure corresponding to its fundamental frequency. As you can see, both the shell and the largest ring are subject to vibrations, that is, a further decrease in the thickness of the shell will lead to a decrease in the first frequency of the «shell» series of frequencies, and a decrease in thickness or an increase in the height of the ring will lead to a decrease in the first frequency of the «plate» series of frequencies. Maximizing the fundamental frequency is important for several reasons. Firstly, it minimizes the likelihood of resonance, which can occur when the frequency of the applied load coincides with the structure's natural frequency, leading to excessive vibrations and destruction of the structure. Secondly, having equal or nearly equal

Table 1.3 — Optimal values of the parameter u and the corresponding values of the function f_s for a simply supported shell with linear (f_{lin}), parabolic (f_{parab}) and exponential (f_{exp}) height distributions n_s rings.

		u	f_s	u	f_s	u	f_s	u	f_s	u	f_s
	n_s	$k=1$		$k=1{,}5$		$k=2$		$k=2{,}5$		$k=3$	
f_{lin}	4	14,52	4,32	9,74	4,31	7,35	4,29	5,92	4,28	4,96	4,27
	5	6,57	4,72	4,61	4,69	3,63	4,67	3,03	4,65	2,64	4,63
	6	6,59	5,12	4,60	5,10	3,61	5,07	3,02	5,05	2,62	5,03
	7	4,43	5,33	3,22	5,30	2,61	5,27	2,25	5,25	2,00	5,23
	8	4,50	5,63	3,26	5,60	2,64	5,58	2,26	5,55	2,01	5,53
f_{parab}	4	14,52	4,32	9,74	4,31	7,35	4,29	5,92	4,28	4,96	4,27
	5	9,93	4,72	6,77	4,70	5,18	4,69	4,23	4,67	3,60	4,66
	6	8,69	5,10	5,95	5,08	4,59	5,06	3,77	5,04	3,22	5,03
	7	7,17	5,35	4,97	5,34	3,88	5,32	3,22	5,30	2,77	5,29
	8	6,51	5,61	4,55	5,59	3,57	5,57	2,97	5,55	2,58	5,54
f_{exp}	4	14,52	4,32	9,74	4,31	7,35	4,29	5,92	4,28	4,96	4,27
	5	3,81	4,78	2,82	4,74	2,33	4,71	2,03	4,67	1,83	4,65
	6	3,87	5,19	2,85	5,16	2,35	5,12	2,04	5,10	1,84	5,07
	7	1,81	5,44	1,53	5,37	1,39	5,31	1,30	5,26	1,25	5,22
	8	1,85	5,79	1,55	5,72	1,40	5,67	1,31	5,62	1,25	5,58

natural frequencies for different vibration modes can lead to a more balanced response of the structure to external loads, leading to increased stability and durability, reducing the risk of destruction. Finally, equal or nearly equal natural frequencies can also lead to improved dynamic characteristics, such as reducing the level of vibrations and noise. This is particularly important in applications such as aerospace, transportation, and

Table 1.4 — Optimal values of the parameter u and the corresponding values of the function f_c for a rigidly fixed shell with linear (f_{lin}), parabolic (f_{parab}) and exponential (f_{exp}) height distributions n_s rings.

		u	f_c	u	f_c	u	f_c	u	f_c	u	f_c
	n_s	$k = 1$		$k = 1{,}5$		$k = 2$		$k = 2{,}5$		$k = 3$	
f_{lin}	4	15,64	3,39	10,47	3,38	7,88	3,38	6,33	3,37	5,29	3,36
	5	6,76	3,78	4,72	3,76	3,70	3,75	3,09	3,74	2,68	3,73
	6	6,60	4,14	4,60	4,13	3,61	4,11	3,01	4,10	2,61	4,09
	7	4,31	4,38	3,14	4,36	2,55	4,34	2,19	4,32	1,95	4,30
	8	4,32	4,66	3,13	4,63	2,54	4,61	2,18	4,60	1,94	4,58
f_{parab}	4	15,64	3,39	10,47	3,38	7,88	3,38	6,33	3,37	5,29	3,36
	5	10,21	3,77	6,94	3,76	5,31	3,75	4,33	3,74	3,67	3,74
	6	8,68	4,12	5,94	4,11	4,57	4,10	3,74	4,09	3,20	4,08
	7	6,97	4,38	4,83	4,37	3,77	4,36	3,12	4,34	2,70	4,33
	8	6,22	4,63	4,35	4,62	3,41	4,60	2,85	4,59	2,47	4,57
f_{exp}	4	15,64	3,39	10,47	3,38	7,88	3,38	6,33	3,37	5,29	3,36
	5	3,93	3,82	2,90	3,80	2,38	3,77	2,07	3,76	1,86	3,74
	6	3,90	4,19	2,87	4,17	2,35	4,15	2,04	4,13	1,83	4,11
	7	1,79	4,46	1,52	4,41	1,38	4,37	1,29	4,33	1,23	4,30
	8	1,82	4,77	1,53	4,73	1,38	4,69	1,30	4,65	1,24	4,62

industrial equipment, where structures must operate reliably and quietly under high loads.

The obtained distribution function parameters can be used for preliminary design. Refinement of parameters can be conducted using the finite element method.

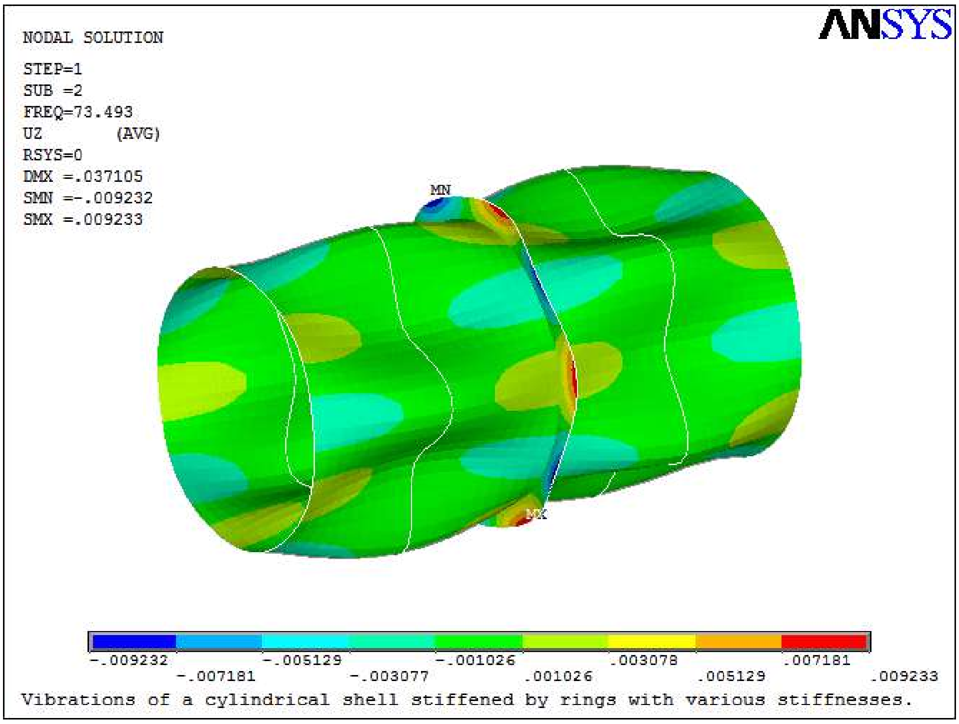


Fig 1.12 — The vibration form of a cylindrical shell supported by three unequal rings.

Table 1.5 — Analytical and numerical values of vibration frequencies of ω stiffened shells.

	$n_s = 3$		$n_s = 4$		$n_s = 5$		$n_s = 6$	
	ω_{asymp}	ω_{fea}	ω_{asymp}	ω_{fea}	ω_{asymp}	ω_{fea}	ω_{asymp}	ω_{fea}
f_{lin}	67,01	73,49	72,72	78,07	76,59	82,02	80,16	86,71
f_{parab}	67,01	73,49	72,72	78,07	76,54	84,16	79,94	87,06
f_{exp}	67,01	73,49	72,72	78,07	77,2	82,58	80,72	87,97

2 Natural vibrations of a cylindrical shell joined with a plate at the edge

In this chapter, the lowest natural frequencies and vibration modes of a structure consisting of a closed circular cylindrical shell with an attached end cap are investigated. Numerical and asymptotic methods are used to study caps in the form of a circular plate and a shallow spherical segment. Three types of natural vibrations of the structure are identified. The natural frequencies and vibration modes of the first type are close to the frequencies and modes of vibrations of a shallow spherical shell, the second type to those of a cylindrical shell, and the third type to the frequencies and modes of vibrations of a cantilever beam with a load at the end. Approximate values for the frequencies of all types are found using asymptotic methods.

Two optimization problems are solved. In the first, the optimal ratio of the thicknesses of the plate and shell is evaluated to ensure the maximum value of the fundamental frequency of the structure with a given mass. In the second optimization problem, the values of the relative thickness of the structural elements and the curvature of the end cap are found, at which the fundamental natural frequency of the structure is maximized.

2.1 Numerical results

In this chapter, the frequencies and modes of vibrations of a structure consisting of a thin circular cylindrical shell of radius R, length L, and thickness H_s with a rigidly attached end cap in the form of a spherical segment of thickness H_p and curvature radius R_p are determined using asymptotic and numerical methods. The other edge of the shell is rigidly fixed. The shell and cap are made of isotropic material with Young's modulus E, Poisson's ratio ν, and density ρ. At $R_p = \infty$, the shell degenerates into a plate, and at $R_p = R$, into a shell of hemispherical shape.

The solution begins with a numerical study, which will serve as a benchmark for asymptotic analysis. Numerical analysis of the structure was performed using the finite element package *Comsol v*5.6 with the following parameters: $R = 1$ m, $L = 4$ m, $H = 0.01$ m — the thickness of the shells, $R_p = 10$ m, $E = 110$ GPa, $\nu = 0.35$, $\rho = 8{,}960$ kg/m^3. The convergence of the method on different meshes was considered. The best convergence is provided by a mapped mesh with element sizes in the range $[0.008, 0.08]$. Further reduction in element size has a minimal effect on the results: the relative change in the frequency spectrum does not exceed 1%.

The values of the natural vibration frequency parameter

$$\Omega = \sqrt{\omega}\left(\frac{12\rho\sigma R^4}{EH^2}\right)^{1/4},$$

where $\sigma = 1-\nu^2$ and f, $\omega = 2\pi f$ — respectively the frequency and cyclic frequency of vibrations, are shown in Figure 2.1 for caps in the form of a circular plate ($1/R_p = 0$) and in the form of a spherical segment ($R_p = 10$ m).

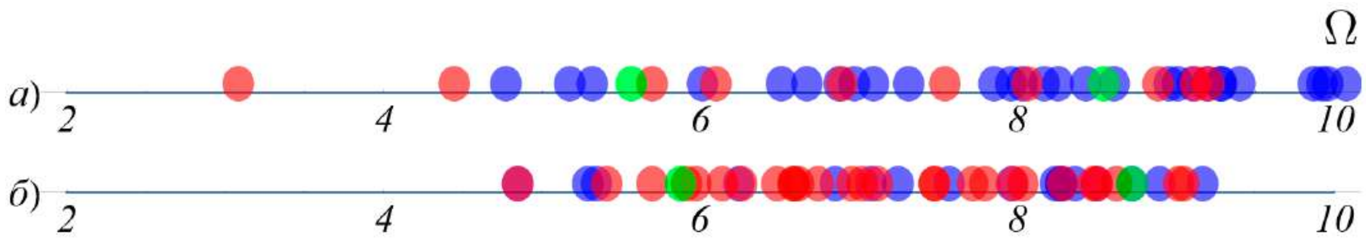


Fig 2.1 — The lowest natural frequencies of the structure with a cap in the form of a) a plate, b) a spherical segment.

"Shell"frequencies are marked in blue, cap frequencies in red, and "beam-like"frequencies in green.

The corresponding vibration modes for the structure with a cap in the form of a spherical segment are shown in Figure 2.2

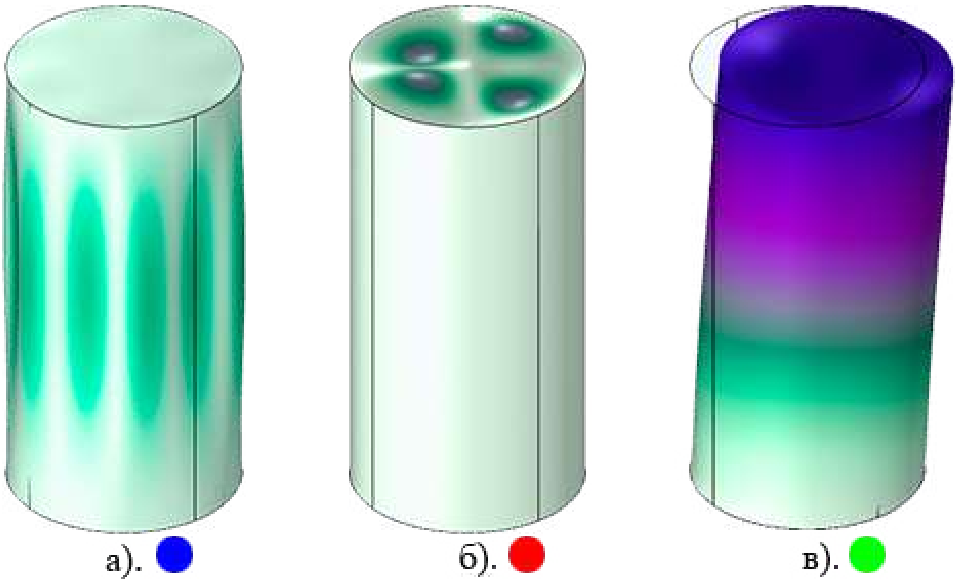


Fig 2.2 — Types of natural vibration modes:

a) "shell b) "cap c) "beam".

2.2 Problem statement

Let's consider small free low-frequency vibrations of a thin circular cylindrical shell, coupled at the edge with a shallow spherical shell. Non-dimensional parameters are used, related to dimensional ones as follows: $l = L/R$, $r_p = R_p/R$, $h_s = H_s/R$, $h_p = H_p/R$, where H_s and H_p are the dimensional thicknesses of the cylindrical and spherical shells, respectively.

