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Control algorithms constructing for nonlinear controlled systems

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Contents

Introduction

Relevance of the topic.

One of the fields of study in mathematical control theory is research on the boundary problems for controlled systems of ordinary differential equations (ODEs). For the first time, the solution to these problems for linear non-stationary systems in the class of square-integrable control functions was obtained by Kalman [1].

There are several main research areas on the boundary problems for controlled ODE systems.

The first area of study is related to finding the necessary and sufficient conditions for the right part of controlled systems that guarantee the transfer of a controlled system to a given point in the phase space. Works of Zubov V. I., Krasovskiy N. N., Potapov A. P., Leps N. L., Komarov V. A., Walczak S., Ohta Y., Maeda H., Dirk A., Jersy S., Nistri P., and others were devoted to these researches.

The second research area includes the study of the final states set to which transfer of a controlled system from some initial state is possible. The main results of this area of research were presented in the scientific works of Kalman R., Chernous'ko F. L., Panasuk A. I., Berdyshev Yu. I. and others.

The third direction of research concerns to development of accurate and approximate methods of control function construction and the corresponding trajectories that are connecting given points of the phase space. The most significant results in the field of study are given in the works of Krasovskiy N. N., Zubov V. I., Chernous'ko F. L., Krischenko A. P., Kvitko A. N., and others.

All the aspects mentioned above of the research on controlled systems have been studied enough for linear stationary, non-stationary, and non-linear systems of a special type.

Let us note some methods that are beeing applied for solving the control problems for non-linear systems:

– Pontryagin's maximum principle [2], etc.;

– differential-geometric approach [3—6], etc.;

– method of inverse dynamics problems [7—9], etc.;

– method of inverse spectral problems [10], etc.;

– approximate methods of solving [11; 12], etc.;

– methods of intelligent control [13], etc.;

– neural network methods [14], etc.;
– reinforcement learning [15], etc.;
– classical methods of control theory [16—21], etc.

It follows from the above theory for solving boundary value problems for non-linear systems of general form is fundamental, and it is far from being full developed.

Analysis of the publication activity in the referative database of the peer-reviewed scientific literature from Scopus since 2012, before 2022 years on the topic "nonlinear systems control"that is presented on the graph (see fig. 1) shows growth of publication quantity.

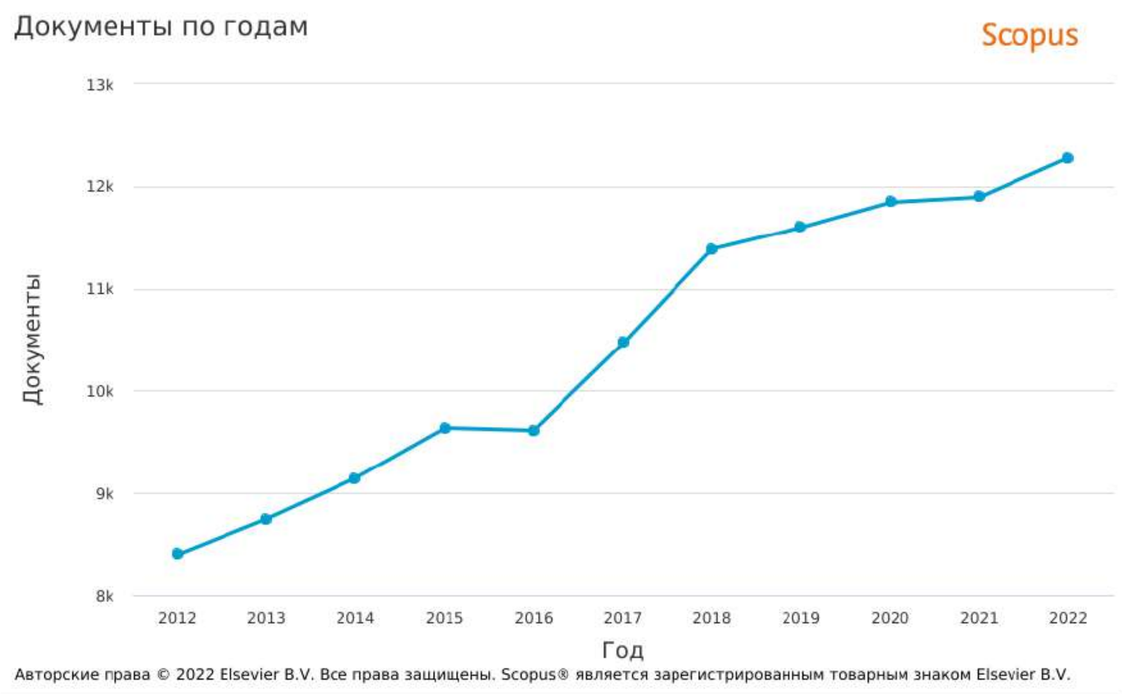

Figure 1 — Analysis of the publication activity in Scopus since 2012 before 2022 years. Keywords: "control" and "nonlinear system" (online; accessed: 13.12.2022).

Thus, the development of methods for constructing of control functions for non-linear systems of ordinary differential equations, as also, algorithms and software development based on the methods, are actual problems of control theory.

Aims and objectives. The main **goals** of the present work are:

1. development of a method for constructing control functions that guarantee the transfer of the control object from an initial state to the given final state at a finite time interval with account for discreteness and limitation of a controlling action and also sufficient simple for numeric realisation and stable to computation errors;

2. development of an algorithm for constructing a control function that guarantees transfer for a wide class of stationary non-linear controlled systems from an initial state to the origin of the coordinate system, taking account of the possibility of the computer system's operability checking;
3. study of dynamics for controlled non-linear systems of ODEs that are describing identical and non-identical Josephson junction arrays by the well-known optimal control method.

To achieve the goals, it is necessary to solve a number of tasks:

1. development of the algorithm for constructing the discrete control function that is restricted by a norm and provides transfer from initial to the given final state for a sufficiently wide class of non-linear non-stationary systems of ordinary differential systems;
2. development of the control algorithm with consideration of operability checking for computer systems based on the control function construction that is providing the transfer of a wide class of non-linear stationary systems of ordinary differential systems from an initial state to the origin of the coordinate system;
3. realisation of the algorithm of discrete control functions constructing for non-linear problem like a set of functions in the Python programming language;
4. testing of the constructed algorithms on the concrete examples and them analysis;
5. solution and numerical modelling of the optimal control problem for arrays of identical and non-identical Josephson junctions.

Methods of the research. In the work, methods of mathematical control theory, stability theory, differential equations theory, mathematical and functional analysis, complexity theory of computations, informational technologies, linear Algebra and numerical methods for systems of ordinary differential equations are applied.

The theoretical and practical significance of the work. New constructing method of the discrete control function is developed that is restricted by a norm and provides transfer from the initial to the given final state for a sufficiently wide class of non-linear non-stationary systems of ordinary differential equations. In addition, constructively sufficient Kalman type condition that guarantees this transfer is obtained.

For the implementation of the discrete control algorithm library of software modules that may be applied for the development of the mathematical packages that are intended for control problem solving and in the learning process is developed.

Algorithm for control function construction that guarantees transfer of non-linear stationary system to the origin of the coordinate system from some initial state for a wide class of non-linear stationary systems of ordinary differential equations with account of the possibility of computer system operability checking is suggested. Constructive sufficient conditions that guarantee the existence of the solution for this problem are found.

Application of the algorithm for computer system checking is possible at the step of control system development, and also in the process of control signal form. The suggested checking method may supplement or substitute traditional engineering-technical approaches. In addition, this algorithm may be used for solving the practical problem of the integration step choice for the Cauchy problem solution of ODE system that describes the mathematical model of the control object.

Josephson junctions are perspectives for quantum bit (qubit) construction. This research area is in sufficiently active progress at the present time. The solution of the control problem for Josephson junction arrays may be applied to the solving of qubits constructing technical problems.

The reliability of the obtained results is ensured by correct application of mathematical control theory, computational mathematics and information technologies. The basic provisions are confirmed by numerical modeling of the practical problems.