Figure 2.3 shows a section of this structure cut through its axis of symmetry. The radius R of the cylindrical shell is taken as the unit of length. After separating the variables, the non-dimensional differential equations describing the free vibrations

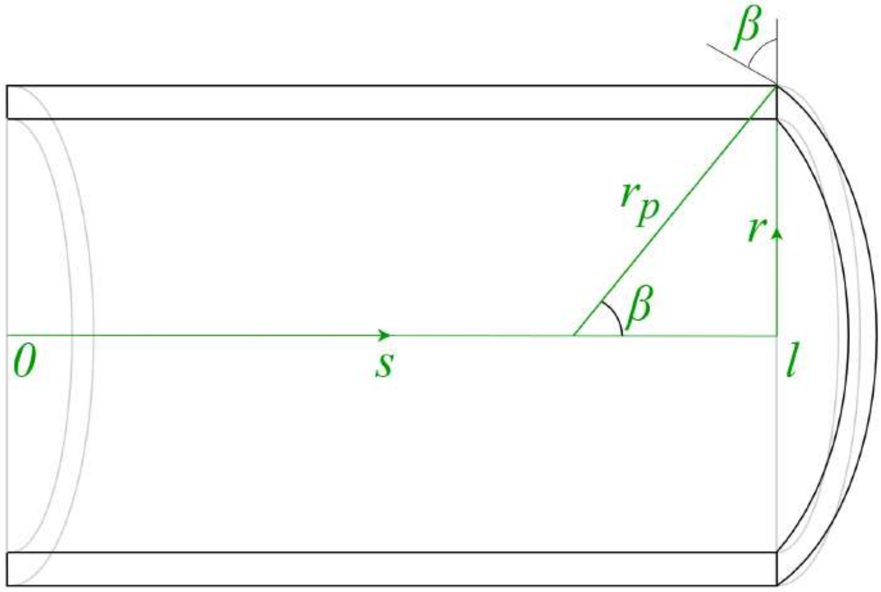


Fig 2.3 — Section of the structure through a plane.

of the cylindrical shell take the form [30]

$$
\begin{gathered}
T_1' + mS + \lambda u = 0, \quad S' - mT_2 + Q_2 + 2H' + \lambda v = 0, \\
Q_1' + mQ_2 - T_2 + \lambda w = 0, \quad Q_1 = M_1' + mH, \quad Q_2 = -mM_2 + H', \\
T_1 = u' + \nu(w + mv), \quad T_2 = w + mv + \nu u', \quad 2S = (1-\nu)(v' - mu), \\
M_1 = \mu^4(\vartheta_1' + \nu m\vartheta_2), \quad M_2 = \mu^4(m\vartheta_2 + \nu\vartheta_1'), \quad H = \mu^4(1-\nu)\vartheta_2', \\
\vartheta_1 = -w', \quad \vartheta_2 = mw + v,
\end{gathered}
\tag{2.1}
$$

where (') denotes the derivative with respect to the non-dimensional longitudinal coordinate $s \in [0, l]$, m is the wave number along the parallel, u, v and w are the displacement components, T_1, T_2, S, Q_1, Q_2 are the forces, M_1, M_2, H are the moments, ϑ_1 and ϑ_2 are the rotation angles, ν is the Poisson's ratio, $\sigma = 1 - \nu^2$, E is the Young's modulus, ρ is the density, f is the vibration frequency, $\omega = 2\pi f$ is the angular frequency, $\lambda = \omega^2\sigma\rho R^2 E^{-1}$ is the frequency parameter, $\mu^4 = h_s^2/12$ is a small parameter.

The equations for the vibrations of a shallow spherical shell [14] in non-dimensional variables take the form

$$
\begin{gathered}
(rT_{1p})' - T_{2p} + \frac{m}{r}S_p = 0, \quad (rS_p)' + S_p - mT_{2p} = 0, \\
(rQ_{1p})' + mQ_{2p} - \frac{r}{r_p}(T_{1p} + T_{2p}) + \lambda r w_p = 0, \\
rQ_{1p} = (rM_{1p})' - M_{2p} + mH_p, \quad rQ_{2p} = -mM_{2p} + (rH_p)' + H_p, \\
T_{1p} = \varepsilon_{1p} + \nu\varepsilon_{2p}, \quad T_{2p} = \varepsilon_{2p} + \nu\varepsilon_{1p}, \quad 2S_p = (1-\nu)\varepsilon_{12p}, \qquad (2.2) \\
\varepsilon_{1p} = u_p' + \frac{w_p}{r_p}, \quad \varepsilon_{2p} = \frac{u_p}{r} + m\frac{v_p}{r} + \frac{w_p}{r_p}, \quad \varepsilon_{12p} = v_p' - \frac{v_p}{r} - \frac{m}{r}u_p, \\
M_{1p} = \mu^4\left(\vartheta_{1p}' + \nu\frac{m}{r}\vartheta_{2p} + \frac{\nu}{r}\vartheta_{1p}\right), \quad M_{2p} = \mu^4\left(\frac{m}{r}\vartheta_{2p} + \frac{1}{r}\vartheta_{1p} + \nu\vartheta_{1p}'\right), \\
H_p = \mu^4(1-\nu)\vartheta_{2p}', \quad \vartheta_{1p} = -w_p', \quad \vartheta_{2p} = \frac{m}{r}w_p.
\end{gathered}
$$

Here (’) denotes the derivative with respect to the non-dimensional radial coordinate $r \in [0,1]$, u_p, v_p, and w_p are the displacement components, T_{1p}, T_{2p}, S_p, Q_{1p}, Q_{2p} are the forces, M_{1p}, M_{2p}, H_p are the moments, ϑ_{1p} and ϑ_{2p} are the rotation angles, $\mu_p^4 = h_p^2/12$ is a small parameter. If the moment-free forces are expressed through the non-dimensional function Φ by the formulas

$$
T_{1p} = r_p\mu^4\left(\frac{\Phi'}{r} - \frac{m^2}{r^2}\Phi\right), \quad T_{2p} = r_p\mu^4\Phi'', \quad S_p = -\frac{r_p\mu^4}{r}\left(m\Phi' - \frac{\Phi}{r}\right),
$$

then the system of equations (2.2) can be written in the following compact form [14, 24]:

$$
\Delta^2 w_p + \Delta\Phi - \Omega_p^4 w = 0, \quad \Delta^2\Phi = k^2\Delta w_p, \qquad (2.3)
$$

where

$$
\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}, \quad \Omega_p^4 = \frac{\lambda}{\mu_p^4}, \quad k^2 = \frac{\sigma}{\mu_p^4 r_p^2}.
$$

Assume that the shell and plate are made of the same material. In this case, the following 8 continuity conditions must be satisfied at the coupling parallel $s = l$,

$r = 1$ [30]:

$$
\begin{gathered}
u_p = -u \sin\beta + w \cos\beta, \quad w_p = -w \sin\beta - u \cos\beta, \\
h_p T_{1p} = h_s (T_1 \sin\beta - Q_1 \cos\beta), \quad h_p Q_{1p} = h_s (Q_1 \sin\beta + T_1 \cos\beta), \\
v_p = v, \quad \vartheta_{1p} = \vartheta_1, \quad h_p S_p = h_s S, \quad h_p M_{1p} = -h_s M_1,
\end{gathered}
\tag{2.4}
$$

where $\sin\beta = 1/r_p$ (see Fig. 2.3).

For a shallow spherical shell $r_p \gg 1$. Therefore, $\sin\beta \ll 1$, $\beta \ll 1$ and the coupling conditions (2.4) can be replaced by approximate conditions

$$
\begin{gathered}
u_p = w, \quad w_p = -u, \quad h_p T_{1p} = -h_s Q_1, \quad h_p Q_{1p} = h_s T_1, \\
v_p = v, \quad \vartheta_{1p} = \vartheta_1, \quad h_p S_p = h_s S, \quad h_p M_{1p} = -h_s M_1.
\end{gathered}
\tag{2.5}
$$

At the edge of the shell $s = 0$, four homogeneous boundary conditions should be set. For example, let's consider the conditions of rigid fixation:

$$
u = w = v = \vartheta_1 = 0 \quad \text{at} \quad s = 0. \tag{2.6}
$$

If at $\lambda = \lambda_k$ the equations (2.1), (2.2) have a non-trivial solution satisfying the boundary conditions (2.5), (2.6), then λ_k is an eigenvalue of the corresponding boundary problem. The smallest positive eigenvalue λ_1 corresponds to the first vibration frequency.

2.3 Frequencies of the first type of vibrations (cap vibrations)

To approximate the calculation of the frequencies of the first type of vibrations, which are close to the frequencies of vibrations of a shallow spherical shell, we use the analogy from [14] between an elastically supported plate and a shallow shell. We replace the shallow spherical shell with a circular plate lying on an elastic foundation with non-dimensional stiffness

$$
k^2 = \frac{\sigma}{\mu_p^4 r_p^2}.
$$

Consequently, the original problem of vibrations of a cylindrical shell coupled with a shallow spherical shell is reduced to the problem of vibrations of a cylindrical shell coupled with a circular plate on a Winkler foundation. Using the asymptotic approach, which was employed in the problem of vibrations of a cylindrical shell connected with a ring plate [57, 90], we find that in the first approximation, the frequencies of the first type of vibrations can be determined by solving the problem of vibrations of a circular plate with a rigidly fixed edge.

The equation describing the transverse vibrations of a circular plate lying on a Winkler foundation with stiffness k^2 is

$$\Delta^2 w_p - \varkappa^4 w_p = 0, \quad \varkappa^4 = \Omega_p^4 - k^2. \tag{2.7}$$

Substituting the exact solution of this equation

$$w_p = C_1 J_m(\varkappa s_p) + C_2 I_m(\varkappa s_p),$$

where C_1 and C_2 are arbitrary constants, J_m is the Bessel function, I_m is the modified Bessel function, into the boundary conditions of rigid fixation

$$w_p(1) = w_p'(1) = 0$$

yields a system of linear algebraic equations to determine C_1 and C_2. Setting the determinant of this system to zero

$$J_m(\varkappa) I_{m-1}(\varkappa) - J_{m-1}(\varkappa) I_m(\varkappa) = 0, \tag{2.8}$$

serves as the equation for determining the parameters $\varkappa(m, n)$, where n is the number of the positive root of equation (2.8). The roots are numbered in ascending order. From the second formula (2.7), it follows that the approximate value of the frequency parameter Ω_p can be found using Zedel's formula [28].

$$\Omega_p^4 = \varkappa^4 + k^2. \tag{2.9}$$

When $k = 0$, we obtain vibration frequencies $\Omega_p = \varkappa$, corresponding to the frequencies of the first type of vibrations for a cylindrical shell coupled with a circular plate. These frequencies turn out to be lower than the frequencies of vibrations of the shell coupled with a shallow spherical segment. As the radius of curvature of the spherical segment r_p increases, the parameter k decreases, and the difference between the frequencies diminishes.

Consider a structure with the following parameter values: $h_p = h_s = h = 0.01$, $r_p = 20$, $\nu = 0.35$. In the third and fourth columns of Table 2.1 for different values of m and n, the values of the parameter $\varkappa$ and the approximate values of the frequency parameter Ω_p obtained using the formula (2.9) are presented. The fifth column contains the values of Ω_p found using the finite element method (FEM), and the sixth column shows the error of the approximate frequency value.

Table 2.1 — Frequency parameters $\Omega_p(m,n)$

m	n	$\varkappa$	Formula (2.9)	FEM (Comsol)	Error, %
0	1	3.20	4.38	3.94	11
1	1	4.61	5.17	4.97	4
2	1	5.91	6.20	6.03	3
0	2	6.31	6.55	6.36	3
3	1	7.14	7.32	7.11	3
1	2	7.80	7.93	7.71	3

For the considered example, the error in calculating the lower frequencies using Zedel's formula is small, except for the lowest frequency, which corresponds to the axisymmetric form of vibration. Let's consider axisymmetric vibrations of a cylindrical shell with a cap in the form of a shallow spherical shell to obtain more accurate approximate formulas for the lower frequencies of the first type.

The equations of axisymmetric vibrations of a cylindrical shell are derived from the equations (2.1) by setting $m = 0$:

$$\begin{gathered} T_1' + \lambda u = 0, \quad Q_1' - T_2 + \lambda w = 0, \quad Q_1 = M_1', \\ T_1 = u' + \nu w, \quad T_2 = w + \nu u', \quad M_1 = \mu^4 \vartheta_1', \quad \vartheta_1 = -w'. \end{gathered} \tag{2.10}$$

From the equations (2.2) with $m = 0$, we obtain the equations for axisymmetric vibrations of a shallow spherical shell:

$$\begin{aligned} (rT_{1p})' - T_{2p} = 0, \quad T_{1p} = \varepsilon_{1p} + \nu\varepsilon_{2p}, \quad T_{2p} = \varepsilon_{2p} + \nu\varepsilon_{1p}, \\ \varepsilon_{1p} = u_p' + \frac{w_p}{r_p}, \quad \varepsilon_{2p} = \frac{u_p}{r} + \frac{w_p}{r_p}, \\ (rQ_{1p})' - \frac{r}{r_p}(T_{1p} + T_{2p}) + \lambda r w_p = 0, \quad rQ_{1p} = (rM_{1p})' - M_{2p}, \\ M_{1p} = \mu_p^4 \vartheta_{1p}', \quad M_{2p} = \mu^4 \left(\frac{1}{r}\vartheta_{1p} + \nu\vartheta_{1p}' \right), \quad \vartheta_{1p} = -w_p'. \end{aligned} \tag{2.11}$$

Introduce a function Φ such that

$$T_{1p} = \frac{r_p \mu^4 \Phi'}{r}, \quad T_{2p} = r_p \mu^4 \Phi''.$$

Then the system of equations (2.11) can be written in the form

$$\Delta^2 w_p + \Delta\Phi - \Omega_p^4 w = 0, \quad \Delta^2 \Phi = k^2 \Delta w_p, \tag{2.12}$$

where

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}, \quad \Omega_p^4 = \frac{\lambda}{\mu_p^4}, \quad k^2 = \frac{\sigma}{\mu_p^4 r_p^2}.$$

When $k = 0$, the equations (2.12) turn into the equations of axisymmetric vibrations of a circular plate.

Eliminating the unknown function Φ from the equations (2.12) leads to an equation for determining w_p:

$$\Delta(\Delta^2 w_p - \varkappa^4 w_p) = 0, \quad \varkappa^4 = \Omega_p^4 - k^2.$$

For the axisymmetric boundary problem at $s = l$, $r = 1$, out of the eight approximate coupling conditions (2.5), six remain:

$$u_p = w, \quad w_p = -u, \quad h_p T_{1p} = -h_s Q_1,$$
$$h_p Q_{1p} = h_s T_1, \quad \vartheta_{1p} = \vartheta_1, \quad h_p M_{1p} = -h_s M_1, \tag{2.13}$$

and the conditions of rigid fixation become

$$u = w = \vartheta_1 = 0 \quad \text{at} \quad s = 0. \tag{2.14}$$

Assume that $\mu_p \sim \mu \ll 1$, $\lambda \sim \mu^4$, and look for a solution to the system of equations (2.11) in the form of a sum of the membrane solution y_0 and edge effect functions y_1, y_2:

$$y = \mu^{I_0(y)} y_0 + \mu^{I_1(y)+2} y_1 + \mu^{I_1(y)} y_2, \tag{2.15}$$

Here, the letter y replaces any of the unknown functions in the system (2.11), $I_1(y)$ and $I_2(y)$ are intensity indicators, given in Table 2.2, with their selection taking into account the results of [11].

Table 2.2 — Intensity Indicators

	Functions					
Indicators	u	w	ϑ	T_1	M_1	Q_1
I_0	2	4	6	4	∞	∞
I_1	3	2	1	4	4	3

The functions y_0 are solutions to the membrane equations obtained from the equations (2.10) when $\mu = 0$:

$$T_1' + \lambda u = 0, \quad T_2 = \lambda w, \quad T_1 = u' + \nu w, \quad T_2 = w + \nu u'. \tag{2.16}$$

The edge effect functions y_1 and y_2 take the form

$$y_1 = \sum_{j=1}^{2} D_j \hat{y}_j \exp\left(\frac{r_j s}{\mu}\right), \quad y_2 = \sum_{j=3}^{4} D_j \hat{y}_j \exp\left(\frac{r_j (s-l)}{\mu}\right), \tag{2.17}$$

where $D_j \sim 1$ are arbitrary constants,

$$r_{1,2} = \frac{\sigma^{1/4}(-1 \pm i)}{\sqrt{2}}, \quad r_{3,4} = \frac{\sigma^{1/4}(1 \pm i)}{\sqrt{2}}.$$

In particular, $\hat{w}_j = 1$, $j = 1, 2, 3, 4$.

The function y_1 rapidly decreases as s increases from 0 to l, and the function y_2 decreases as s decreases from l to 0. Assuming $l \gg \mu$, we get $y_1(l) \ll 1$, and $y_2(0) \ll 1$.

The equations (2.10) and (2.11) include two small parameters μ, μ_p, and one large parameter r_p. The asymptotic expansions of the solutions to the boundary problem depend on relative order of these parameters. In obtaining the asymptotics of the solutions to the system (2.10), it was assumed that $\mu \sim \mu_p$, meaning that the thicknesses of the shells do not differ significantly from each other. Additionally, let's assume that

$$\frac{1}{r_p} \sim \mu^2,$$

and consider the case where the variables included in the system (2.11) have the following orders

$$w_p \sim \vartheta_{1p} \sim 1, \quad u_p \sim T_{1p} \sim \mu^2, \quad M_{1p} \sim Q_{1p} \sim \mu^4.$$

Substituting the solutions (2.15) into the conditions (2.13) and (2.14), we obtain the boundary conditions for the first approximation

$$u_0 = 0, \quad w_1 = -w_0, \quad \vartheta_{11} = 0, \quad s = 0,$$

$$w_2 = -u_p, \quad w_p = 0, \quad T_{1p} = 0, \quad h_p Q_{1p} = h_s T_1,$$

$$\vartheta_{1p} = 0, \quad h_p M_{1p} = -h_s M_{12}, \quad s = l, \quad r = 1.$$

In the first approximation, the boundary problem for the equations (2.10), (2.11) or (2.12) with boundary conditions (2.13) and (2.14) is reduced to solving four independent problems:

1. The eigenvalue boundary problem for the equations (2.11) or (2.12) with boundary conditions

$$w_p(1) = 0, \quad w_p'(1) = 0, \quad T_{1p}(1) = 0, \quad \text{or} \quad \Phi' = 0.$$

2. The system of linear algebraic equations

$$w_2(l) = -u_p(1), \quad h_s M_{12}(l) = -h_p M_{1p}(1)$$

to determine the constants D_3 and D_4.

3. The system of linear algebraic equations

$$w_1(0) = -w_0, \quad \vartheta_{11} = 0$$

to determine the constants D_1 and D_2.

4. The non-homogeneous boundary problem for the membrane system (2.16) with boundary conditions

$$u_0(0) = 0, \quad h_s T_{10} = h_p Q_{1p}(1) - h_s T_{11}(0).$$

Let's consider only problem 1, as its solution allows us to find an approximate value of the parameter λ and the main part of the vibration shape. The displacements of the cylindrical shell $u \sim w \sim \mu^2$ are small compared to the normal displacement of the spherical shell $w_p \sim 1$, which is also confirmed by FEM calculations (see Fig. 2.2).

The solution to the system of equations (2.12) with boundary conditions

$$w_p = 0, \quad w_p' = 0, \quad \Phi' = 0 \quad r = 1 \tag{2.18}$$

is given in the reference [24]. After substituting its general solution

$$w_p = C_1 + C_2 J_0(\varkappa r) + C_3 I_0(\varkappa r), \quad \Phi = C_1 \frac{r^2}{4} \Omega^4 - \frac{k^2}{\varkappa^2}(C_2 J_0(\varkappa r)) - C_3 I_0(\varkappa r))$$

into the boundary conditions (2.18) and equating to zero the determinant of the system of linear homogeneous equations for the arbitrary constants C_1, C_2, and C_3, we obtain the following equation for determining the approximate value of the parameter $\varkappa$

$$(\varkappa^4 + k^2)[J_0(\varkappa)I_1(\varkappa) + I_0(\varkappa)J_1(\varkappa)] - \frac{k^2}{\varkappa}I_1(\varkappa)J_1(\varkappa) = 0. \qquad (2.19)$$

The root $\varkappa$ of the equation (2.19) is related to the frequency parameter Ω_p by the formula (2.9).

Table 2.3 presents a comparison of the values of the parameters for the first two axisymmetric vibration frequencies $\Omega_p(0,1) = 4.03$ and $\Omega_p(0,2) = 6.52$, found using the equation (2.19), with the results of FEM provided in Table 2.1 and the approximation obtained using the equation (2.9) for different values of the curvature radius of the cap.

Table 2.3 — Frequency parameter for the two lowest axisymmetric vibration frequencies of a spherical cap with different radii of curvature, found analytically and numerically

r_p	(2.19)		(2.9)		Comsol	
	$\Omega_p(0,1)$	$\Omega_p(0,2)$	$\Omega_p(0,1)$	$\Omega_p(0,2)$	$\Omega_p(0,1)$	$\Omega_p(0,2)$
∞	3.196	6.306	3.196	6.306	3.196	6.303
20	4.028	6.517	4.379	6.554	3.939	6.364
10	5.696	7.076	5.833	7.164	5.003	6.971

For $r_p = 20$ for $\Omega_p(0,1)$, using equation (2.19) instead of equation (2.9) reduces the error from 11% to 2%. For $\Omega_p(0,2)$, replacing equation (2.9) with equation (2.19) leads to a slight reduction in error, amounting to 3%. For $r_p = 10$ for $\Omega_p(0,1)$, the error of both equations increases to 13% and 16%, whereas for $\Omega_p(0,2)$, the error of the equations does not increase, remaining at 2% and 3% respectively.

2.4 Frequencies of the first type of vibrations for a circular plate

The study of the case $1/r_p = 0$, where the segment of the spherical shell degenerates into a circular plate (Figure 2.4), is of particular interest. After separating variables, the non-dimensional differential equations describing the free vibrations of a cylindrical shell [62] are written in the form (2.1).

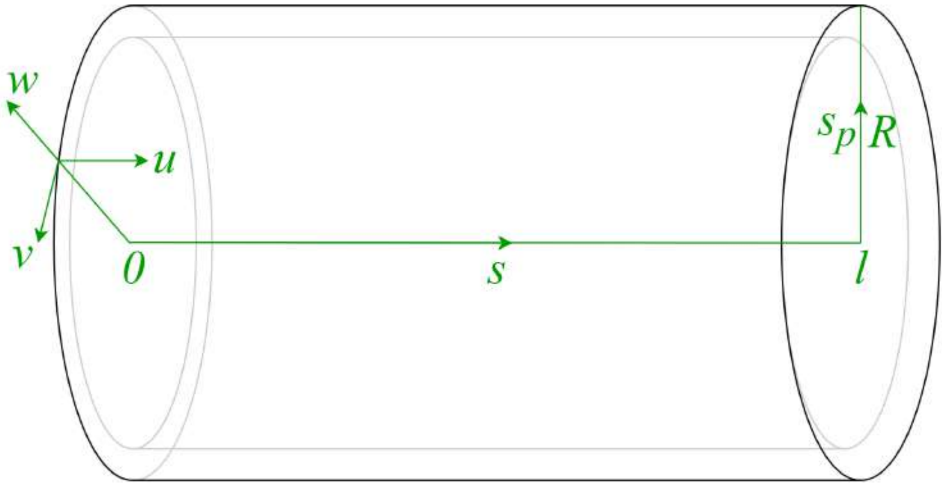


Fig 2.4 — Shell coupled with a circular plate.

The non-dimensional differential equations describing the free vibrations of a circular plate can be written as

$$\begin{aligned} &(s_p Q_{1p})' + mQ_{2p} + \lambda s_p w_p = 0, \\ &s_p Q_{1p} = (s_p M_{1p})' - M_{2p} + 2mH_p, \quad s_p Q_{2p} = -mM_{2p} + 2H_p, \\ &s_p M_{1p} = \mu_p^4[s_p \vartheta_{1p}' + \nu(m\vartheta_{2p} + \vartheta_{1p})], \quad s_p M_{2p} = \mu_p^4(m\vartheta_{2p} + \vartheta_{1p} + \nu s_p \vartheta_{1p}'), \\ &H_p = \mu_p^4(1-\nu)\vartheta_{2p}', \quad \vartheta_{1p} = -w_p', \quad s_p\vartheta_{2p} = mw_p. \end{aligned} \tag{2.20}$$

Here (') denotes the derivative with respect to the radial coordinate, $s_p \in [0,1]$, w_p is the transverse deflection, Q_{1p}, Q_{2p}, M_{1p}, M_{2p}, H_p are non-dimensional resultant stresses and stress couples, ϑ_{1p} and ϑ_{2p} are the rotation angles of the normal, h_p is the non-dimensional thickness of the plate, $\mu_p^4 = h_p^2/12$ is a small parameter.

The following equations [62] can be used to find the tangential (in-plane) deformation of the plate:

$$
\begin{gathered}
(s_p T_{1p})' - T_{2p} + mS_p + \lambda s_p u_p = 0, \quad s_p S_p' + 2S_p - mT_{2p} + \lambda v_p = 0, \\
s_p T_{1p} = s_p u_p' + \nu(m v_p + u_p), \quad s_p T_{2p} = u_p + m v_p + \nu s_p u_p', \\
2 s_p S_p = (1-\nu)(s_p v_p' - m u_p - v_p),
\end{gathered}
\tag{2.21}
$$

where u_p and v_p are the tangential displacement components, T_{1p}, T_{2p}, S_p are non-dimensional resultant stresses.

If the shell and the plate are made of the same material, then the following 8 coupling conditions must be satisfied around the circumference $s = l$, $s_p = 1$

$$
\begin{gathered}
w = u_p, \quad u = -w_p, \quad v = v_p, \quad \vartheta_1 = \vartheta_{1p}, \\
h_s Q_1 = -h_p T_{1p}, \quad h_s T_1 = h_p Q_{1p}, \quad h_s S = -h_p S_p, \quad h_s M_1 = -h_p M_{1p}.
\end{gathered}
\tag{2.22}
$$

At the edge of the shell $s = 0$, four homogeneous boundary conditions must be introduced. For example, let's consider the shell with a clamped edge, i.e.

$$
u = w = v = \vartheta_1 = 0 \quad \text{at} \quad s = 0. \tag{2.23}
$$

As in the case with the shallow shell, assume that $\mu_p \sim \mu$, $\lambda \sim \mu^4$, and seek an approximate solution to the system (2.1) in the form of a sum of the membrane state and edge effect functions:

$$
y = \mu^{I_0(y)} y_0 + \mu^{I_1(y)+1} y_1 + \mu^{I_1(y)} y_2, \tag{2.24}
$$

Here y also denotes any unknown function, $I(y)$ and $I_1(y)$ are intensity indices. The corresponding choice of intensity indices allows satisfying boundary conditions in the first and subsequent approximations and ensures the existence of non-trivial solutions to the corresponding eigenvalue problems. Usually, the estimation of intensity indices is based on one of the enumeration methods and/or intuitive considerations.

The functions u_0, v_0, T_{10}, and S_0 satisfy the membrane equations derived from (2.1) under the assumption $\mu = 0$ and neglecting small terms λu, λv, and λw:

$$T'_{10} + mS_0 = 0, \quad S'_0 = 0, \quad 2S_0 = (1-\nu)(v'_0 - mu_0), \quad T_{10} = \sigma u'_0. \qquad (2.25)$$

The edge effect functions y_1 and y_2 take the form (2.17) with intensity indices listed in Table 2.4.

Table 2.4 — Intensity indices for the first (plate-type) mode of vibration

	Functions							
Indices	u	v	w	ϑ	T_1	S	M_1	Q_1
I_0	3	3	3	3	3	3	7	7
I_1	3	4	2	1	4	3	4	3

We assume that

$$w_p \sim \vartheta_{1p} \sim 1, \quad M_{1p} \sim Q_{1p} \sim \mu^4, \quad u_p \sim v_p \sim T_{1p} \sim S_p \sim \mu^3. \qquad (2.26)$$

Substituting the solutions (2.24) and (2.17) into equations (2.1, 2.20, 2.21), the continuity conditions (2.22), and the boundary conditions (2.23), the eigenvalue problem (2.1, 2.20–2.23) in the first approximation can be divided into five separate problems:

1) Eigenvalue problem for equations (2.20) describing the deflection deformation of the plate with boundary conditions

$$w_p = \vartheta_{1p} = 0, \quad s_p = 1. \qquad (2.27)$$

2) Linear algebraic equations

$$w_2(l) = 0, \quad h_s M_{12}(l) = -h_p M_{1p}(1) \qquad (2.28)$$

for the unknown constants D_3, D_4. Solving these equations, we obtain edge effect functions on the parallel $s = l$.

3) Non-homogeneous boundary problem for the membrane shell equations (2.25) with boundary conditions

$$u_0(0) = v_0(0), \quad T_{10}(l) = 0, \quad h_s[S_0(l) + S_2(l)] = -h_p S_p(1). \tag{2.29}$$

4) Linear algebraic equations

$$w_1(0) = -w_0(0), \quad \vartheta_{11}(0) = 0. \tag{2.30}$$

for unknown constants D_1 and D_2.

5) Non-homogeneous boundary problem for the plate equations (2.21) with boundary conditions

$$v_p(1) = v_0(l), \quad h_p T_{1p}(1) = -h_s Q_{12}(l).$$

Thus, the approximate solution of the eigenvalue problem (2.1, 2.20–2.23) is reduced to sequentially solving five simple problems.

First, it is necessary to solve the eigenvalue problem 1) for the deflection deformation of the plate. Then, the solution of equations (2.28) is used to find edge effect functions on the parallel $s = s_k$. Next, the solution of the membrane problem 3) can be obtained using boundary conditions (2.29), and edge effect integrals on the parallels $s = 0$ and $s = l$ are found using equations (2.30). Finally, boundary problem 5), describing the tangential displacements of the plate, can be solved.

We consider only problem 1), as its solution provides the frequency parameter λ and the main mode of vibration. The displacements of the cylindrical shell and the tangential displacements of the plate are very small compared to the deflection of the plate (see relationship (2.26) and Table 2.4).

To obtain an improved first approximation, the solution to problem 1) can be refined by replacing boundary condition (2.27) with a more accurate condition. From the relationships

$$w_2(l) = 0, \quad \vartheta_{1p}(1) = \vartheta_{12}(l), \quad h_s M_{12}(l) = -h_p M_{1p}(1)$$

we find

$$D_3 + D_4 = 0, \quad \vartheta_{1p}(1) = -\mu(r_3 D_3 + r_4 D_4), \quad h_p M_{1p}(1) = \mu^4 h (r_3^2 D_3 + r_4^2 D_4). \quad (2.31)$$

From (2.31) it follows that

$$M_{1p}(1) = -\sqrt{2} g \mu^3 \frac{h_s}{h_p} \vartheta_{1p}(1).$$

A refined value of the frequency parameter λ is obtained using the following condition

$$w_p(1) = 0, \quad M_{1p}(1) = -\sqrt{2} g \mu^3 \frac{h_s}{h_p} \vartheta_{1p}(1) \quad (2.32)$$

instead of the boundary condition (2.27).

As a result, the equations (2.20) describing the free transverse vibrations of a circular plate are reduced to the following equation

$$\Delta^2 w - \beta^4 w = 0, \quad \beta^4 = \frac{\lambda}{\mu_p^4}, \quad (2.33)$$

where

$$\Delta = \frac{1}{s_p} \frac{d}{ds_p} \left(s_p \frac{d}{ds_p} \right) - \frac{m^2}{s_p^2}.$$

The exact solution of equation (2.33) is

$$w = C_1 J_m(\beta s_p) + C_2 I_m(\beta s_p), \quad (2.34)$$

where C_1 and C_2 are arbitrary constants, J_m is the Bessel function, and I_m is the modified Bessel function. Substituting the solution (2.34) into the boundary conditions

(2.32) leads to a system of linear algebraic equations for the unknowns C_1 and C_2. The system has non-trivial solutions if its determinant equals zero:

$$J_m(\beta)I_{m-1}(\beta) - J_{m-1}(\beta)I_m(\beta) - \frac{2\beta J_m(\beta)I_m(\beta)}{1-\nu-k} = 0, \qquad (2.35)$$

where

$$k = \frac{\sqrt{2}\sigma^{1/4}}{\mu\delta^3}, \qquad \delta = \frac{h_p}{h_s}.$$

For a plate connected to a shell, the roots $\beta(m,n)$ $(m = 0, 1, 2, \ldots, n = 1, 2, \ldots)$ of equation (2.35) are the first approximation of the non-dimensional frequency of plate vibrations. For $m = 0$, the vibrations are axisymmetric. The case $k = 0$ corresponds to vibrations of a plate with a hinged edge. The case $1/k = 0$ corresponds to vibrations of a plate with a clamped edge. In this case, equation (2.35) becomes

$$J_m(\beta)I_{m-1}(\beta) - J_{m-1}(\beta)I_m(\beta) = 0. \qquad (2.36)$$

Consider a structure with the following parameters: $h_p = h_s = h = 0.01$, $\nu = 0.35$. The first roots of equations (2.36) and (2.35) and the frequency parameter values obtained by the finite element method (FEM) for a plate connected to a cylindrical shell are presented in Table 2.5. The last column contains the values of β for a plate with a hinged edge. The formula for the relationship between angular frequency and frequency parameter is

$$\omega^2 = \beta^4 \frac{h^2 E}{12\sigma\rho R^2} \qquad (2.37)$$

2.5 Frequencies of the second type of vibrations (shell type)

For low-frequency vibrations of the second type, the vibration modes of the cylindrical shell with a cover are similar to the vibration modes of an unreinforced shell with a large number of waves along the parallel ($m \sim \mu^{-1/2}$). Following [11], we seek a solution

Table 2.5 — Frequency parameter β for the lowest frequencies of the "plate-type" mode of vibration

m	n	(2.36)	(2.35)	FEM	$k=0$
0	1	3.196	3.086	3.070	2.238
1	1	4.611	4.460	4.422	3.736
2	1	5.906	5.722	5.658	5.067
0	2	6.306	6.111	6.058	5.457
3	1	7.144	6.932	6.835	6.326
1	2	7.799	7.570	7.480	6.967
4	1	8.347	8.111	7.978	7.543
2	2	9.197	8.939	8.805	8.377
0	3	9.439	9.176	9.002	8.615

to the system (2.1) as a sum of a semi-momentless solution and an edge effect:

$$y = \mu^{I_0(y)} y_0 + \mu^{I_1(y)} (y_1 + y_2), \qquad (2.38)$$

The values of the intensity indicators I_0 and I_1 are given in Table 2.6.