Approbation of the research results. The results presented in this dissertation have been presented and discussed at conferences:

1. Litvinov N. Global variables control of a Josephson junctions array. The 10th International Scientific Conference on Physics and Control PHYSCON'2021, Fudan University, Shanghai, China, 04.10.2021-08.10.2021
2. Litvinov N. N. Discrete control of a single-link robot-manipulator with account perturbations //Conference SPISOK-2022, 27–29 april 2022.
3. Litvinov N. N. Optimal control of a single-link robot-manipulator with account perturbations //Conference SPISOK-2022, 27–29 april 2022.

4. Litvinov N. N. On the computational complexity of a discrete control algorithm. LIV International Scientific Conference on Control Processes and Stability (CPS'23), 4–7 april 2023. [22]

Publications. The results of the dissertation were presented by two articles in journals included in the list of editions of the Higher Attestation Commission and indexed in the Scopus database, one paper that has been accepted, and also two computer programmes that have been registered:

1. Kvitko A. N., Litvinov N. N. Solution of a local boundary problem for a non-linear non-stationary system in the class of discrete controls. // Vestnik of Saint Petersburg University. Applied Mathematics. Computer Sciences. Control Processes, 2022, vol. 18, iss. 1, pp. 18–36. (In Russian). [23]
2. Litvinov N. Control of global variables for identical and non-identical Josephson junctions arrays // Cybernetics and Physics, vol. 10, No 3, pp. 138-142, 2021 https://doi.org/10.35470/2226-4116-2021-10-3-138-142. [24]
3. The Certificate on Official Registration of the Computer Program № 2023616889 in Russian Federation. «A library of functions for solving of discrete control problems» (DiscrControlLib) : № 22023615862: req. 24.03.2023: published 03.04.2023 / N. N. Litvinov, A. N. Kvitko; applicant Federal State Budgetary Educational Institution of Higher Education "St. Petersburg State University". (in Russian). [25]
4. The Certificate on Official Registration of the Computer Program № 2023616890 in Russian Federation. «A library of functions for solving of optimal control LQ-problems» (DiscrControlLib) : № 2023615863 : req. 24.03.2023: published 03.04.2023 / N. N. Litvinov; applicant Federal State Budgetary Educational Institution of Higher Education "St. Petersburg State University". (in Russian). [26]
5. Kvitko A. N., Litvinov N. N. Solution of the Local-Boundary-Value Problem of Control for a Nonlinear Stationary System Taking into Account Computer System Verification. Vestnik of Saint-Petersburg University. Mathematics. Mechanics. Astronomy, 2024, Vol. 57, No 2. pp. 202 – 212. Accepted. [27]

Personal contribution of the author to publications. In the collaborative publications, problem statements, suggestions of the solution concept, and discussions of the results belong to scientific advisor Kvitko A. N.

Overview of the thesis. In the **introduction**, the actuality of the dissertation is substantiated, a brief literature review is provided, and the main targets, problems, methods, and research results are presented. The scientific novelty, theoretical and practical significance of the work are discussed.

The **first chapter** of the thesis is devoted to the solution of the discrete control problem for a non-linear, non-stationary system of ordinary differential equations. Formulation and proof of the theorem that is substantiating the method of the problem solving are provided. In addition, estimation of the reachability set for the considered problem is provided.

In the **second chapter**, computational complexity analyses of the discrete control algorithm, numerical modeling for various options of control for robot-manipulator with the help of this algorithm is carried out. Also, a comparison of the constructed algorithm with an optimal control algorithm is done.

In the **third chapter** of the present work, the algorithm for the control function constructing that guarantees the transfer of a non-linear stationary system to the origin of the coordinate system from some initial state for a wide class of non-linear stationary systems of ordinary differential equations with account of the possibility of the computer system operability checking is provided. Constructive, sufficient conditions that guarantee the existence of the solution for the problem are found. Recomendations for the algorithm construction are given, and estimation of the theoretical complexity is provided. The efficiency of the algorithm is demonstrated by numerical modeling of the interorbital flying problem.

The **fourth chapter** is devoted to the solution of the optimal control problem for ODE systems that give a description of the identical and nonidentical Josephson junction arrays, numerical simulation, and analysis of the dynamics with account of control for these models.

In the **conclusion**, a brief discussion of the obtained results and possible future research directions are provided.

In the **appendices**, programme code of the software package for solving the robot-manipulator discrete and optimal control problems are presented.

Structure and contents of the work. The thesis consists of an introduction, 4 chapters, conclusion and 2 appendices. The total volume of the dissertation is 103 pages, including 27 figures. The reference list contains 82 titles.

Main scientific results.

1. The algorithm for constructing the discrete control functions for a sufficiently wide class of non-linear, non-stationary systems is developed [23].
2. The constructively sufficient condition that gives the possibility of transfer from an initial state to the given final state for a wide class of non-linear non-stationary systems are obtained [23].
3. The programme module for solving discrete control problems in the Python programming language is developed [25].
4. The algorithm of the computer system operability checking by solving of boundary value problem for non-linear stationary systems is developed and constructive sufficient condition that guarantees the existence of this solution is found [27].
5. Dynamics of equation systems that are describing arrays of identical and non-identical Josephson junctions with an account of control action have been studied by the optimal control method [24].

Thesis statements are to be defended.

1. Algorithm of piecewise-constant control functions, constructing that provides the transfer of the ODE system from an initial state to the given final state for a wide class of non-linear, non-stationary systems at a finite time interval.
2. Algorithm of boundary problem solution for non-linear stationary system with account of the computer systems operability checking.
3. Finding of constructively sufficient conditions that guarantee the transfer of non-linear stationary system to the origin of the coordinate system from some neighbourhood of an initial state in the discrete and continuous-time controls classes.
4. A package of application programmes for solving the discrete control problems in the Python programming language.
5. Solution of the optimal control problem for arrays of identical and non-identical Josephson junctions.

Chapter 1. Solution of a local boundary problem for a non-linear non-stationary system in the class of discrete controls

1.1 Brief Literature Review

The use of digital computing technology in the formation of a control effect dictates the need to solve the problems of managing ODE systems by constructing piecewise constant or discrete control functions. This circumstance justifies the relevance of the study of boundary value problems for controlled ODE systems in the class of these controls.

The main approaches to solving the boundary problems in the discrete control class for a finite period of time include issues related to finding the necessary and sufficient conditions that guarantee the existence of the solutions [28—34], construction or evaluation of the reachability set, and also development of accurate and approximate methods for construction of control functions [30; 31; 34—38].

Stabilisation problems for linear and non-linear ODE systems in the discrete control class represent considerable interest. These problems may be considered as boundary problems for an infinite period of time [37—42].

Local and global boundary problems in the discrete control class sufficient for linear, quasi-linear, and non-linear systems of a special kind are studied at the present time [28—53].

Solutions of the piecewise constant stabilisation and control with incomplete information for an infinite period of time are presented in the works [17; 37].

In the monograph [52] the synthesis method of discrete control algorithms for non-linear nonstationary systems based on the discretization of the continuous algorithms (that are considered in this work) is considered.

In the publication [53] necessary and sufficient conditions of local optimality in the class of the piecewise-constant controls are suggested.

The construction of algorithms for solving control problems of nonlinear systems of ordinary differential equations using piecewise Lyapunov functions of various types is presented in the works [54; 55] and others.

In the [54] method of solution of the stability problem for nonlinear systems, using piecewise-polynomial Lyapunov functions is suggested.

In the article [55] the stabilisation algorithm for nonlinear systems using piecewise-polynomial Lyapunov functions is considered.

The results of this chapter are published in the paper [23] and included in the work [56].

1.2 Problem staitment and main theorem

Let us consider the controlled system of the ordinary differential equations

$$\dot{x} = f(x,u,t), \tag{1.1}$$

where $x \in R^n$, $x = (x_1, \ldots, x_n)^T$, $u \in R^r$, $u = (u_1, \ldots, u_r)^T$, $r \leqslant n$, $t \in [0,1]$, $f = (f_1, \ldots, f_n)^T$, $f \in C^{(n)}(R^n \times R^r \times R^1; R^n)$;

$$\|u\| \leqslant N, N > 0, N = \text{const}. \tag{1.2}$$

In the next text, we will assume a value $\|x\| = \sqrt{\sum x_i^2}$ as a norm of the vector x and a matrix norm is a norm that is corresponding to the norm of the vector x.

The right part of the system (1.1) sutisfies the conditions

$$f(0,0,t) \equiv 0, \tag{1.3}$$

$$A_0 = \tfrac{\partial f}{\partial x}(0,0,1),\ B_0 = \tfrac{\partial f}{\partial u}(0,0,1),\ S_0 = (B_0,A_0B_0, \ldots ,A_0^{n-1}B_0),$$

$$\text{rank } S_0 = n. \tag{1.4}$$

Let us consider following matrices:

$$\begin{aligned} P &= \alpha e^{-\alpha\tau}A_0 + \alpha e^{-2\alpha\tau}A_1 + \ldots + \alpha e^{-(n-1)\alpha\tau}A_{n-2}, \\ Q &= \alpha e^{-\alpha\tau}B_0 + \alpha e^{-2\alpha\tau}B_1 + \ldots + \alpha e^{-(n-1)\alpha\tau}B_{n-2}, \end{aligned} \tag{1.5}$$

where
$A_i = \frac{(-1)^i}{i!}\frac{\partial^{i+1} f}{\partial x \partial t^i}(0,0,1), i = 1, \ldots , n-1$, $B_i = \frac{(-1)^i}{i!}\frac{\partial^{i+1} f}{\partial u \partial t^i}(0,0,1), i = 1, \ldots , n-1.$

Let's construct a matrix: $S = \{L_1(\tau), \ldots , L_n(\tau)\}$, здесь $L_1(\tau) = Q(\tau)$, $L_i(\tau) = P(\tau)L_{i-1}(\tau) - \frac{dL_{i-1}}{d\tau}$, $i = 2, \ldots ,n.$

Let

$$\operatorname{rank} S(\tau) = n, \;\; \tau \in [0{,}\infty), \;\; \alpha > 0. \tag{1.6}$$

Let us consider splitting the interval $[0, 1]$ by the infinite number of points:

$$0 = t_0 < t_1 < \ldots < t_k < 1,$$

where $t_k \to \infty$ when $k \to \infty$.

Definition 1.1. *Control function $u(t)$ is discrete if $u(t) = u_k$, $u_k \in R^r$, $\forall t \in [t_k, t_{k+1})$, $k = 0{,}1, \ldots$*

Problem 1.1. Find a discrete control $u(t)$ that is defined on an infinite splitting of the interval $[0, 1]$ and an absolutely continuous function $x(t)$ that is almost everywhere satisfying to the system (1.1) and conditions

$$x(0) = x_0, x(1) = 0, x_0 = (x_0^1, \ldots, x_0^n)^T. \tag{1.7}$$

Problem 1.2. Find a discrete control $u(t)$ that is defined on a finite splitting $0 = t_0 < t_1 < \ldots < t_m < 1$ of the interval $[0{,}1]$, $t \in [0{,}t_m]$ and an absolutely continuous function $x(t)$ that is almost everywhere satisfying (1.1) and conditions

$$x(0) = x_0, \; \| x(t_m) \| \leqslant \varepsilon_1, \, |t_m - 1| < \varepsilon_2, \tag{1.8}$$

where t_m – unknown value of the time, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ – fixed numbers.

Theorem 1.1. *Let us suppose that for the right part of the equations system* (1.1) *conditions* (1.3), (1.4), *and* (1.6) *are satisfied, there exists such $\varepsilon > 0$ when for all x_0 such that $\| x_0 \| < \varepsilon$, there exist such solutions of the problems 1.1 and 1.2 which may be found with the help of the stabilisation problem solving for linear non-stationary system of ODEs with exponential coefficients followed by a solution of the Cauchy problem for an auxiliary system.*

1.3 Auxiliary system construction

Let us consider a problem.

Find an absolutely continuous function $x(t)$ and a discrete control function $\bar{u}(t)$, that are almost everywhere satisfying to the system (1.1) and conditions

$$x(0) = x_0, \, x(t) \to 0 \text{ as } t \to 1. \tag{1.9}$$

We call this pair of functions $x(t)$, $\bar{u}(t)$ the solution of the problem (1.1), (1.9).

Obviously, if we have the solution of the problem (1.1), (1.9), we can obtain the solution of the Problem 1.1 with the help of the passage to the limit as $t \to 1$.

For solving the problem (1.1), (1.9), we perform a transformation of the independent variable t by a formula

$$t(\tau) = 1 - e^{-\alpha\tau}, \ \tau \in [0,\infty), \tag{1.10}$$

where $\alpha > 0$ is some constant subjected to definition.

Then the original system (1.1) and condition (1.9) will take the form

$$\frac{dc}{d\tau} = \alpha e^{-\alpha\tau} f(c, d, t(\tau)), \tag{1.11}$$

$$c(0) = x_0, \ c(\tau) \to 0 \quad \text{при} \quad \tau \to \infty,$$

$$c(\tau) = x(t(\tau)), \, c = (c_1 \ldots c_n)^T, \ d(\tau) = u(t(\tau)), \ d = (d_1 \ldots d_r)^T. \tag{1.12}$$

Let us consider a discrete control function in the form

$$\bar{d}(\tau) = d(kh), \, \tau \in [kh, (k+1)h), \, h > 0, \, k = 0,1, \ldots$$

Problem 1.3. Find a discrete control $\bar{d}(\tau)$ and an absolutely continuous function $c(\tau)$ that are almost everywhere satisfying the system (1.11) and the following conditions:

$$c(0) = x_0,\, c(\tau) \to 0 \text{ при } \tau \to \infty. \tag{1.13}$$

Pair functions $c(\tau)$, $\bar{d}(\tau)$, we will call the solution of the problem (1.11), (1.13). Let us introduce the notations: $\tilde{x} = \theta_i c$, $\tilde{d} = \theta_i d$, $\tilde{t}(\tau) = 1 - \theta_i e^{-\alpha\tau}$, $\theta_i \in [0{,}1]$. Let $k_1, \ldots, k_n$, $m_1, \ldots, m_r$ – arbitrary natural numbers then

$$|k| = \textstyle\sum_{j=1}^n k_j,\ |m| = \sum_{j=1}^r m_j,\ k! = k_1! \ldots k_n!,\ m! = m_1! \ldots m_r!.$$

We will imagine the right part of the system (1.11) in the neighbourhood of the point (0,0,1) in Taylor series form:

$$\begin{aligned}
\frac{dc_i}{d\tau} &= \alpha e^{-\alpha\tau} \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(0{,}0{,}1) c_j + \alpha e^{-\alpha\tau} \sum_{j=1}^r \frac{\partial f_i}{\partial u_j}(0{,}0{,}1) d_j + \\
&+ \frac{1}{2}\alpha e^{-\alpha\tau} \Bigg(\sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 f_i}{\partial x_j \partial x_k}(0{,}0{,}1) c_j c_k + \sum_{j=1}^r \sum_{k=1}^r \frac{\partial^2 f_i}{\partial u_j \partial u_k}(0{,}0{,}1) d_j d_k + \\
&+ 2 \sum_{j=1}^n \sum_{k=1}^r \frac{\partial^2 f_i}{\partial x_j \partial u_k}(0{,}0{,}1) c_j d_k - 2\alpha e^{-\alpha\tau} \sum_{j=1}^n \frac{\partial^2 f_i}{\partial x_j \partial t}(0{,}0{,}1) c_j - \\
&- 2\alpha e^{-\alpha\tau} \sum_{j=1}^r \frac{\partial^2 f_i}{\partial u_j \partial t}(0{,}0{,}1) d_j \Bigg) + \ldots + \\
&+ \alpha e^{-\alpha\tau} \sum_{\substack{|k|+|m|+l=n-1,\\ |k|+|m|\geqslant 1}} \frac{1}{k!m!l!} \frac{\partial f_i^{|k|+|m|+l}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial u_1^{m_1} \ldots \partial u_r^{m_r} \partial t^l}(0{,}0{,}1) \times \\
&\times c_1^{k_1} \ldots c_n^{k_n} d_1^{m_1} \ldots d_r^{m_r} (-1)^l e^{-l\alpha\tau} + \\
&+ \alpha e^{-\alpha\tau} \times \sum_{\substack{|k|+|m|+l=n,\\ |k|+|m|\geqslant 1}} \frac{1}{k!m!l!} \frac{\partial f_i^{|k|+|m|+l}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial u_1^{m_1} \ldots \partial u_r^{m_r} \partial t^l}(\tilde{c}, \tilde{d}, \tilde{t}(\tau)) \times \\
&\times c_1^{k_1} \ldots c_n^{k_n} d_1^{m_1} \ldots d_r^{m_r} (-1)^l e^{-l\alpha\tau},\ i = 1, \ldots, n.
\end{aligned} \tag{1.14}$$