Table 2.6 — Intensity Indicators

	Functions							
Indicators	u	v	w	ϑ	T_1	S	M_1	Q_1
I_0	1	1/2	0	0	1	3/2	3	3
I_1	2	5/2	1	0	2	3/2	3	2

The function v_0 satisfies the semi-momentless equation

$$\frac{d^4 v_0}{ds^4} - \alpha_s^4 v_0 = 0, \qquad (2.39)$$

where

$$\alpha_s^4 = \frac{\lambda m^4 - \mu^4 m^8}{\sigma},$$

$$w_0 = -v_0, \quad u_0 = \frac{dv_0}{ds}, \quad T_{10} = \frac{d^2 v_0}{ds^2}, \quad S_0 = -\sigma \frac{d^3 v_0}{ds^3}. \tag{2.40}$$

The edge effect functions y_1 and y_2 are as in (2.17).

Suppose that

$$h \sim h_p, \quad \frac{1}{r_p} \sim \mu^2, \quad v_p \sim \mu^5, \quad u_p \sim \mu^2, \quad M_{1p} \sim Q_{1p} \sim \mu^4.$$

Considering the relations

$$v \sim \mu^{1/2}, \quad w \sim 1, \quad T_1 \sim \mu, \quad M_1 \sim \mu^3,$$

and disregarding minor terms in the first, fourth, fifth, and eighth conjunction conditions (2.5), we obtain approximate boundary conditions at the edge $s = l$ for the equations of vibration of the cylindrical shell (2.1):

$$v = w = T_1 = M_1 = 0. \tag{2.41}$$

These conditions correspond to a hinged support of the shell edge.

Thus, in the first approximation for a cylindrical shell with a cover, we have an eigenvalue boundary problem for the system of equations (2.1) with the boundary conditions of rigid fixation (2.6) and hinged support (2.41). It is shown in the monograph [11] that the boundary conditions (2.6) and (2.41) for equation (2.39) degenerate into the conditions

$$v_0(0) = \frac{dv_0}{ds}(0) = 0, \quad v_0(l) = \frac{d^2 v_0}{ds^2}(l) = 0. \tag{2.42}$$

The boundary problem (2.39), (2.42) describes the vibrations of a beam with a clamped edge at $s = 0$ and a hinged edge at $s = l$. The solution to this problem is well known (see [3]). The eigenvalues for the problem (2.39), (2.42) are given by the formula $\alpha_{sn} = \varkappa_n/l$, where $\varkappa_n$ are the roots of the equation

$$\operatorname{tg} \varkappa = \tanh \varkappa. \tag{2.43}$$

with $\varkappa_1 = 3.927$, $\varkappa_2 = 7.069$.

The frequency parameter

$$\lambda(m, n) = \frac{\sigma \varkappa_n^4}{m^4 l^4} + \mu^4 m^4 \tag{2.44}$$

takes the minimum value corresponding to the fundamental frequency if $n = 1$ and m is close to m_0 where

$$m_0^4 = \frac{\sqrt{\sigma} \varkappa_1^2}{\mu^2 l^2}.$$

Let's consider a cylindrical shell of thickness $h = 0.01$, to which a flat cover of the same thickness is attached.

The values of the parameter

$$\eta = \frac{(12\lambda)^{1/4}}{\sqrt{h}} \tag{2.45}$$

are given in Table 2.7 for $\nu = 0.35$ and three values of the length of the shell.

2.6 Frequencies of the third type of vibrations (beam type)

Among the lowest natural frequencies of the shell, there can be frequencies of beam vibrations, during which the structure oscillates similarly to a cantilever beam with a load at the end (see Figure 2.2). The equation of transverse vibrations of a beam with one clamped end and loaded with a concentrated mass at the other end is given by

$$w^{(IV)}\left(\frac{s}{l}\right) - \alpha^4 w\left(\frac{s}{l}\right) = 0, \quad \alpha^4 = \frac{\rho S l^4 R^4}{EJ}\omega^2, \quad 0 \le s \le l, \tag{2.46}$$

and the boundary conditions are as follows:

$$w(0) = w'(0) = 0, \quad w''(1) = 0, \quad \frac{EJw'''(1)}{l^3} = -m\omega^2 w(1).$$

Table 2.7 — Comparison of numerical and asymptotic lower values of the frequency parameter for the shell-type vibrations.

$l=4$				$l=6$				$l=8$			
n	m	Num.	Asymp.	n	m	Num.	Asymp.	n	m	Num.	Asymp.
1	4	4.770	5.026	1	3	3.933	4.228	1	3	3.315	3.537
1	5	5.173	5.287	1	4	4.106	4.267	1	4	3.926	4.090
1	3	5.315	5.991	1	5	4.940	5.061	1	2	3.962	4.466
1	6	6.010	6.086	1	2	5.044	5.914	2	4	4.561	4.744
2	5	6.508	6.901	2	4	5.353	5.690	1	5	4.888	5.019
2	6	6.669	6.760	2	5	5.434	5.551	2	3	4.978	5.436
1	2	6.872	8.848	1	6	5.917	6.017	2	5	5.095	5.194
2	4	7.089	8.082	2	6	6.124	6.173	3	5	5.599	5.722
1	7	6.968	7.030	2	3	6.249	7.061	3	4	5.676	6.058
2	7	7.311	7.293	3	5	6.408	6.719	1	6	5.898	6.005
1	8	7.956	8.012	3	6	6.610	6.668	2	6	5.985	6.057

Here, J is the moment of inertia of the shell's section around its diameter, $m=\pi R^2 h_p \rho$ is the mass of the cover (plate), and $S=2\pi Rh$ is the cross-sectional area of the shell. Substituting the solution of equation (2.46)

$$w\left(\frac{s}{l}\right)=C_1\sin\frac{\alpha s}{l}+C_2\cos\frac{\alpha s}{l}+C_3\sinh\frac{\alpha s}{l}+C_4\cosh\frac{\alpha s}{l}$$

into the boundary conditions and setting the determinant of the homogeneous linear system to zero, we obtain an equation for determining the value of α

$$\gamma\alpha(\cos(\alpha)\sinh(\alpha)-\cosh(\alpha)\sin(\alpha))+\cosh(\alpha)\cos(\alpha)+1=0, \tag{2.47}$$

where $\gamma=\frac{m}{M}=\frac{h_p}{2lh}$, and M is the mass of the shell.

In the case where the cover is a circular plate, $\gamma = 1/(2l)$. For such structures, the results of analytical and finite element analysis for various lengths of shells are presented in Table 2.8, where the frequency parameter Ω is related to α by the relation

$$\Omega = \frac{3\sigma\alpha}{l}\sqrt[4]{1+\frac{4}{h^2}}.$$

Table 2.8 — Dependency of the frequency parameter Ω on the length of the shell l for "beam-like"vibrations

l	Analytical	FEM	Error
4	6.411	5.561	15.3 %
8	3.357	3.203	4.8 %
12	2.277	2.223	2.3 %
16	1.723	1.701	1.8 %
20	1.387	1.375	0.8 %

For shells with a relatively short length, the solution of equation (2.47) gives significantly overstated frequencies, but the accuracy of the analytical formula increases with the length of the shell. It should be noted that as the length of the shell increases, the frequencies of beam vibrations decrease rapidly and for shells with $l > 8$, the fundamental frequency of the structure becomes the first frequency of "beam-like"vibrations.

2.7 Single-parameter optimization of the eigenfrequency spectrum

The obtained results for the spectrum can be used in solving optimization problems. The most common optimization problem of the spectrum is to maximize the value of the lowest eigenfrequency of vibrations by changing the system parameters, both geometric and physical.

First, let's consider the influence of the thickness of the plate ($1/r_p = 0$) on the eigenfrequencies while mass conservation of the structure. Since the lowest eigenfrequencies of the structure belong to the series of "plate-like"frequencies, to increase the fundamental eigenfrequency, it is necessary to increase the thickness of the plate so that its lowest frequency coincides with the lowest frequency of the shell.

For a shell and plate with thicknesses h and $h_p = \delta h$ respectively, the angular frequency ω is given by a formula similar to (2.37):

$$\omega^2 = \beta_\delta^4 \frac{h^2 E}{12\sigma\rho R^2}, \quad \beta_\delta^4 = (\beta\sqrt{\delta})^4. \tag{2.48}$$

The second row of Table 2.9 shows the first root β of equation (2.36), and the third row shows the value of β_δ. The fourth row contains the values of β_δ corresponding to the fundamental frequency found using the Finite Element Method (FEM). The difference between asymptotic and numerical results is less than 1.1%.

Table 2.9 — Dependency of the frequency parameter of the fundamental frequency on the thickness of the plate.

δ	1	1.5	2	2.5	3	3.5	4
β	3.086	2.906	2.715	2.565	2.461	2.393	2.348
β_δ	3.086	3.560	3.840	4.055	4.263	4.477	4.697
FEM	3.069	3.537	3.815	4.028	4.231	4.439	4.648

The condition of mass conservation of the structure allows for the determination of the thickness of the shell h_s, which decreases with increasing thickness of the plate. Indeed, from the condition of volume conservation

$$2\pi hl + \pi h = 2\pi h_s l + \pi h\delta,$$

the thickness of the shell is

$$h_s = \left(1 + \frac{1-\delta}{2l}\right) h = \delta_s h.$$

When $\delta = 1$, the thicknesses of the plate and the shell are equal, $h_s = h_p = h$.

Figure 2.5 a) shows the analytical dependencies of the frequency parameter Ω on δ for the lowest "plate-like"frequency ($m = 0$), two lowest "shell-like"frequencies ($m = 4$ and $m = 5$), and the first "beam-like"frequency for $h = 0.01, l = 4$. The frequency parameter Ω is related to the parameters α, β, and η as follows:

$$\Omega = \begin{cases} \beta_\delta & \text{for "plate-like"frequencies (see (2.48))}, \\ \sqrt[4]{\frac{12\lambda}{h_s^2}} & \text{for "shell-like"frequencies (see (2.45))}, \\ \frac{3\sigma\alpha}{l}\sqrt[4]{1+\frac{4}{h_s^2}} & \text{for "beam-like"frequencies (see (2.46))}. \end{cases}$$

At $\delta = 1$, the fundamental frequency is the lowest "plate-like"frequency. As δ increases, h_s decreases and the mass of the plate increases, thus reducing both the first "shell-like"and the first "beam-like"frequencies. As the length of the shell l increases, the lowest "beam-like"frequency decreases and becomes the fundamental frequency of the structure for $l > 8$.

When changing the parameter δ, frequencies can become multiples. When "plate-like"and "shell-like"frequencies collide, distortions in the corresponding vibration forms are negligible. When "plate-like"and "beam-like"frequencies are close, the vibration form represents a superposition of "beam-like"and "plate-like"vibration forms, which does not allow for determining the type of vibration. The collision of the first "beam-like"and second "plate-like"frequencies explains the non-monotonicity of the green line in Figure 2.5.

Table 2.10 shows the values of the optimal thicknesses of the structural elements and the corresponding frequency parameter values, found analytically and by the finite element method.

2.8 Two-parameter optimization of the eigenfrequency spectrum

Consider the problem of vibrations of a structure consisting of a cylindrical shell of length l, radius 1, and thickness h_s, coupled with a spherical segment of radius r_p and thickness $h_p = h_s/\delta$. As $r_p \to \infty$, the spherical shell turns into a flat circular plate.

Reducing the radius of curvature of the spherical segment leads to a rapid increase in its eigenfrequencies, while reducing the thickness of the spherical segment leads to a less significant decrease in its eigenfrequencies. At the same time, to maintain the mass of the structure, the "excess"material can be used to increase the thickness of the cylindrical shell, which leads to an increase in its eigenfrequencies.

The condition for mass conservation of the structure is given by

$$2\pi hLR + \pi hR^2 = 2\pi h_s LR + \pi h_p R_p \left(R_p - \sqrt{R_p^2 - R^2}\right),$$

from which, with $h_s = \delta h_p$, follows the formula for dimensionless parameters

$$h_p = h \cdot \frac{2l+1}{2l\delta + 2r_p^2 - 2r_p\sqrt{r_p^2 - 1}}.$$

In the first optimization problem, for a given mass of the structure, we find the curvature of the cover k ($k = 1/r_p$) and the thickness ratio δ, at which the fundamental frequency of the structure is the highest.

For a fixed-mass structure, Figure 2.6 presents analytically derived dependencies of the lowest eigenfrequencies on parameters $\delta(1 < \delta < 5)$ and $k(0.01 < k < 0.15)$. In the graph, the values of "plate-like"frequencies are denoted in green, "shell-like"frequencies in yellow, and the lowest eigenfrequency of the cylindrical shell in the limit case, when the thickness of the cover tends to zero, in red. It is noted that as the curvature of the cover increases, the lowest eigenfrequency of "shell-like"vibrations rapidly approaches the limit value.

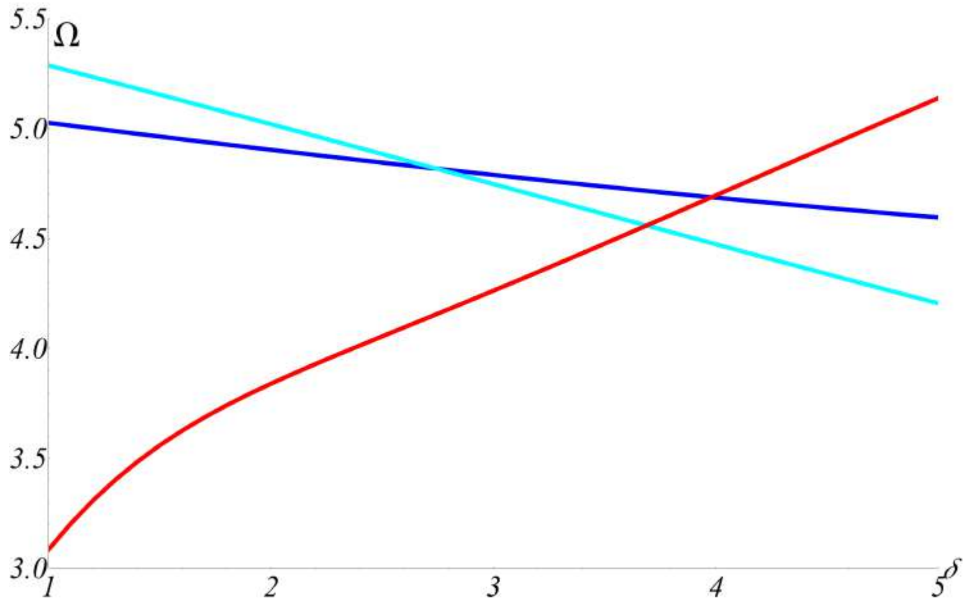
Figure 2.7 demonstrates a good match between the values of the lowest eigenfrequencies of vibrations obtained analytically (green, yellow, and red surfaces) and numerically (dots) in the *Comsol* package for different values of δ and k.