All the arguments given below we will conduct with account restrictions on the function $c(\tau)$:

$$\|c(\tau)\| < C_1, C_1 > 0, \tau \in [0{,}\infty). \tag{1.15}$$

Combining in the right part of the system (1.14) all summands that are linear by vector components c and d with coefficients $e^{-i\alpha\tau}$, $i = 1, \dots n$ system (1.14) may be written in the following form:

$$\begin{gathered}\frac{dc}{d\tau} = P \cdot c + Q \cdot d + R_1(c, d,\tau) + R_2(c, d, \tau) + R_3(c,d, \tau),\\ R_1 = (R_1^1 \dots R_1^n)^T;\ R_2 = (R_2^1 \dots R_2^n)^T,\ R_3 = (R_3^1 \dots R_3^n)^T,\end{gathered} \tag{1.16}$$

where P and Q are defined by formulae (1.5).

R_i^1 are summands of the right part of the system (1.16) that are linearly dependent from components of the vector c with coefficients $e^{-n\alpha\tau}$, R_i^2 are summands that are linearly dependent from components of the vector d with coefficients $e^{-n\alpha\tau}$. In R_i^3 all summands that are nonlinear by components of the vectors c and d are included.

From constructing of the functions R_1, R_2, R_3 with account (1.2), (1.15), the next estimations are following:

$$\|R_1(c,d, \tau)\| \leqslant e^{-n\alpha\tau} L_1 \|c\|, L_1 > 0, \tag{1.17}$$

$$\|R_2(c,d, \tau)\| \leqslant e^{-n\alpha\tau} L_2 \|d\|, L_2 > 0, \tag{1.18}$$

$$\|R_3(c,d, \tau)\| \leqslant e^{-\alpha\tau} L_3 (\|c\|^2 + \|d\|^2), L_3 > 0. \tag{1.19}$$

Let us introduce an auxiliary control function $\upsilon(\tau)$ that is linked with $d(\tau)$ by the next differential equation:

$$\frac{dd(\tau)}{d\tau} = \upsilon(\tau),\ \upsilon = (\upsilon_1, \dots, \upsilon_r)^T. \tag{1.20}$$

Let

$$d(0) = 0. \tag{1.21}$$

Then system (1.16), (1.20), and initial conditions (1.13), (1.21) may be written as

$$\frac{d\bar{c}}{d\tau} = \bar{P} \cdot \bar{c} + \bar{Q} \cdot \upsilon + \bar{R}_1(c, d,\tau) + \bar{R}_2(c, d, \tau) + \bar{R}_3(c,d, \tau), \tag{1.22}$$

$$\bar{P} = \begin{pmatrix} P & Q \\ O_1 & O_2 \end{pmatrix}_{n+r \times n+r},\ \bar{Q} = \begin{pmatrix} O_3 \\ E \end{pmatrix}_{n+r \times r}, \tag{1.23}$$

where $\bar{c} = (c, d)^T_{n+r\times1}$, $\bar{R}_1 = (R_1^1, \dots, R_1^n, 0, \dots, 0)^T_{n+r\times1}$, $\bar{R}_2 = (R_2^1, \dots, R_2^n, 0, \dots, 0)^T_{n+r\times1}$, $\bar{R}_3 = (R_3^1, \dots, R_3^n, 0, \dots, 0)^T_{n+r\times1}$, O_1, O_2, O_3 – matrices with corresponding dimentions that consist of zero elements, E is the unity matrix,

$$\bar{c}(0) = \bar{c}_0,\ \bar{c}_0 = (c(0), 0, \dots, 0)^T_{n+r\times1}. \tag{1.24}$$

1.4 Solution of the stabilisation problem for the auxiliary system

Let us consider the linear part of the system (1.22):

$$\frac{d\bar{c}}{d\tau} = \bar{P} \cdot \bar{c} + \bar{Q} \cdot \upsilon. \tag{1.25}$$

Lemma 1.1 *Let conditions* (1.4), (1.6) *for system of differential equations* (1.1) *be satisfied, then there exists an auxiliary control function* $\upsilon(\tau)$

$$\upsilon(\tau) = M(\tau)\bar{c}, \tag{1.26}$$

$$\|M(\tau)\| = O(e^{n\alpha\tau}) \text{ as } \tau \to \infty, \tag{1.27}$$

that provides exponential decay of the fundamental matrix of system (1.25) *closed by function* (1.26).

Here is a brief description of the main steps of the problem solution.

After the solution of the stabilisation problem for system (1.25), we find the solution of the Cauchy ptoblem for system (1.22) closed by found auxiliary control (1.26) with initial conditions (1.24). As a result, we obtain functions $c(\tau)$, $d(\tau)$, $\upsilon(\tau)$ with $\tau \in [0, \infty)$. Then, we construct the function

$$\bar{d}(\tau) = d(kh), \tau \in [kh, (k+1)h), k = 0,1, \dots, \tag{1.28}$$

and solve the Cauchy problem for system (1.22) with initial conditions (1.24) after substituting of the function $\bar{d}(\tau)$ into the right-hand sides of it's first n equations. As a result, we obtain a function $\bar{c}(\tau)$. In addition, the first component of the $\bar{c}(\tau)$ gives functions $c(\tau)$. If to do transition to original independent variables by formulae (1.10), (1.12), then from the construction of the system (1.22) we will have functions

$x(t)$, $\bar{u}(t)$, that are solution to Problem 1.3. Passing to limit in functions $x(t)$, $\bar{u}(t)$ as $t \to 1$, we will obtain a solution for Problem 1.1. In this case, the time switching points t_k for discrete control are defined by a formula

$$t_k = 1 - e^{-\alpha kh}, k = 0,1,\dots . \tag{1.29}$$

If to choose values of t_m in the solution of Problem 1.1 so that $\| x(t_m) \| \leqslant \varepsilon_1$, $1 - t_m < \varepsilon_2$, then narrowing of functions $x(t)$, $\bar{u}(t)$ on the interval $[0,t_m]$ give solution of Problem 1.2.

P r o o f o f t h e l e m m a. Let us denote L_1^j, $j = 1,\dots,r$ as j-th column of the matrix $\bar{Q}$. Let us consider the matrix

$$\begin{gathered} S_1 = \{L_1^1, L_2^1, \dots, L_{k_1}^1, L_1^2, L_2^2, \dots, L_{k_2}^2, \dots, L_1^r, L_2^r, \dots, L_{k_r}^r\}, \\ L_i^j = \bar{P} L_i^{j-1} - \frac{dL_i^{j-1}}{d\tau},\ j = 1,\dots,r,\ i = 2,\dots,k_j, \end{gathered} \tag{1.30}$$

where $k_j, j = 1,\dots,r,$ – the maximum columns number of the matrix $L_1^j,\dots,L_{k_j}^j$, $j = 1,\dots,r$, such that vectors $L_1^1, L_2^1, \dots, L_{k_1}^1, L_1^2, L_2^2, \dots, L_{k_2}^2, \dots, L_1^r, L_2^r, \dots, L_{k_r}^r$ are linear independent.

Remark 1.1. *It is easy to see that matrix S_1 up to the exact column permutation has the next structure:*

$$S_1 = \begin{pmatrix} O_{n\times r} & L_1 & \dots & L_n \\ E_{r\times r} & O_{r\times r} & \dots & O_{r\times r} \end{pmatrix},$$

where $O_{r\times r}$ is null-matrix of the $r \times r$ size,

$$L_1 = Q,\ L_i = PL_i - \tfrac{dL_i}{d\tau},\ i = 2,\dots,n.$$

Let $\bar{L}_1^j$, $j = 1,\dots,r$ is j-th column of the matrix $\alpha e^{-\alpha\tau} B_0$. Consider the matrix

$$\begin{gathered} S_2 = \{\bar{L}_1^1, \bar{L}_2^1, \dots, \bar{L}_{k_1}^1, \bar{L}_1^2, \bar{L}_2^2, \dots, \bar{L}_{k_2}^2, \dots, \bar{L}_1^r, \bar{L}_2^r, \dots, \bar{L}_{k_r}^r\}, \\ \bar{L}_i^j = \alpha e^{-\alpha\tau} A_0 \bar{L}_i^{j-1} - \tfrac{d\bar{L}_i^{j-1}}{d\tau},\ j = 1,\dots,r,\ i = 2,\dots,k_j. \end{gathered}$$

From one point of view, with the help of contrudiction reasoning in account (1.4), we can verify the validity of the condition

$$\operatorname{rank} S_2 = n,\ \tau \in [0,\infty). \tag{1.31}$$

From another point of view

$$\begin{aligned}&A_0 + e^{-\alpha\tau}A_1 + \ldots + e^{-(n-2)\alpha\tau}A_{n-2} \to A_0, \text{при } \tau \to \infty,\\ &B_0 + e^{-\alpha\tau}B_1 + \ldots + e^{-(n-2)\alpha\tau}B_{n-2} \to B_0, \text{при } \tau \to \infty.\end{aligned} \tag{1.32}$$

From (1.31), (1.32) estimation is following

$$\|S_2^{-1}\| = O(e^{n\alpha\tau}), \tau \to \infty. \tag{1.33}$$

From condition (1.6) and the structure of the matrix S_1 (see Remark 1.1), a condition is the following:

$$\operatorname{rank} S_1 = n + r,\ \tau \in [0,\infty). \tag{1.34}$$

From the construction of the columns matrix S_2 it follows that its elements are decreasing not faster then $e^{-n\alpha\tau}$. From here and structure of matrices $\bar{P}$ и $\bar{Q}$ follows that elements of the matrix S_2^{-1} will be increasing not faster than $e^{n\alpha\tau}$ (see (1.33)). As a result, we have an estimation:

$$\|S_1^{-1}\| = O(e^{n\alpha\tau}), \tau \to \infty. \tag{1.35}$$

Using (1.34), we provide replacing of the variables

$$\bar{c} = S_1(\tau)y. \tag{1.36}$$

As a result, the system (1.25) takes a form

$$\frac{dy}{d\tau} = S_1^{-1}\left(\bar{P}S_1 - \frac{dS_1}{d\tau}\right)y + S_1^{-1}\bar{Q}\upsilon. \tag{1.37}$$

In accordance with the [57] matrix of the right-hand side of the system (1.37), we can write as follows:

$$S_1^{-1}\left(\bar{P}S_1 - \frac{dS_1}{d\tau}\right) = \{e_2, \ldots, e_{k_1}, \varphi_{k_1}(\tau), \ldots, e_{k_1+\ldots+k_{r-1}+2}, \ldots, e_{k_1+\ldots+k_r}, \varphi_{k_r}(\tau)\}. \tag{1.38}$$

In (1.38) $e_i = (0, \ldots, 1, \ldots, 0)^T_{n+r\times 1}$ is a mstrix column in which 1 is at the i-th place.

Components of the vector $\varphi_{k_j}(\tau)$ have a form:

$$\varphi_{k_j}(\tau) = (-\varphi^1_{k_1}(\tau), \ldots, -\ \varphi^{k_1}_{k_1}(\tau), \ldots, -\ \varphi^1_{k_j}(\tau), \ldots, -\ \varphi^{k_j}_{k_j}(\tau), 0, \ldots, 0)^T_{n+r\times 1},$$

where $-\varphi^i_{k_j}(\tau)$ are coefficients of the decomposition of the vector $L^j_{k_j+1}$ by vectors L^1_i,

$i = 1, \ldots, k_1;\ L^2_i,\ i = 1, \ldots, k_2;\ L^j_i,\ i = 1, \ldots, k_j,\ j = 1, \ldots, r,\ \sum_{j=1}^r k_j = n + r$.

$$L^j_{k_j+1} = -\sum_{i=1}^{k_1} \varphi^i_{k_1}(\tau) L^1_i - \ldots - \sum_{i=1}^{k_j} \varphi^i_{k_j}(\tau) L^j_i,$$
$$S_1^{-1} Q = \{e_1, \ldots, e_{k_j+1}, \ldots, e_{\gamma+1}\},\ \gamma = \sum_{i=1}^{r-1} k_i. \tag{1.39}$$

Consider the stabilisation problem of the system

$$\frac{dy_{k_j}}{d\tau} = \{e^{k_j}_2, \ldots, e^{k_j}_2, \bar{\varphi}_{k_j}\} y_{k_j} + e^{k_j}_1 d_j,\ j = 1, \ldots, r, \tag{1.40}$$

where $y_{k_j} = (y^1_{k_j}, \ldots, y^{k_j}_{k_j})^T_{k_j\times 1}$, $e^{k_j}_1 = (0, \ldots, 1, \ldots, 0)^T_{k_j\times 1}$, 1 stands in the i-th place, $\bar{\varphi}^i_{k_j} = (-\varphi^1_{k_j}, \ldots, -\ \varphi^{k_j}_{k_j})^T_{k_i\times 1}$.

Let $y^{k_j}_{k_j} = \psi$. Equalities

$$y^{k_j}_{k_j} = \psi,\ y^{k_j-1}_{k_j} = \psi^{(1)} + \varphi^{k_j}_{k_j}\psi,$$
$$y^{k_j-2}_{k_j} = \psi^{(2)} + \varphi^{k_j}_{k_j}\psi^{(1)} + \left(\frac{d\varphi^{k_j}_{k_j}}{d\tau} + \varphi^{k_j-1}_{k_j}\right)\psi, \tag{1.41}$$
$$y^1_{k_j} = \psi^{(k_j-1)} + r_{k_j-2}(\tau)\psi^{(k_j-2)} + \ldots + r_1(\tau)\psi^{(1)} + r_0(\tau)\psi.$$

follow from the matrix structure of the right-hand side of the system (1.40). After differentiating the last equality (1.41) and substituting the last expression in the first equation of the system (1.40), we will have following system

$$\psi^{(k_j)} + \varepsilon_{k_j-1}(\tau)\psi^{(k_j-1)} + \ldots + \varepsilon_0(\tau)\psi = \upsilon_j,\ j = 1, \ldots, r. \tag{1.42}$$

Remark 1.2. *Limitation of the functions* $\varphi^{k_j}_{k_j}, \ldots, \varphi^2_{k_j}, \varphi^1_{k_j}$*, their derivatives and functions* $r_{k_j-2}(\tau), \ldots, r_1(\tau), r_0(\tau)$ *arise from the construction of matrices* $\bar{P}$ *and* $\bar{Q}$*, and also from formulae* (1.39).