Table 2.11 lists the values of the first frequencies of structures for different values of δ and r_p.

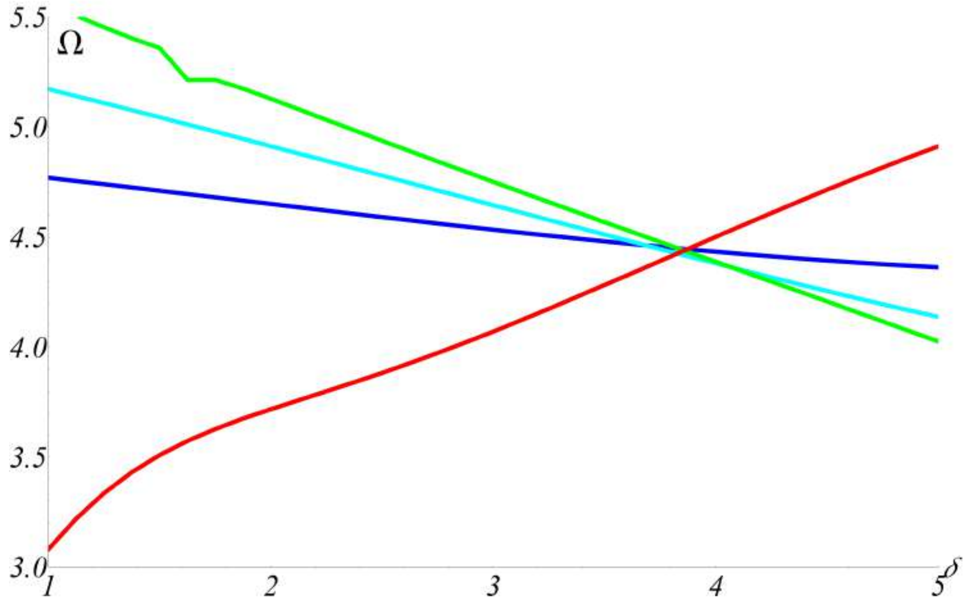
The optimal solution, in which the fundamental frequency reaches the maximum value while mass conservation of the structure, is achieved with parameters $r_p = 9.851$ and $\delta = 30$. However, such a setup is not justified, as for a cylindrical shell with a radius of $R = 1$ m and a thickness of $H = 1$ cm, the cover would have a thickness of $H_p = 0.3$ mm, which is unacceptable for real structures. In particular, most engineering standards, for example, for designing railway tanks, specify a minimum thickness of the end cover of 3 mm.

Therefore, consider the second optimization problem: for a given mass of the structure and a given minimum thickness of the cover $H_p = 3$ mm, find the radius of curvature of the cover at which the fundamental frequency of the structure is the highest.

For a structure with parameters defined above, the maximum fundamental frequency of 40.36 Hz is achieved at $R_p = 9.851$ m, which significantly exceeds the fundamental frequency of the structure with a flat cover of 16.318 Hz.



a)



b)

Fig 2.5 — Dependency of the frequency parameter of the lowest eigenfrequencies on the thickness of the plate while maintaining the overall mass of the structure. a) analytical results, b) numerical (FEM). "Plate-like"frequency — red, "beam-like"frequency — green, "shell-like"frequencies: for $m = 4$ — blue, $m = 5$ — light blue.

Table 2.10 — Optimal thicknesses of structural elements and the corresponding frequency parameter

	Analytical	FEM
Ω	4.50	4.43
h_p	3.90	3.83
h_s	0.64	0.65

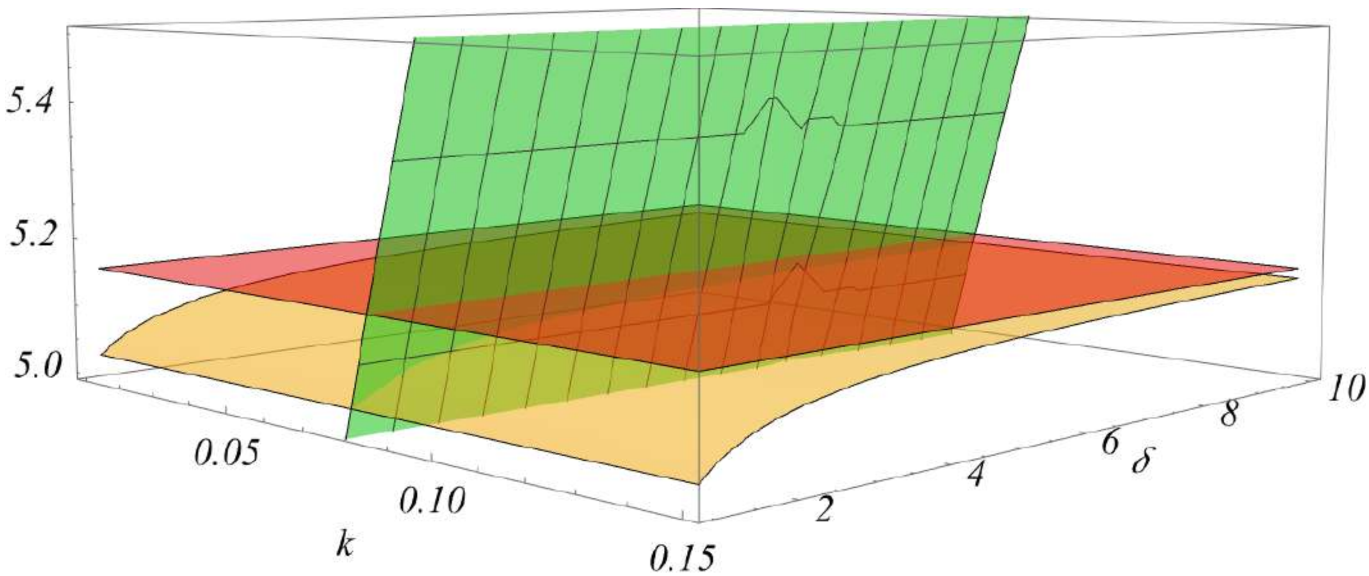


Fig 2.6 — Analytical dependencies of the lowest eigenfrequencies on parameters δ and k: "plate-like"(green), "shell-like"(yellow), and the lowest eigenfrequency of the cylindrical shell (red) in the limit case.

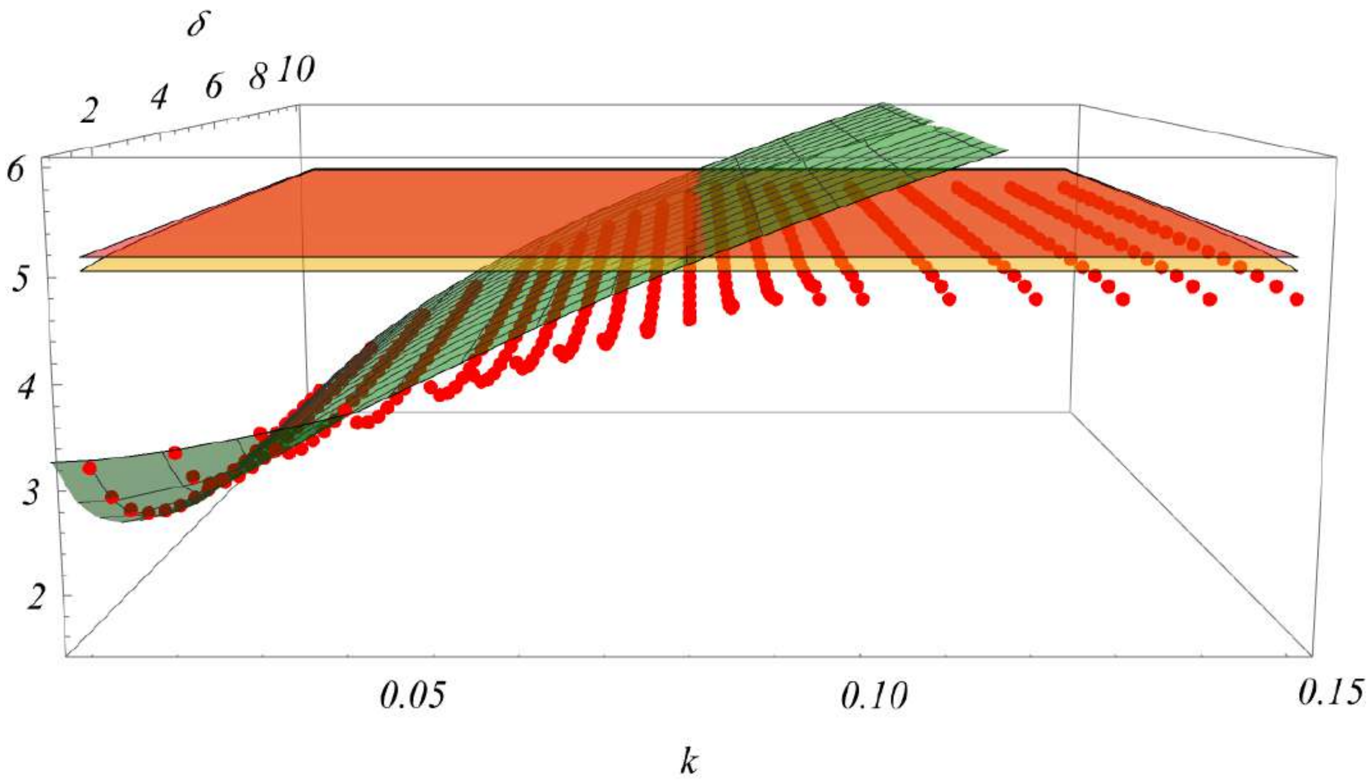


Fig 2.7 — Lowest eigenfrequencies of vibrations found analytically (green, yellow, and red surfaces) and numerically (dots).

Table 2.11 — Frequency parameter of a shell coupled with a shallow spherical segment

δ \ ρ	0.01	0.03	0.05	0.07	0.09	0.11	0.13
1	3.162	3.500	3.940	4.386	4.768	4.768	4.767
2	2.446	3.032	3.640	4.171	4.633	4.823	4.823
3	2.122	2.892	3.564	4.118	4.594	4.843	4.842
4	1.957	2.850	3.554	4.127	4.619	4.852	4.852
5	1.869	2.845	3.573	4.164	4.673	4.858	4.857

3 Buckling of a thin cylindrical shell stiffened with rings of varying stiffness

This chapter is devoted to study of the buckling of a structure consisting of a thin-walled elastic cylindrical shell, stiffened with rings of varying stiffness along its parallels. An example of such a stiffened shell is shown in Figure 1.1.

The Rayleigh-Ritz method is used to derive an asymptotic formula for evaluation of the critical load value of the structure. Numerical and asymptotic methods are used to study the effect of varying the stiffness distribution law of the frames along the generatrix on the critical pressure of the shell.

Two optimization problems are solved. In the first, the coefficients of the distribution functions are chosen such that the structure of a given mass achieves maximum critical pressure. In the second problem, the mass of the structure is minimized for a given critical pressure.

3.1 Buckling of a stiffened cylindrical shell

3.1.1 Problem statement

The problem of buckling in a thin-walled elastic cylindrical shell under the action of uniform external pressure p is considered. To increase the critical pressure, the shell is stiffened with n_s transverse ribs of varying stiffness (rings) with zero eccentricity.

rings whose stiffness varies along the generatrix of the shell are considered. It is assumed that the optimal choice of ring stiffnesses will lead to the maximum increase in the critical pressure of the structure. The solution of the optimization problem involves finding the geometrical parameters of the rings corresponding to the maximum critical pressure for a given mass of the structure.

To study the buckling, we will use the dimensionless equations of the technical theory of shells [25]:

$$\varepsilon^8 \Delta^2 w^{(i)}(s) - \Delta_k \Phi^{(i)}(s) - \lambda m^2 w^{(i)}(s) = 0, \quad \Delta^2 \Phi^{(i)}(s) + \Delta_k w^{(i)}(s) = 0, \tag{3.1}$$

where

$$\Delta_k = \frac{d^2}{ds^2}, \quad \Delta = \Delta_k - m^2, \quad \sigma = 1 - \nu^2, \quad \lambda = \frac{p}{Eh}, \quad \varepsilon^8 = \frac{h^2}{12\sigma}.$$

Here ε is a small parameter, s is the coordinate along the generatrix of the cylinder, $w^{(i)}(s)$ is the projection of displacement in the direction normal to the median surface between rings, $\Phi^{(i)}(s)$ is the force function, m is the number of waves along the parallel, h is the dimensionless thickness of the shell, l is the dimensionless length of the shell, ν is Poisson's ratio, and E is Young's modulus. The unit of length is chosen to be the radius R of the median surface of the cylinder.

We represent the solution of the system of equations (3.1) as a sum of the basic semi-momentless state and the simple edge effect of the shell and parallels where the rings are installed. In the first approximation, we get

$$\left(w_0^{(i)}\right)^{IV} - \alpha^4 w_0^{(i)} = 0, \quad \alpha^4 = m^6 \lambda_0 - \varepsilon^8 m^8, \tag{3.2}$$

where $w_0^{(i)}$ is the approximate solution of the system (3.1), λ_0 is the approximate value of λ (see [14]). Further, only the first term of the asymptotic expansion of the solution is considered, and $w_0^{(i)}$ and λ_0 are replaced by $w^{(i)}$ and λ, respectively.

The boundary conditions for equation (3.2) in the case of simply supported edges of the shell are as follows:

$$\begin{aligned} &w^{(1)} = 0, \quad \frac{d^2}{ds^2} w^{(1)} = 0, \quad \text{at} \quad s = 0, \\ &w^{(n)} = 0, \quad \frac{d^2}{ds^2} w^{(n)} = 0, \quad \text{at} \quad s = l, \end{aligned} \tag{3.3}$$

and in the case of clamped edges

$$w^{(1)} = 0, \quad \frac{d}{ds}w^{(1)} = 0, \quad \text{at} \quad s = 0,$$
$$w^{(n)} = 0, \quad \frac{d}{ds}w^{(n)} = 0, \quad \text{at} \quad s = l. \tag{3.4}$$

Assume that the characteristic size of the cross-section of the ring $a \ll \varepsilon$. Then, on the parallels stiffened with rings, the conjugation conditions are satisfied [30]:

$$w^{(i)} = w^{(i+1)}, \quad w^{(i)\prime} = w^{(i+1)\prime},$$
$$w^{(i)\prime\prime} = w^{(i+1)\prime\prime}, \quad w^{(i)\prime\prime\prime} - w^{(i+1)\prime\prime\prime} = -c_i w^{(i+1)}, \tag{3.5}$$
$$\text{for} \quad s = s_i, \quad (i = 1, 2, \ldots, n-1),$$

where

$$c_i = \frac{m^8 \varepsilon^8 l \eta_i}{n}, \quad \eta_i = \frac{12\sigma n E_c J_i}{h^3 E l}.$$

Here E_c is Young's modulus of the ring material, η_i is the effective stiffness of the i-th ring, proportional to the ratio of the bending stiffnesses of the ring and the shell, J_i is the moment of inertia of the cross-section of the i-th ring relative to the generatrix of the cylinder.

The approximate value of the critical pressure parameter λ for the stiffened shell is determined by the formula [14]:

$$\lambda(\eta) = \min_m \left[\frac{\alpha^4(\mathbf{c})}{m^6} + \varepsilon^8 m^2 \right], \mathbf{c} = \{c_i\}_{i=1}^n, \tag{3.6}$$

where $\alpha(\mathbf{c})$ is the eigenvalue of the boundary problem (3.2), (3.5) with the boundary conditions (3.3) in the case of simply supported edges of the shell and (3.4) in the case of their rigid clamping.

3.1.2 Vibrations of a stiffened Beam

The method of reinforcing the shell with rings is discussed in detail in Section 2 of the first chapter of the dissertation. Figure 3.1 shows the shell with rings in a sectional view along the shell generatrix.

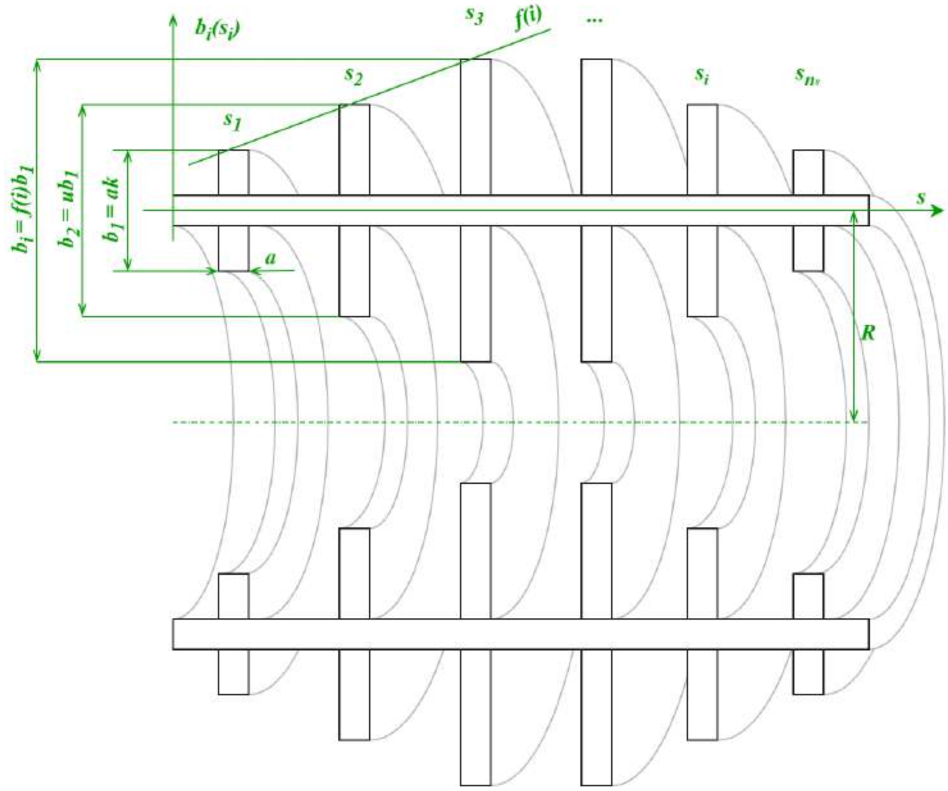


Fig 3.1 — Axial section of a shell stiffened with rings with zero eccentricity.

The eccentricity of a ring is defined as the distance between the center of gravity of the ring's cross-section and the median surface of the shell. For the case of stiffenning with rings of zero eccentricity, the moment of inertia of the i-th ring is given by the formula:

$$J_i = Jf^3(i), \quad J = \frac{a^4k^3}{12}.$$

where a is the width of the rings, $k = b/a$ is the ratio of the height of the first ring to its width, $f(i)$ is a function describing the profile of the structure. Then the dimensionless stiffness (c_i) and the relative stiffness (η_i) of the i-th ring can be written as:

$$c_i = c \cdot f^3(i), \quad \eta_i = \eta \cdot f^3(i), \quad \text{where} \quad c = \frac{m^8\varepsilon^8 l\eta}{n}, \qquad \eta = \frac{12\sigma nJ}{h^3 l}. \tag{3.7}$$

The profile function of the structure $f(i)$ can take any form, however, it is advisable to stiffen the shell with rings, the heights of which are symmetrical relative to the middle.

In particular:

- for linear distribution of ring heights (Figure 3.2 (a)), the function $f(i)$ has the form

$$f_{lin}(i) = (\kappa(i) - 1)(u - 1) + 1, \quad u = \frac{b_2}{b_1}, \tag{3.8}$$

- for distribution of ring heights by a parabola (Figure 3.2 (b))

$$f_{parab}(i) = a_p\kappa^2(i) - na_p\kappa(i) + na_p - a_p + 1, \quad \text{where} \quad a_p = \frac{1-u}{n-3}, \tag{3.9}$$

- for exponential distribution of ring heights (Figure 3.2 (c))

$$f_{exp}(i) = \frac{u-1}{e^2-e}e^{\kappa(i)} + \frac{e-u}{e-1}. \tag{3.10}$$

- when reinforcing the shell with identical rings

$$f_0(i) = 1. \tag{3.11}$$

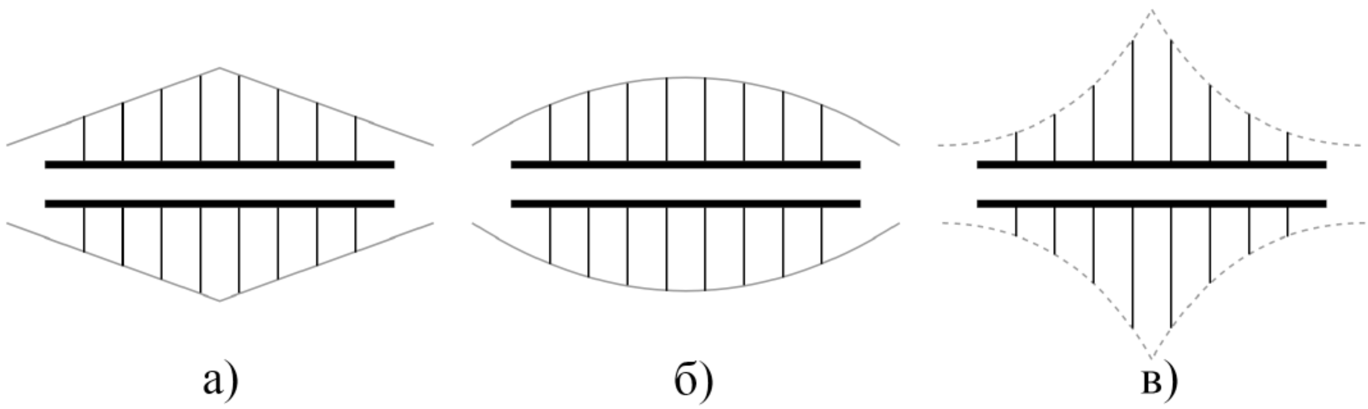


Fig 3.2 — Construction profiles for the case of

a). linear b). parabolic c). exponential

functions of distribution of ring heights.

In formulas (3.8, 3.9, 3.10) the function

$$\kappa(i) = \frac{n}{2} - \left|\frac{n}{2} - i\right| = \begin{cases} i, & i < \frac{n}{2} \\ n - i, & i \geqslant \frac{n}{2} \end{cases}$$

ensures the symmetry of the construction profile functions, and the parameter $u = b_2/b_1$ characterizes the amplitude of the distribution function.

The boundary problems (3.2, 3.3, 3.5) and (3.2, 3.4, 3.5) are equivalent to problems of determining the lowest frequencies of transverse vibrations of simply supported and clamped beams (Figure 3.3), stiffened with springs of stiffness c_i at points $s = s_i$.

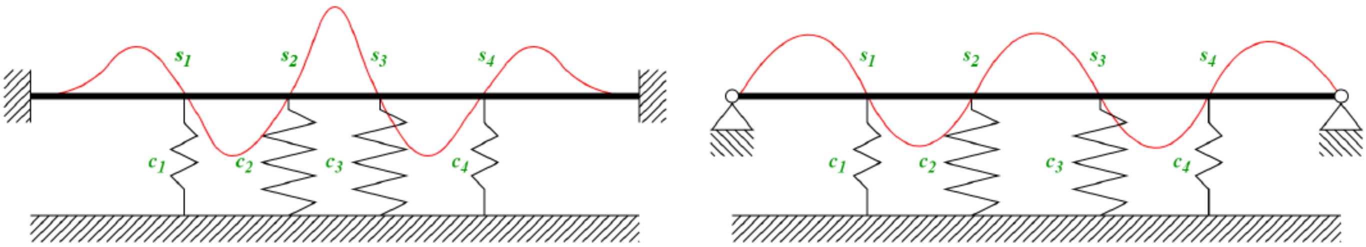


Fig 3.3 — Vibration modes of a beam, stiffened with springs with

a). clamped, b). simply supported

ends.

The choice of optimal coordinates for installing the springs s_i is considered in the first chapter of the dissertation. Springs should be installed at the nodes of the vibration modes of the unstiffened beam:

- for simply supported edges (3.2, 3.3, 3.5)

$$w_n(s) = \sin(\alpha_n s), \quad \alpha_n = \frac{n}{l} \cdot \pi, \tag{3.12}$$

- for clamped edges (3.2, 3.4, 3.5)

$$w_n(s) = U(\alpha_n s) - \varkappa_n V(\alpha_n s), \quad \alpha_n = \frac{z_n}{l}, \quad \varkappa_n = \frac{U(z_n)}{V(z_n)}, \tag{3.13}$$

where

$$U(x) = \operatorname{ch} x - \cos x, \quad V(x) = \operatorname{sh} x - \sin x$$

are Krylov functions, and the values

$$z_n \simeq (2n+1)\frac{\pi}{2}, \quad n = 1,2,\ldots$$

are the roots of the equation $\operatorname{ch} z \cdot \cos z = 1$. At the same time, the points of optimal placement of the springs coincide with the roots of the equation $w_n(s) = 0$.

Subsequently, for simplicity of notation, we omit the indices, and thus $\alpha = \frac{\pi}{2l}$, $\varkappa = U(\frac{3\pi}{2})/V(\frac{3\pi}{2})$, and the first form of vibrations of the unstiffened beam is denoted $w(s)$.

The formula for estimating the first eigenvalue by the Rayleigh-Ritz method of the boundary problem (3.2, 3.5) with boundary conditions of simply supported ends (3.3) of the beam is as follows

$$\alpha_s^4 = \frac{\pi^4}{l^4} + c\frac{2T_s(n)}{l}, \quad T_s(n) = \sum_{i=1}^{n-1} f^3(i)\sin^2\left(\frac{\pi i}{n}\right), \tag{3.14}$$

and the value of the approximate parameter of the first frequency of vibrations of the beam with clamped ends (3.4) can be calculated by the formula

$$\alpha_c^4 = \left(\frac{3\pi}{2l}\right)^4 + c\frac{T_c(n)}{I_0}, \tag{3.15}$$

where

$$I_0 = \int_0^l \left(U\left(\frac{\pi s}{2l}\right) - \varkappa \cdot V\left(\frac{\pi s}{2l}\right)\right)^2 ds, \quad T_c(n) = \sum_{i=1}^{n-1} f^3(i) w^2(s_i).$$

3.1.3 Finding eigenvalues in the problem of buckling of a stiffened shell

Consider the problem of determining the value of the critical pressure parameter λ_1 for a cylindrical shell with simply supported (3.3) edges. The shell is stiffened with rings of stiffness c_i along parallels with coordinates s_i, which are the nodes of the vibration form of the unstiffened simply supported shell (3.12). Denote by $\alpha_s(\eta, m)$ the eigenvalue of the boundary problem for the case of simply supported (3.2, 3.3, 3.5), corresponding to the critical pressure. The corresponding value of the critical pressure parameter $\lambda_1(\eta)$ is determined from formula (3.6) and denoted as $\lambda^s(\eta)$

$$\lambda^s(\eta) = \min_m \left[\frac{\alpha_s^4(\eta, m)}{m^6} + \varepsilon^8 m^2\right]. \tag{3.16}$$

Taking into account the value of ring stiffness c, which is defined in (3.7), the eigenvalue α_s considering the formula (3.14) can be written as

$$\alpha_s^4 = \frac{\pi^4}{l^4} + c\frac{2T_s(n)}{l} = \frac{\pi^4}{l^4} + \frac{m^8\varepsilon^8 l\eta}{n}\frac{2T_s(n)}{l} = \frac{\pi^4}{l^4} + \frac{2T_s(n)\eta\varepsilon^8}{n}m^8.$$

Substitution of the obtained α_s^4 in (3.16) gives the following expression for the eigenvalue of the critical pressure parameter of the simply supported shell

$$\lambda^s(\eta) = \min_m \left[\left(\frac{\pi^4}{l^4} + \frac{2T_s(n)\varepsilon^8\eta}{n}m^8\right)m^{-6} + \varepsilon^8 m^2\right] =$$
$$= \min_m \left[\frac{\pi^4}{l^4}m^{-6} + \frac{2T_s(n)\varepsilon^8\eta}{n}m^2 + \varepsilon^8 m^2\right]$$

In general, the minimized function $\lambda^s(\eta, m)$ takes the form

$$\lambda(\eta, m) = Xm^{-6} + Ym^2, \quad \text{where} \quad X = \frac{\pi^4}{l^4}, \quad Y = \varepsilon^8\left(1 + \frac{2T_s(n)\eta}{n}\right). \tag{3.17}$$

Minimize the obtained function $\lambda(\eta, m)$ over m:

$$-6Xm^{-7} + 2Ym = 0, \quad m^2 = \sqrt[4]{\frac{3X}{Y}}, \quad \lambda_{min} = \sqrt[4]{\frac{256}{27}XY^3}. \tag{3.18}$$

then

$$\lambda^s(\eta) = \sqrt[4]{\frac{256}{27}\frac{\pi^4}{l^4}\left(\varepsilon^8\left(1 + \frac{2T_s(n)\eta}{n}\right)\right)^3} = \frac{4\pi\varepsilon^6}{l\sqrt[4]{27}}\sqrt[4]{\left(1 + \frac{2T_s(n)\eta}{n}\right)^3}.$$

Since $\varepsilon^8 = h^2/(12\sigma)$, the value of λ^s for the case of simply supported edges of the shell can be written as

$$\lambda^s(\eta) = \lambda^s(0)\sqrt[4]{\left(1 + \frac{2T_s(n)\eta}{n}\right)^3}, \quad \text{where} \quad \lambda^s(0) = \frac{\sqrt{6}\pi h^{3/2}}{9l\sigma^{3/4}}. \tag{3.19}$$

In the case of simply supported ends of the shell, the boundary problem (3.2, 3.4) has an eigenvalue independent of η, which is also the eigenvalue of the boundary problem (3.2, 3.3, 3.5). To find the limiting value of η, substitute the eigenvalue α_n from (3.12) of the boundary problem (3.2, 3.3) into expression (3.6):

$$\lambda_n^s = \min_m \left[\frac{\alpha_n^4}{m^6} + \varepsilon^8 m^2\right] = \sqrt[4]{\frac{256}{27}\alpha_n^4\varepsilon^{24}} = \alpha_n\varepsilon^6\frac{4}{\sqrt[4]{27}} = \frac{\pi n}{l}\frac{h^{3/2}}{12^{3/4}\sigma^{3/4}}\frac{4}{\sqrt[4]{27}} = \lambda^s(0)n.$$

The parameter η_s^*, which is the root of the equation

$$\lambda^s(0)\sqrt[4]{\left(1+\frac{2T_s(n)\eta_s^*}{n}\right)^3}=\lambda_n^s,$$

is referred to as the *effective stiffness of the ring.* The critical pressure parameter does not increase for values of the ring stiffness η greater than η_s^*. Therefore,

$$\frac{\lambda^s(\eta)}{\lambda^s(0)}=\begin{cases}\sqrt[4]{\left(1+\frac{2T_s(n)}{n}\eta\right)^3}, & \eta\leqslant\eta_s^*\\ n, & \eta>\eta_s^*\end{cases},\quad \eta_s^*=\frac{n(n^{4/3}-1)}{2T_s(n)}. \tag{3.20}$$