Let

$$\upsilon_j = \sum_{i=1}^{k_j} (\varepsilon_{k_j-i}(\tau) - \gamma_{k_j-i})\psi^{(k_j-i)},\ j = 1,\ldots,r. \tag{1.43}$$

Coefficients γ_{k_j-i} are chosen such that the roots of the characteristic equation

$$\lambda^{k_i} + \gamma_{k_i-1}\lambda^{k_i-1} + \ldots + \gamma_0 = 0,\ i = 1,\ldots,r$$

satisfy the conditions

$$\lambda^i_{k_i} \neq \lambda^j_{k_i},\ i \neq j;\ \lambda^j_{k_i} < -(2n+1)\alpha - 1, j = 1,\ldots,k_i,\ i = 1,\ldots,r. \tag{1.44}$$

Returning to the original variables, we obtain

$$\upsilon_j = \delta_{k_j} T^{-1}_{k_j} S^{-1}_{1k_j}\bar{c},\ \ j = 1,\ldots,r, \tag{1.45}$$

where $\delta_{k_j} = (\varepsilon_{k_j-1}(\tau) - \gamma_{k_j-1},\ldots,\varepsilon_0(\tau) - \gamma_0)$, T_{k_j} is a matrix from equalities (1.41) such that $y_{k_j} = T_{k_j}\bar{\psi}$, $\bar{\psi} = (\psi^{k_j-1},\ldots,\psi)^T$, $S^{-1}_{1k_j}$ is a matrix that consists of the corresponding k_j strings of S^{-1}_1.

The resulting auxiliary control function may be written in the form (1.26), where $M(\tau) = \delta_k T^{-1}_k S^{-1}_{1k} = (\delta_{k_1} T^{-1}_{k_1} S^{-1}_{1k_1},\ldots,\delta_{k_r} T^{-1}_{k_r} S^{-1}_{1k_r})^T$.

Let $\Psi(\tau)$ is a fundamental matrix of the system (1.42) closed by auxiliary control (1.43). From (1.44), it follows that $\Psi(\tau)$ is the fundamental matrix of the exponential stable linear system of the differential equations with constant coefficients. It follows that

$$\|\Psi(\tau)\Psi(t)^{-1}\| \leqslant \bar{M}e^{-\lambda(\tau-t)},\ \bar{M} > 0,\ \lambda > 0. \tag{1.46}$$

Consider the system (1.25) closed by the auxiliary control function (1.45):

$$\frac{d\bar{c}}{d\tau} = D(\tau)\bar{c},\ D(\tau) = \bar{P}(\tau) + \bar{Q}(\tau)M(\tau). \tag{1.47}$$

Let $\Phi(\tau)$ ($\Phi(0) = E$) be a fundamental matrix of the system (1.47). E is the identity matrix. Let us introduce a block diagonal matrix $T(\tau)$ with matrices T_{k_j}, $j = 1,\ldots,r$, on its diagonal. From formulae (1.36) and (1.41), we obtain equality

$$\Phi(\tau) = S_1(\tau)T(\tau)\Psi(\tau)\Psi^{-1}(0)T^{-1}(0)S_1^{-1}(0). \tag{1.48}$$

Further, estimations

$$\begin{gathered}\|\Phi(\tau)\| \leqslant \bar{K}e^{-\lambda\tau},\, \lambda > 0,\, \bar{K} > 0,\\ \|\Phi(\tau)\Phi^{-1}(t)\| \leqslant \bar{K}_1 e^{-\lambda(\tau-t)}e^{(n-1)\alpha t}, \tau \geqslant t,\, \bar{K}_1 > 0,\\ \|M(\tau)\| = O(e^{n\alpha\tau}),\, \tau \to \infty.\end{gathered} \tag{1.49}$$

follow from (1.35), (1.36), (1.41), (1.46), (1.48) and Remark 1.2.

The lemma is proved.

1.5 Continuation of the theorem proof

Consider the system (1.22) closed by found auxiliary control (1.26):

$$\frac{d\bar{c}}{d\tau} = D(\tau)\bar{c} + \bar{R}_1(c,d,\tau) + \bar{R}_2(c,d,\tau) + \bar{R}_3(c,d,\tau). \tag{1.50}$$

Let us to make the following replace of the variables:

$$\begin{gathered}\bar{c} = z(\tau)e^{-n\alpha\tau}, z = (z_1, z_2)^T, \bar{c}(0) = z(0);\\ c = z_1(\tau)e^{-n\alpha\tau}, d = z_2(\tau)e^{-n\alpha\tau}.\end{gathered} \tag{1.51}$$

As a result, the system will have a form

$$\begin{gathered}\frac{dz}{d\tau} = C(\tau)z + e^{n\alpha\tau}(\bar{R}_1(z_1e^{-n\alpha\tau}, z_2e^{-n\alpha\tau},\tau) +\\ \bar{R}_2(z_1e^{-n\alpha\tau}, z_2e^{-n\alpha\tau}, \tau)\ ++\ \bar{R}_3(z_1e^{-n\alpha\tau}, z_2e^{-n\alpha\tau}, \tau)),\\ C(\tau) = D(\tau) + n\alpha E.\end{gathered} \tag{1.52}$$

Let us show that all solutions of the system (1.52) with initial conditions (1.51), which are beginning in a sufficiently small neighbourhood of zero, are decreasing exponentially.

Let $\Phi_1(\tau)$, and $\Phi_1(0) = E$ is a fundamental matrix of the system $\frac{dz}{d\tau} = C(\tau)z$. Then, according to (1.49), (1.51):

$$\|\Phi_1(\tau)\| \leqslant \bar{K}e^{-\beta\tau},\ \|\Phi_1(\tau)\Phi_1^{-1}(t)\| \leqslant K_1 e^{-\beta(\tau-t)}e^{(n-1)\alpha t},$$
$$\beta = \lambda - n\alpha,\ \tau \geqslant t. \tag{1.53}$$

Let us choose value α such that the condition

$$\beta > 0. \tag{1.54}$$

is fulfilled.

Let us present solutions of the system (1.50) with initial conditions (1.24), (1.51) as follows:

$$z(\tau) = \Phi_1(\tau)\Phi_1^{-1}(\tau_1)z(\tau_1)+$$
$$\int\limits_{\tau_1}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{n\alpha t}(R_1(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t) + \tag{1.55}$$
$$R_2(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t) + R_3(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t))dt;\ \tau \in [\tau_1,\infty),$$

$$z(\tau) = \Phi_1(\tau)\bar{c}(0) + \int\limits_{0}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{n\alpha t}(R_1(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t) +$$
$$+ R_2(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t) + R_3(z_1e^{-n\alpha t}, z_2e^{-n\alpha t}, t))dt;\ \tau \in [0,\tau_1]. \tag{1.56}$$

Estimations

$$\| z(\tau) \| \leqslant \bar{K}e^{-\beta\tau} \| \Phi_1^{-1}(\tau_1)z(\tau_1) \| + \int\limits_{\tau_1}^{\tau} K_1 e^{-\beta(\tau-t)}Le^{-\alpha t}\|z\|dt,$$
$$\tau \in [\tau_1,\infty), \tag{1.57}$$

$$\| z(\tau) \| \leqslant \bar{K}e^{-\beta\tau} \| c(0) \| + \int\limits_{0}^{\tau} K_1 e^{-\beta(\tau-t)}Le^{-\alpha t}\|z\|dt,$$
$$\tau \in [0, \tau_1], \tag{1.58}$$

in the domain (1.2), (1.15) are following from (1.55), (1.56) taking into account (1.17) - (1.19), (1.53), and (1.54). Here, $L > 0$ is a constant that is dependent on domain (1.2), (1.15).

Using theorem 6.1 [58], we obtain inequalities:

$$\|z(\tau)\| \leqslant \bar{K}e^{-\gamma\tau}\|z(\tau_1)\|,\ \tau \in [\tau_1,\infty), \tag{1.59}$$

where $\gamma = \beta - K_1Le^{-\alpha\tau_1}$,

$$\|z(\tau)\| \leqslant \bar{K}e^{-\mu\tau}\|c(0)\|,\ \tau \in [0,\tau_1], \tag{1.60}$$

$\mu = \beta - K_1L$, from (1.57), (1.58).

Using condition (1.54), let us choose $\tau_1 > 0$ so that inequality $\gamma > 0$ has been satisfied. Estimations (1.59), (1.60) we can write as follows:

$$\|z(\tau)\| \leqslant K_1e^{-\gamma\tau}\|c(0)\|, \tau \in [0,\infty). \tag{1.61}$$

All solutions of the system (1.52) that are beginning in the domain

$$\|x_0\| \leqslant \frac{C_1}{\bar{K}_1} \tag{1.62}$$

are exponentially decreasing. This fact is arising from (1.61).

With the help of formulae (1.51), (1.26), we obtain the functions $\bar{c}(\tau) = (c(\tau), d(\tau))^T$, $\upsilon(\tau)$. The second component of the vector $\bar{c}(\tau)$ gives the function $d(\tau)$. Estimation

$$\| \bar{c}(\tau) \| \leqslant K_2e^{-(\gamma+n\alpha)\tau},\ \tau \in [\tau,\infty),\ K_2 > 0. \tag{1.63}$$

in the domain (1.2), (1.15) arises from (1.51), (1.59)

Constant K_2 depends from domain (1.2), (1.15).

Substitution of $d(\tau)$ in the (1.28) gives control $\bar{d}(\tau)$. Consider system (1.25) that is closed by auxiliary control system $\upsilon(\tau)$ after substitution into its right-hand side the first n equations of the function $\bar{d}(\tau)$, that was introduced in the statement of Problem 1.3. This system may be written in the form:

$$\begin{aligned}\frac{d\bar{c}}{d\tau} = D(\tau)\bar{c} + \bar{Q}(\bar{d} - d) + \bar{R}_1(c, d, \tau) + \bar{R}_2(c, d, \tau) + \bar{R}_3(c, d, \tau) +\\ (\bar{R}_2(c, \bar{d}, \tau) - \bar{R}_2(c, d, \tau)) + (\bar{R}_3(c, \bar{d}, \tau) - \bar{R}_3(c, d, \tau)),\\ \tau \in [kh, (k+1)h),\ k = 0,1,\ldots, \bar{Q} = (Q,O_4)^T_{n+r\times r},\end{aligned} \tag{1.64}$$

Here, O_4 is a null-matrix of the appropriate dimension.

From the middle theorem, we obtain the following equalities:

$$\begin{aligned}
R_2^i(c,\bar{d},\tau) - R_2^i(c,d,\tau) &= \left(\left(\frac{\partial R_2^i}{\partial d}(c,\tilde{d},\tau)\right)^T, (d-\bar{d})\right) = \\
&= \left(\left(\frac{\partial R_2^i}{\partial d}(c,\tilde{d},\tau)\right)^T, \frac{d}{d\tau}d(\bar{\tau})h\right), \\
R_3^i(c,\bar{d},\tau) - R_3^i(c,d,\tau) &= \left(\left(\frac{\partial R_3^i}{\partial d}(c,\tilde{d},\tau)\right)^T, (d-\bar{d})\right) = \\
&= \left(\left(\frac{\partial \bar{R}_3^i}{\partial d}(c,\tilde{d},\tau)\right)^T, \frac{d}{d\tau}d(\bar{\tau})h\right),
\end{aligned} \tag{1.65}$$

where $\tilde{d}$ is a middle point in the domain (1.2), $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_r)^T$.

Estimations

$$\begin{aligned}
\parallel \upsilon(\tilde{\tau}) \parallel = \parallel \frac{d}{d\tau}d(\tilde{\tau}) \parallel \leqslant \parallel M(\tilde{\tau}) \parallel \parallel \bar{c}(\tilde{\tau}) \parallel \leqslant K_4 e^{-\gamma\tilde{\tau}} = \\
= K_3 e^{-\gamma\tau} e^{\gamma(\tau-\tilde{\tau})} \leqslant K_3 e^{-\gamma\tau} e^{\gamma h} = K_4 e^{-\gamma\tau}, \\
K_4 > 0, K_4 = K_3 e^{\gamma h}, \ \tilde{\tau} \in [kh,(k+1)h), \tau \in [kh,(k+1)h), \\
\tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_r)^T; \upsilon(\tilde{\tau}) = (\upsilon_1(\tilde{\tau}_1), \ldots, \upsilon_r(\tilde{\tau}_r))^T.
\end{aligned} \tag{1.66}$$

in the domain (1.2), (1.15) are following from (1.20), (1.26), (1.27), (1.63), (1.65).

In (1.66), constant K_4 depends on the domain (1.2), (1.15), but it does not depend on the number k.

From (1.65), (1.66), we obtain enaqulities

$$\begin{aligned}
\parallel \bar{d} - d \parallel \leqslant K_5 e^{-\gamma\tau} h, K_5 > 0, \ \parallel R_2^i(c,\bar{d},\tau) - R_2^i(c,d,\tau) \parallel \leqslant K_6 e^{-\gamma\tau} h, \\
K_6 > 0, \quad \parallel R_3^i(c,\bar{d},\tau) - R_3^i(c,d,\tau) \parallel \leqslant K_7 e^{-\gamma\tau} h, K_7 > 0, \\
\tau \in [kh,(k+1)h), \ k = 0,1,\ldots.
\end{aligned} \tag{1.67}$$

In (1.67), constants K_5, K_6, and K_7 depend on domain (1.2), (1.15), but do not depend on value k. Let us show that all solutions of the system (1.64) that are beginning in the sufficiently small neghbourhood of zero are decreasing exponentially. Let us replace the variable $\bar{c}$ in the system (1.64) by the formula (1.51). As a result, we have enaqulities

$$\begin{aligned}\frac{dz}{d\tau} = C(\tau)z + e^{n\alpha\tau}(\bar{Q}(\bar{d}-d) + \bar{R}_1(z_1e^{-n\alpha\tau},\bar{d},\tau) +\\ \bar{R}_2(z_1e^{-n\alpha\tau},d,\tau) \ + \bar{R}_3(z_1e^{-n\alpha\tau},d,\tau) +\\ (\bar{R}_2(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_2(z_1e^{-n\alpha\tau},d,\tau)) +\\ + (\bar{R}_3(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_3(z_1e^{-n\alpha\tau},d,\tau))),\\ d(\tau) = z_2(\tau)e^{-n\alpha\tau},\ \bar{d}(kh) = z_2(kh)e^{-n\alpha kh},\\ \tau \in [kh,(k+1)h),\ k = 0,1,\ldots\ .\end{aligned} \tag{1.68}$$

The solution of the system (1.68) with the initial conditions (1.51), (1.24) has a form

$$\begin{aligned}z(\tau) = \Phi_1(\tau)\Phi_1^{-1}(kh)z(kh) + \int\limits_{kh}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{n\alpha t}(\bar{Q}(\bar{d}-d) +\\ +\bar{R}_1(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) + \bar{R}_2(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) +\\ R_3(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) + (\bar{R}_2(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_2(z_1e^{-n\alpha\tau},d,\tau)) +\\ + (\bar{R}_3(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_3(z_1e^{-n\alpha\tau},d,\tau)))dt, \tau \in [kh,(k+1)h),\end{aligned} \tag{1.69}$$

$$\begin{aligned}z(\tau) = \Phi_1(\tau)\bar{c}(0) + \int\limits_{0}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{n\alpha t}(\bar{Q}(\bar{d}-d) +\\ \bar{R}_1(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) + \bar{R}_2(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) +\\ R_3(z_1e^{-n\alpha t},z_2e^{-n\alpha t},t) + (\bar{R}_2(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_2(z_1e^{-n\alpha\tau},d,\tau)) +\\ +(\bar{R}_3(z_1e^{-n\alpha\tau},\bar{d},\tau) - \bar{R}_3(z_1e^{-n\alpha\tau},d,\tau)))dt,\ \tau \in [0,kh).\end{aligned} \tag{1.70}$$

Estimations

$$\begin{aligned}\| z(\tau) \| \leqslant \bar{K}e^{-\beta(\tau-kh)} \| z(kh) \| + \int\limits_{kh}^{\tau} Ke^{-\beta(\tau-t)}(L\|z\| + K_8h)e^{-\alpha t}dt,\\ \tau \in [kh,(k+1)h),\end{aligned} \tag{1.71}$$

$$\| z(\tau) \| \leqslant Ke^{-\beta\tau} \| c(0) \| + \int\limits_{0}^{\tau} Ke^{-\beta\tau}(L\|z\| + K_8h)e^{-\alpha t}dt, \tau \in [0,kh], \tag{1.72}$$

are following from (1.69), (1.70) with account (1.17), (1.18), (1.19), (1.51), (1.53), (1.63), (1.66), (1.67). In (1.71), (1.72) $\bar{K} = Ke^{(n-1)\alpha kh}$.