Consider the problem of determining the value of the critical pressure parameter λ^c for other boundary conditions, namely, for a cylindrical shell with clamped (3.4) edges. In this case, the coordinates of the parallels where the shell is stiffened with rings of stiffness c_i are the nodes of the vibration form of the unstiffened clamped shell (3.13). Taking into account the expression for c (3.7), the eigenvalue of the boundary problem (3.2, 3.4, 3.5) can be written as:

$$\alpha_c^4=\left(\frac{3\pi}{2l}\right)^4+c\frac{T_c(n)}{I_0}=\left(\frac{3\pi}{2l}\right)^4+\frac{m^8\varepsilon^8 l\eta}{n}\frac{T_c(n)}{I_0}=\left(\frac{3\pi}{2l}\right)^4+\frac{\varepsilon^8 lT_c(n)\eta}{nI_0}m^8.$$

The corresponding value of the critical pressure parameter $\lambda^s(\eta)$, taking into account the minimum value of the function (3.18), is

$$\lambda^c=\min_m\left[\frac{\alpha_c^4}{m^6}+\varepsilon^8m^2\right]=\min_m\left[\left(\frac{3\pi}{2l}\right)^4m^{-6}+\frac{\varepsilon^8 lT_c(n)\eta}{nI_0}m^2+\varepsilon^8m^2\right]=$$
$$=\sqrt[4]{\frac{256}{27}\left(\frac{3\pi}{2l}\right)^4\varepsilon^{24}\left(1+\frac{lT_c(n)\eta}{nI_0}\right)^3}=\frac{6\pi\varepsilon^6}{l3^{3/4}}\sqrt[4]{\left(1+\frac{lT_c(n)\eta}{nI_0}\right)^3}$$

Taking into account the expression for the small parameter $\varepsilon^8=h^2/(12\sigma)$, the value of λ^c for the case of clamped edges of the shell can be written as

$$\lambda^c(\eta)=\lambda^c(0)\sqrt[4]{\left(1+\frac{lT_c(n)\eta}{nI_0}\right)^3},\quad\text{where}\quad\lambda^c(0)=\frac{\pi h^{3/2}}{\sqrt{6}l\sigma^{3/4}}. \tag{3.21}$$

As in the case of simply supported edges of the shell, the boundary problem (3.2, 3.4) has an eigenvalue independent of η, which is also the eigenvalue of the boundary problem (3.2, 3.4, 3.5). To find the limiting value of η, substitute the eigenvalue α_n from (3.13) of the boundary problem (3.2, 3.4) into expression (3.6):

$$\lambda_n^c = \min_m \left[\frac{\alpha_n^4}{m^6} + \varepsilon^8 m^2\right] = \sqrt[4]{\frac{256}{27}\alpha_n^4 \varepsilon^{24}} = \frac{4}{3^{3/4}} \frac{(2n+1)\pi}{2l} \frac{h^{3/2}}{12^{3/4}\sigma^{3/4}} = \frac{2n+1}{3}\lambda^c(0).$$

The effective stiffness of the ring η_c^* in this case is the root of the equation

$$\lambda^c(0)\sqrt[4]{\left(1 + \frac{lT_c(n)\eta_c^*}{nI_0}\right)^3} = \lambda_n^c.$$

Increasing the stiffness of the ring η after reaching the value η_c^* does not lead to an increase in the critical pressure parameter. Therefore,

$$\frac{\lambda^c(\eta)}{\lambda^c(0)} = \begin{cases} \sqrt[4]{\left(1 + \frac{lT_c(n)\eta}{nI_0}\right)^3}, & \eta \leqslant \eta_c^* \\ \frac{2n+1}{3}, & \eta > \eta_c^* \end{cases}, \quad \eta_c^* = \frac{nI_0}{lT_c(n)}\left(\left(\frac{2n+1}{3}\right)^{4/3} - 1\right). \quad (3.22)$$

3.2 Maximum increase in critical pressure of a cylindrical shell stiffened with rings of varying stiffness

Let the mass of the stiffened shell be fixed. Consider the problem of determining the optimal distribution of mass between the rings and the shell (cladding), corresponding to the highest value of critical pressure.

The critical pressure p_0 for a smooth shell of length l and thickness h_0 can be found using the approximate Southwell-Papkovich formula [30]:

$$p = \lambda E h_0, \quad (3.23)$$

where in the case of simply supported edges of the shell

$$\lambda = \lambda^s(0) = \frac{\sqrt{6}\pi h_0^{3/2}}{9l\sigma^{3/4}}, \quad p_0^s = \frac{\sqrt{6}\pi h_0^{5/2} E}{9l\sigma^{3/4}}. \quad (3.24)$$

and in the case of clamped edges

$$\lambda = \lambda^c(0) = \frac{\sqrt{6}\pi h^{3/2}}{6l\sigma^{3/4}}, \quad p_0^c = \frac{\sqrt{6}\pi h_0^{5/2} E}{6l\sigma^{3/4}}. \tag{3.25}$$

Assume that by reducing the thickness of the shell to h, $n-1$ rings of rectangular cross-section with width a and height b_i, $i = 1 \ldots n-1$, are installed on it. Let the mass of the shell be

$$M(h) = \rho \cdot 2\pi R \cdot Rh \cdot Rl,$$

and the mass of the rings

$$M_r = \rho \sum_{i=1}^{n-1} 2\pi R \cdot Ra \cdot Rakf(i),$$

where $k = b/a$ is the ratio of the height of the first ring to its width, and $f(i)$ is the function of the construction profile. Then from the condition of the mass of the stiffened shell equaling the mass of the smooth shell

$$\rho \cdot 2\pi R \cdot Rh_0 \cdot Rl = \rho \cdot 2\pi R \cdot Rh \cdot Rl + \rho \sum_{i=1}^{n-1} 2\pi R \cdot Ra \cdot Rakf(i)$$

the dependence of the width of the rings a on the ratio of the thicknesses of the stiffened and smooth shells $d = h/h_0$ can be expressed as:

$$a^2 = \frac{1-d}{A}, \quad \text{where} \quad A = \frac{kP(n)}{h_0 l}, \quad P(n) = \sum_{i=1}^{n-1} f(i). \tag{3.26}$$

For the optimal distribution of the construction mass between the shell and the rings, for the case of simply supported edges, introduce the function f_s of the ratio of the critical pressure of the stiffened shell to the critical pressure of the smooth shell. Taking into account the relationship (3.20), write f_s as

$$f_s = \frac{p^s}{p_0^s} = \frac{\lambda^s(\eta) E h}{\lambda^s(0) E h_0} = \begin{cases} d^{5/2} \sqrt[4]{\left(1 + \frac{2T_s(n)}{n}\eta\right)^3}, & 0 \leqslant \eta \leqslant \eta_s^* \\ d^{5/2} n, & \eta > \eta_s^* \end{cases}, \tag{3.27}$$

and find the maximum value of this function.

Transform the second term under the root expression taking into account the value of the relative stiffness of the ring η (3.7):

$$\frac{2T_s(n)}{n}\eta = \frac{2T_s(n)}{n}\frac{12\sigma nJ}{h^3 l} = \frac{24T_s(n)\sigma}{h^3 l}\frac{a^4k^3}{12} = B_s\frac{a^4}{d^3}, \quad \text{where} \quad B_s = \frac{2T_s(n)\sigma k^3}{h_0^3 l}.$$

Then the function f_s can be written as

$$f_s = \begin{cases} d^{5/2}n, & 0 \leqslant d \leqslant d_* \\ d^{5/2}\sqrt[4]{\left(1+B_s\frac{a^4}{d^3}\right)^3}, & d_* < d \leqslant 1 \end{cases}, \tag{3.28}$$

where d_* is the optimal ratio of the thicknesses of the stiffened and smooth shells, at which the maximum value of the function f_s is achieved. Taking into account the condition of equality of the masses of the stiffened and smooth shells (3.26)

$$f_s = \begin{cases} d^{5/2}n, & 0 \leqslant d \leqslant d_* \\ d^{5/2}\sqrt[4]{\left(1+\frac{B_s}{A^2}\frac{(1-d)^2}{d^3}\right)^3}, & d_* < d \leqslant 1 \end{cases}. \tag{3.29}$$

The value d_* is found from the condition of continuity of the function f_s at the point d_*, which we write as

$$d^3 = C_s(d-1)^2, \tag{3.30}$$

where

$$C_s = \frac{B_s}{A^2(n^{4/3}-1)} = \frac{2T_s(n)k^3\sigma}{lh_0^3}\frac{h_0^2l^2}{k^2P^2(n)}\frac{1}{n^{4/3}-1} = \frac{2kl\sigma}{h_0}\frac{T_s(n)}{P^2(n)(n^{4/3}-1)}.$$

Since $C_s > 0$, equation (3.30) has a unique real root in the interval $(0,1)$. The function $g_s(d) = f_s^{4/3}$ increases for $d \in [0; d_*]$, therefore, the maximum of the function is on the interval $[d_*; 1]$. On this interval, the function $g_s(d)$ can be written as:

$$g_s(d) = d^{1/3}\left(d^3 + \frac{B_s}{A^2}(d-1)^2\right).$$

The roots of the equation $g_s'(d) = 0$ are outside the segment $[d_*; 1]$ (see [60]), hence the maximum of the function $f_s(d)$ is achieved at one of the ends $[d_*; 1]$.

$$f_s^* = \max\left[d_*^{\frac{5}{2}} n; 1\right].$$

For small h_0, the inequality $d_*^{\frac{5}{2}} n > 1$ holds, as $d_* \to 1$ when $h_0 \to 0$.

Thus, a cubic equation is obtained, the root of which corresponds to the maximum critical pressure of a clamped shell stiffened with rings of varying stiffness:

$$d_*^3 - \frac{2kl\sigma}{h_0} \frac{T_s(n)}{P^2(n)(n^{4/3} - 1)} (d_* - 1)^2 = 0.$$

The optimal thickness of the rings a^* and the maximum value of the function f_s, corresponding to d_*, can be calculated by the formulas

$$a^* = \sqrt{\frac{1 - d_*}{A}}, \quad f_s^* = d_*^{5/2} n.$$

Finally, consider the case of rigid clamping of the shell edges. For this, we introduce the function f_c of the ratio of the critical pressure of the stiffened shell with clamped edges to the critical pressure of the smooth shell with clamped edges. Use the relationship (3.22) for this purpose:

$$f_c = \frac{p^c}{p_0^c} = \frac{\lambda^c(\eta) h}{\lambda^c(0) h_0} = \begin{cases} d^{5/2} \sqrt[4]{\left(1 + \frac{lT_c(n)}{I_0 n}\eta\right)^3}, & 0 \leqslant \eta \leqslant \eta_c^* \\ d^{5/2}\left(\frac{2n+1}{3}\right), & \eta_c^* < \eta \end{cases}.$$

Taking into account the condition of mass equality of the stiffened and smooth shells (3.26), as well as the expression for the relative stiffness of the rings η (3.7), the second term under the root expression can be written as

$$\frac{lT_c(n)}{I_0 n}\eta = \frac{lT_c(n)}{I_0 n} \frac{12\sigma n}{h^3 l} \frac{a^4 k^3}{12} = \frac{B_c}{A^2} \frac{(1-d)^2}{d^3}, \quad \text{where} \quad B_c = \frac{\sigma T_c(n) k^3}{I_0 h_0^3},$$

then

$$f_c = \begin{cases} d^{5/2}\left(\frac{2n+1}{3}\right), & 0 \leqslant d \leqslant d_* \\ d^{5/2} \sqrt[4]{\left(1 + \frac{B_c}{A^2}\frac{(1-d)^2}{d^3}\right)^3}, & d_* < d \leqslant 1 \end{cases}. \qquad (3.31)$$

where d_* is the root of the cubic equation

$$d_*^3 = C_c(d_* - 1)^2,$$

where

$$C_c = \frac{B_c}{\left(\left(\frac{2n+1}{3}\right)^{4/3} - 1\right) A^2} = \frac{\sigma T_c(n) k^3}{I_0 h_0^3} \frac{h_0^2 l^2}{k^2 P^2(n)} \frac{1}{\left(\frac{2n+1}{3}\right)^{4/3} - 1} =$$
$$= \frac{\sigma k l^2}{h_0 I_0} \frac{T_c(n)}{P^2(n)\left(\left(\frac{2n+1}{3}\right)^{4/3} - 1\right)}.$$

This equation has one root in the interval $(0, 1)$.

The function $g_c = f_c^{4/3}$ increases for $d \in [0; d_*]$, therefore, the maximum of the function is on the interval $[d_*; 1]$. On this interval, the function g can be written as:

$$g_c(d) = d^{1/3}\left(d^3 + B_c A^{-2}(d-1)^2\right).$$

The roots of the equation $g_c'(d) = 0$ are outside the segment $[d_*; 1]$, hence the maximum of the function $f_c(d)$ is achieved at one of the ends $[d_*; 1]$.

$$f_c^* = \max\left[\frac{2n+1}{3} d_*^{\frac{5}{2}}; 1\right].$$

For small h_0, the inequality $d_*^{\frac{5}{2}} \cdot \frac{2n+1}{3} > 1$ holds, as $d \to 1$ when $h_0 \to 0$.

Thus, a cubic equation is obtained, the root of which corresponds to the maximum critical pressure of a clamped shell stiffened with rings of varying stiffness:

$$d_*^3 - \frac{\sigma k l^2}{h_0 I_0} \frac{T_c(n)}{P^2(n)\left(\left(\frac{2n+1}{3}\right)^{4/3} - 1\right)} (d_* - 1)^2 = 0,$$

At the same time, a_* and f_c^*, corresponding to d_*, can be found by the following formulas:

$$a_* = \sqrt{\frac{1 - d_*}{A}}, \quad f_c^* = \left(\frac{2n+1}{3}\right)\sqrt{d_*^5}.$$

3.3 Determining the critical pressure of a structure using analytical and numerical methods

This section investigates the buckling of a cylindrical shell stiffened with rings under normal external pressure, both analytically and numerically. As an example, a copper cylindrical shell of length $l = 4$ and thickness $h_0 = 0.01$ with Young's modulus $E = 11 \cdot 10^{10}$ Pa, Poisson's ratio $\nu = 0.35$, and density $\rho = 8920$ kg/m^3 is considered. The approximate value of the critical pressure of the smooth unstiffened shell (p_0) is calculated using formula (3.24) for simply supported edges, and formula (3.25) for clamped edges. The corresponding values p_s^0 and p_c^0 are

$$p_s^0 = \frac{\sqrt{6}Eh_0^{5/2}}{9l\sigma^{3/4}} \approx 259.3 \text{ kPa}, \quad p_c^0 = \frac{\sqrt{6}Eh_0^{5/2}}{6l\sigma^{3/4}} \approx 389 \text{ kPa}.$$

For a shell stiffened with n_s rings of width a and heights $b_i = kaf(i)(i = 1, \ldots, n_s)$, the function $f(i)$ determines the distribution of ring heights along the generatrix of the shell, and consequently, the distribution of ring stiffnesses and the construction profile. These functions for linear, parabolic, and exponential distributions are defined by formulas (3.8—3.11).

The approximate value of the critical pressure of such a structure can be obtained using the formula $p_s = p_s^0 f_s$ for simply supported edges, and the formula $p_c = p_c^0 f_c$ for clamped edges. The values of f_s and f_c for different construction profiles are provided in tables 3.1 and 3.2. Figures 3.4 and 3.5 show the dependencies of the functions f_s and f_c on the number of rings for various ratios of width and height of the rings.

Based on the obtained results, it can be concluded that the use of non-uniform rings for reinforcing a cylindrical shell leads to a more significant increase in its critical pressure compared to stiffenning with uniform rings. Among the three considered construction profiles, the profile with an exponential law of ring height distribution

Table 3.1 — The value of the function f_s for a simply supported shell supported by n_s rings.

		$f_{lin}(i)$			$f_{parab}(i)$			$f_{exp}(i)$		
	n_s \ u	1	1,5	2	1	1,5	2	1	1,5	2
$k=1$	4	2,94	3,37	3,67	2,94	3,37	3,67	2,94	3,37	3,67
	5	3,11	3,78	4,24	3,11	3,70	4,12	3,11	4,00	4,56
	6	3,22	4,08	4,67	3,22	3,98	4,52	3,22	4,40	5,10
	7	3,28	4,42	5,18	3,28	4,22	4,89	3,28	5,49	6,38
	8	3,32	4,66	5,56	3,32	4,44	5,23	3,32	6,00	7,01
$k=1,5$	4	3,20	3,53	3,78	3,20	3,53	3,78	3,20	3,53	3,78
	5	3,43	3,98	4,38	3,43	3,91	4,26	3,43	4,19	4,68
	6	3,60	4,33	4,85	3,60	4,24	4,71	3,60	4,64	5,25
	7	3,72	4,72	5,39	3,72	4,53	5,11	3,72	5,74	6,52
	8	3,80	5,01	5,81	3,80	4,78	5,49	3,80	6,29	7,18
$k=2$	4	3,37	3,64	3,86	3,37	3,64	3,86	3,37	3,64	3,86
	5	3,65	4,14	4,49	3,65	4,07	4,38	3,65	4,33	4,77
	6	3,87	4,53	4,99	3,87	4,43	4,85	3,87	4,82	5,37
	7	4,03	4,94	5,55	4,03	4,76	5,29	4,03	5,92	6,63
	8	4,15	5,27	6,00	4,15	5,05	5,69	4,15	6,51	7,31

provides the greatest increase in critical pressure. Moreover, it is noticeable that clamping the edges of the shell leads to higher critical pressure values than simply supported edges. These results emphasize the importance of considering both the type of construction profile and boundary conditions in the improvement of cylindrical shell structures.

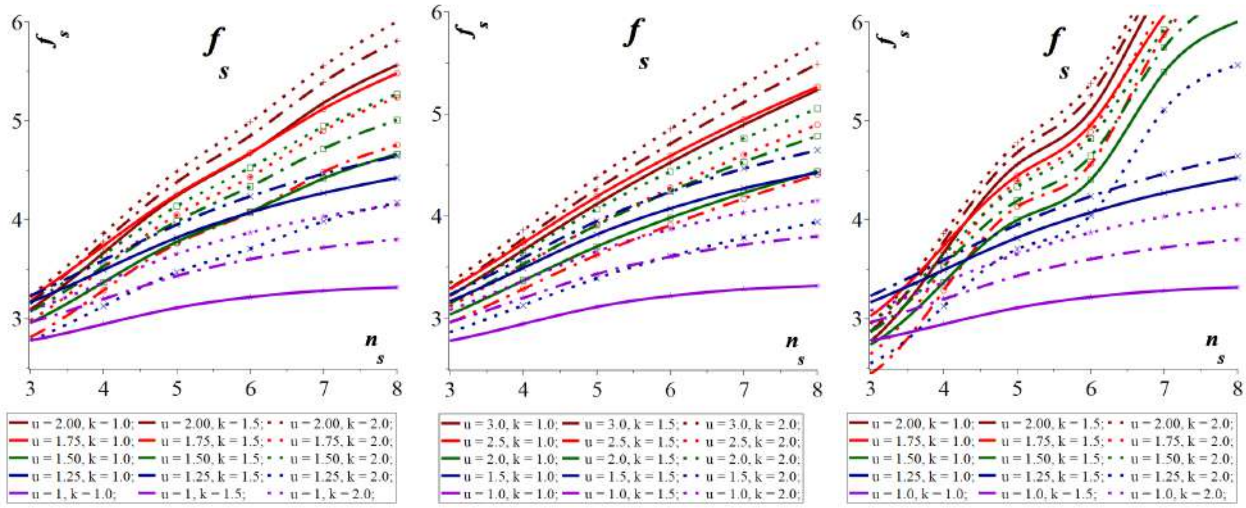


Fig 3.4 — Values of the function f_s for a) linear, b) parabolic, c) exponential construction profiles.

Table 3.3 compares critical pressures obtained by the analytical method (p), described in paragraph 3.3, and the finite element method (p_{fem}) using the *Comsol* package. As an example, several different values of the parameters u and k, characterizing the construction profile, are considered. The clamped shell described above with a linear function of stiffness distribution is analyzed. For a clamped smooth shell of thickness $h = 0.01$, the critical pressure value obtained by the finite element method is $p_{max} = 384840$ Pa.

The good agreement between the analytical and numerical solution results shows that the analytical approach described above can be used for approximate parameter selection before the start of design.

3.4 Minimization of the mass of a cylindrical shell stiffened with rings of varying stiffness

Consider a structure consisting of a cylindrical shell and supporting circular rings of varying stiffness. We seek parameters at which the structure has the lowest mass and

Table 3.2 — The value of the function f_c for a clamped shell stiffened with n_s rings.

	n_s \ u	$f_{lin}(i)$ 1	$f_{lin}(i)$ 1,5	$f_{lin}(i)$ 2	$f_{parab}(i)$ 1	$f_{parab}(i)$ 1,5	$f_{parab}(i)$ 2	$f_{exp}(i)$ 1	$f_{exp}(i)$ 1,5	$f_{exp}(i)$ 2
$k=1$	4	2,42	2,71	2,91	2,42	2,71	2,91	2,42	2,71	2,91
	5	2,57	3,04	3,34	2,57	2,99	3,26	2,57	3,20	3,55
	6	2,68	3,30	3,69	2,68	3,23	3,58	2,68	3,52	3,96
	7	2,75	3,59	4,09	2,75	3,44	3,89	2,75	4,30	4,82
	8	2,80	3,80	4,40	2,80	3,63	4,17	2,80	4,70	5,28
$k=1{,}5$	4	2,59	2,81	2,97	2,59	2,81	2,97	2,59	2,81	2,97
	5	2,79	3,17	3,43	2,79	3,12	3,35	2,79	3,32	3,62
	6	2,94	3,46	3,79	2,94	3,39	3,70	2,94	3,67	4,05
	7	3,06	3,78	4,21	3,06	3,64	4,03	3,06	4,45	4,90
	8	3,15	4,03	4,55	3,15	3,87	4,33	3,15	4,87	5,38
$k=2$	4	2,70	2,89	3,02	2,70	2,89	3,02	2,70	2,89	3,02
	5	2,94	3,27	3,49	2,94	3,22	3,42	2,94	3,40	3,67
	6	3,13	3,58	3,88	3,13	3,52	3,79	3,13	3,78	4,12
	7	3,28	3,92	4,31	3,28	3,79	4,13	3,28	4,55	4,96
	8	3,40	4,20	4,67	3,40	4,04	4,46	3,40	5,00	5,45

does not buckle under the action of external pressure p, where p is the critical pressure of a smooth cylindrical shell of thickness h_0.

The mass of the structure M, consisting of a cylindrical shell of length l and thickness h, with $n-1$ rings of thickness a and width $b_i (i = 1, \ldots, n-1)$, is given by

$$M = 2\pi\rho R^3 lh + 2\pi\rho R^3 \sum_{i=1}^{n-1} ab_i = 2\pi\rho R^3 \left(lh + a^2 k \sum_{i=1}^{n-1} f(i) \right),$$

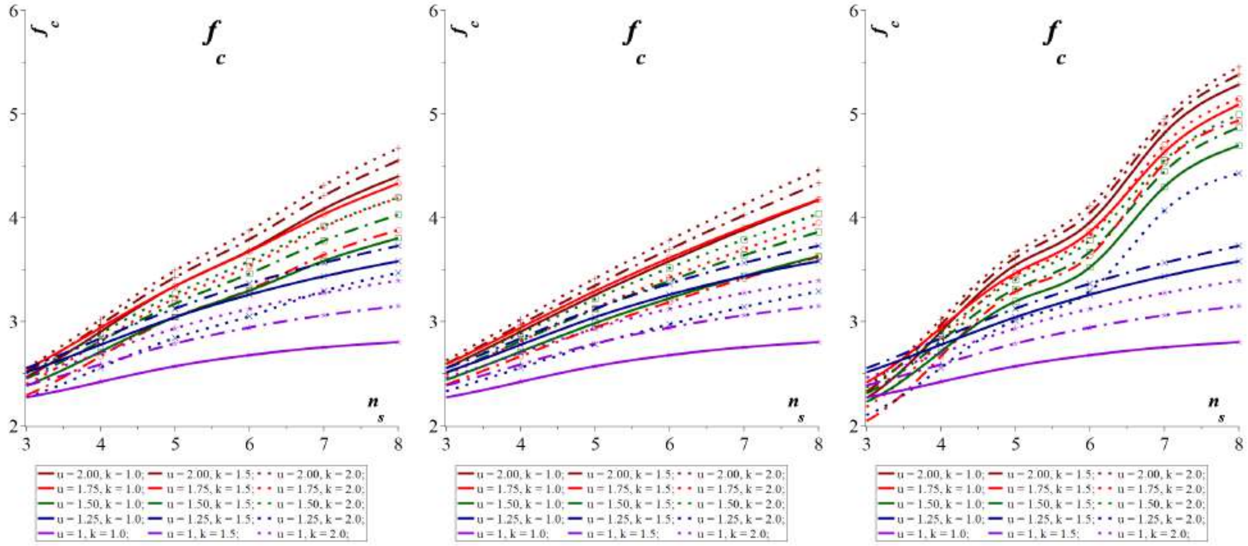


Fig 3.5 — Values of the function f_c for a) linear, b) parabolic, c) exponential construction profiles.

Table 3.3 — Critical pressure values obtained by the analytical method p and the finite element method p_{fem}.

	$k=1$				$k=2$			
	n_s	p,Pa	p_{fem},Pa	d_*	n_s	p,Pa	p_{fem},Pa	d_*
$u=1$	3	628199	609440	0.918	3	666391	645430	0.9399
	4	700574	688990	0.8850	4	760599	747720	0.9145
	5	757109	755760	0.8539	5	839855	839140	0.89
$u=1.5$	3	658402	614370	0.9354	3	689856	642590	0.9530
	4	754081	717130	0.9114	4	803660	762560	0.9349
	5	863682	804290	0.9	5	927847	861660	0.9262
$u=2$	3	681002	616760	0.9481	3	707084	637890	0.9625
	4	787950	733720	0.9276	4	830211	770330	0.9471
	5	918921	811080	0.9226	5	971589	851530	0.9434

and the mass of the smooth shell M_0:

$$M_0 = 2\pi\rho R^3 l h_0,$$

where ρ is the material density, $k = b/a$ is the ratio of the height of the first ring to its width, and $f(i)$ is the construction profile function. Then the ratio of the mass of the stiffened shell to the mass of the smooth shell $F(a, d)$ is written as

$$F(a,d) = \frac{M}{M_0} = d + Aa^2, \quad \text{where} \quad d = \frac{h}{h_0}, \quad A = \frac{kP(n)}{h_0 l}, \quad P(n) = \sum_{i=1}^{n-1} f(i). \tag{3.32}$$

For the case of simply supported edges of the cylindrical shell (3.3), we use the function of the ratio of the critical pressure of the stiffened shell to the critical pressure of the smooth shell (3.27)

$$f_s = \begin{cases} d^{5/2} \sqrt[4]{\left(1 + \frac{2T_s(n)}{n}\eta\right)^3}, & 0 \leqslant \eta \leqslant \eta_s^* \\ d^{5/2} n, & \eta > \eta_s^* \end{cases}, \quad \text{where} \quad \eta_s^* = \frac{n(n^{4/3} - 1)}{2T_s(n)},$$

which, for convenience of analysis, is rewritten as (3.28)

$$f_s = \begin{cases} d^{5/2} n, & 0 \leqslant d \leqslant d_* \\ d^{5/2} \sqrt[4]{\left(1 + \frac{B_s a^4}{d^3}\right)^3}, & d_* < d \leqslant 1 \end{cases},$$

where

$$B_s = \frac{2\sigma k^3}{l h_0^3} T_s(n), \quad T_s(n) = \sum_{i=1}^{n-1} f^3(i) \sin^2\left(\frac{\pi i}{n}\right).$$

The optimal value of the ratio of the thicknesses of the stiffened and smooth shells d_* will be the root of the equation $f_s = 1$ for $\eta = \eta_s^*$:

$$d_*^s = n^{-2/5},$$

and the height of the first ring is

$$a_*^s = \sqrt[4]{\frac{\left(n^{4/3} - 1\right) \cdot d_*^3}{B_s}}.$$

Substituting the obtained optimal values a_*^s and d_*^s into (3.32), we obtain the ratio of the masses of the stiffened shell and smooth shell with equal critical pressure values for the case of simply supported edges:

$$F_s(n) = n^{-2/5} + \frac{A}{\sqrt{B_s}} \sqrt{n^{2/15} - n^{-6/5}}.$$

The same method can be used to solve the problem of minimizing the mass of the structure in the case of clamped edges of the cylindrical shell.

3.5 Analytical and numerical determination of minimum mass of a structure with a given critical pressure

Table 3.4 presents the values of the mass ratio function F_s for the previously described structure with simply supported edges. As the number of rings n_s increases, the mass of the structure decreases. The same trend is observed with an increase in the ratio of the height of the first ring to its width ($k = b/a$). The parameter u, which represents the amplitude of the distribution function ($u = b_2/b_1$), is chosen for each distribution function such that the optimal ratio of the thickness of the stiffened shell to the thickness of the smooth shell ($d_* = h/h_0$) is the same, thereby making it possible to assess the influence of the choice of construction profile function on mass reduction. For all considered values of n_s and k, the greatest reduction in the mass of the structure is observed when using the exponential distribution function.

Table 3.4 — Values of the function F_s for the case of stiffenning of the shell with identical rings (f_0), as well as for linear (f_{lin}), parabolic (f_{parab}), and exponential (f_{exp}) construction profiles.

	$k = 1$				$k = 2$			
	n_s	d_*	a_*	M_s/M_0	n_s	d_*	a_*	M_s/M_0
$f_0(i)$	4	0.525	0.031	0.623	4	0.525	0.019	0.595
	5	0.488	0.030	0.603	5	0.488	0.018	0.570
	7	0.435	0.029	0.580	7	0.435	0.017	0.537
	9	0.398	0.028	0.568	9	0.398	0.016	0.518
	n_s	d_*	a_*	M_s/M_0	n_s	d_*	a_*	M_s/M_0
$f_{lin}(i)$, $u = 2$	4	0.525	0.020	0.585	4	0.525	0.012	0.568
	5	0.488	0.016	0.545	5	0.488	0.010	0.529
	7	0.435	0.012	0.496	7	0.435	0.007	0.478
	9	0.398	0.010	0.461	9	0.398	0.006	0.443
	n_s	d_*	a_*	M_s/M_0	n_s	d_*	a_*	M_s/M_0
$f_{parab}(i)$, $u = 2$	4	0.525	0.010	0.556	4	0.525	0.006	0.547
	5	0.488	0.007	0.517	5	0.488	0.004	0.508
	7	0.435	0.005	0.460	7	0.435	0.002	0.453
	9	0.398	0.002	0.420	9	0.398	0.001	0.414
	n_s	d_*	a_*	M_s/M_0	n_s	d_*	a_*	M_s/M_0
$f_{exp}(i)$, $u = 1.2$	4	0.525	0.008	0.551	4	0.525	0.005	0.544
	5	0.488	0.004	0.503	5	0.488	0.002	0.498
	7	0.435	0.002	0.445	7	0.435	0.001	0.442
	9	0.398	0.001	0.404	9	0.398	0.001	0.403

Conclusion

In the context of further research, the results presented in the dissertation provide a foundation for several promising directions.

For tasks involving vibrations and stability loss of a shell reinforced with frames of varying stiffness, frames with non-zero eccentricity should be considered. In practice, the frame is located either outside or inside the shell.

In the task of vibrations of a cylindrical shell conjugated with a shallow cap, special attention can be given to cases where the materials of the cap and shell differ, possibly having a non-uniform structure or using different methods of attachment. Also, cases where the shell's cap has the shape of an ellipsoid of rotation or other geometric peculiarities.

A detailed analysis of these scenarios will require the development of more complex mathematical models and, possibly, new research methods. These directions open up opportunities for a deeper understanding of the impact of various parameters on the overall strength and stability of the structure.

Additionally, further research may focus on analyzing the effectiveness of various methods of reinforcing connections between the cap and the shell, ultimately contributing to the broader practical application of the obtained results in applied projects.

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