Applying the known result from [58] to formulae (1.71), (1.72), we obtain estimations

$$\|z(\tau)\| \leqslant \bar{K}e^{-\gamma_1(\tau-k_1h)}\|z(k_1h)\| + K_9he^{-\alpha\tau},\ \tau \in [k_1h, (k_1+1)h), \tag{1.73}$$

where $\gamma_1 = \beta - KLe^{-\alpha k_1 h}$,

$$\|z(\tau)\| \leqslant \bar{K}e^{-\mu_1\tau}\|c(0)\| + K_{10}h,\ \tau \in [0,k_1h], \tag{1.74}$$

where $\mu_1 = \beta - KL$.

We choose a value $k = k_1$ such that the condition $\gamma_1 > 0$ is satisfied.

In (1.73), (1.74), constants $K_9, K_{10} > 0$ depend on domain (1.2), (1.15) and do not depend on value k. From one point of view, the fact that all solutions of the system (1.52) that belong to domain (1.2), (1.15) are decreasing exponentially follows from (1.73), (1.74). From another point of view, we can choose so $\varepsilon > 0$, $h_0 > 0$ that for all $x_0, h : ||x_0|| < \varepsilon$, $0 < h < h_0$ solution of the system will be belong to the domain (1.2), (1.15).

Then substitution (1.69), (1.70) in formulae (1.26), (1.51) will give known functions $\bar{c}(\tau)$, $\upsilon(\tau)$. By virtue of the construction of systems (1.16), (1.22), and (1.68), the function $c(\tau)$ that corresponds to the first component of the function $\bar{c}(\tau)$ satisfies the system (1.11) as a substitution in its right-hand side $\bar{d}(\tau)$. From (1.51), (1.73). it follows that function $c(\tau)$ satisfies the boundary conditions (1.13). Therefore, the pair of functions $c(\tau)$, $\bar{d}(\tau)$ is the solution to Problem 1.3.

Returning to the original independent variable t by formulae (1.10), (1.11), and (1.29), we obtain functions $x(t) = c(\tau(t))$, $\bar{u}(t) = \bar{d}(\tau(t))$ and $\bar{\upsilon}(t) = \upsilon(\tau(t))$ that satisfy the system

$$\dot{x} = f(x, \bar{u}(t), t), \dot{u} = \alpha^{-1}(1-t)^{-1}\bar{\upsilon}(t) \tag{1.75}$$

and the initial conditions

$$x(0) = 0, u(0) = 0, \bar{u}(t) = u(t_k), t \in [t_k, t_{k+1}), k = 0,1,\dots\ . \tag{1.76}$$

From the construction of the system (1.75) it follows that the pair functions $x(t)$, $\bar{u}(t)$ is solution to the problem (1.1), (1.9). Further, by passing to the limit as $t \to 1$ in functions $x(t)$, $\bar{u}(t)$, we obtain a solution to Problem 1.1.

Narrowings of functions $x(t)$, $\bar{u}(t)$ in the interval $[0, t_m]$ give the solution to Problem 1.2.

The theorem is proved.

1.6 Estimation of the reachability domain

Definition 1.2. *Reachibility domain is a set of all points* $M = \{x_0\} \in R^n$ *such that, for all* $x_0 \in M$ *there exists pair of functions* $x(t), u(t)$ *that is sutisfying the ODE system* (1.1) *and conditions* (1.2), (1.3), (1.7), (1.8), (1.9).

Estimation of the domain $\|x_0\| < \varepsilon$ in which solutions of the system (1.1) that are transferring state vector to the coordinate system origin will give an estimation of the reachability domain.

The inequality (1.62) describes the reachability domain for a continuous case.

Let us provide estimation of the reachability domain taking into account a discrete control function that acts on the ODE system.

The fairness of the estimation

$$\|x_0\| \leqslant \frac{C_1 - K_{10} \cdot h}{\bar{K}}, \tag{1.77}$$

in the domain (1.2), (1.15) follows from formula (1.74).

In (1.77), $0 < h < h_0$ is the discretization step, $\bar{K}$, K_{10} are constants that are described above.

We obtain

$$K_{10} \leqslant \frac{K_1 \cdot (K_5 + K_6 + K_7)}{(n-1) \cdot \alpha}, \tag{1.78}$$

with account estimations (1.67).

The inequality

$$\|x_0\| \leqslant \frac{C_1 - \frac{K_1 \cdot (K_5 + K_6 + K_7)}{(n-1) \cdot \alpha} \cdot h}{\bar{K}}, \tag{1.79}$$

follows from (1.77), (1.78).

Formula (1.79) gives the estimation of the reachability domain.

Let us provide an estimation of the discretization step from which the reachability domain (1.77) depends. A minimum number of points for the constructed method, in which it is necessary to integrate the function, equals the number of switch points that are defined by formula (1.29).

Further considerations will be conducted, taking into account the application of the explicit numerical integration methods for systems of differential equations.

It follows from (1.44) that all real parts of the eigenvalues of the matrix of the right-hand side of the auxiliary equations system with account control are strictly negative. The value of the integration step for the system of the ordinary differential equations for explicit numerical methods is [59]:

$$\Delta t \leqslant \frac{a}{|\lambda_{max}|}. \tag{1.80}$$

Here, a is a constant that depends on applying the numerical method; λ_{max} is a maximum modulo eigenvalue of the auxiliary system matrix.

It follows from (1.29) and (1.80):

$$1 - e^{-\alpha h_0} \leqslant \tfrac{a}{|\lambda_{max}|}.$$

The estimation of the maximum discretization step, taking into account the conditions (1.44), has the form:

$$h_0 \leqslant \left| \frac{\ln(1 - \frac{a}{(2n+1)\alpha+1})}{\alpha} \right|. \tag{1.81}$$

Let us perform a substitution of the value (1.81) into (1.77). And we will obtain a final estimation of the reachability domain wich takes into account a discreteness of the control:

$$\|x_0\| \leqslant \frac{C_1 - K_{10} \cdot \left| \frac{\ln(1 - \frac{a}{(2n+1)\alpha+1})}{\alpha} \right|}{\bar{K}}. \tag{1.82}$$

It follows from formulae (1.77) and (1.82) that when the discretization step is increasing the reachability domain is decreasing.

1.7 Conclusions for the first chapter

In the present chapter, the piecewise constant control problem for a wide class of non-linear non-stationary systems of ordinary differential equations has been solved. Also, constructive sufficient conditions that guarantee existence of the control function that provides a transition from the initial state to the final state for the specified class of systems for a finite time interval have been obtained.

Based on the theorem proof, algorithm construction is possible, which may be applied to solve problems of control for different technical and physical systems described by ODE systems, the right-hand sides of which satisfy the conditions (1.3), (1.4), and (1.6).

The next chapter is devoted to the construction and detailed study of this algorithm.

Chapter 2. Construction, analysis and application of the discrete control algorithm

The contents of this chapter are published in articles [22],[23] and included in the thesis of the postgraduate programme [56].

2.1 Description of the discrete control algorithm

The solution method of the boundary value problem in the class of discrete controls that is in the First chapter gives an algorithm for constructing the desired control, which consists of the next steps [22; 23]:

Auxiliary system construction.

1. Decomposition of the system (1.1) according to the Taylor formula.
2. Replacing the origin independent variable t by an auxiliary independent variable τ by a fomula

$$t(\tau) = 1 - e^{-\alpha\tau},\ \tau \in [0,\infty),\ \alpha > 0.$$

The result of the calculations of clauses 1 and 2 is a system:

$$\frac{dc}{d\tau} = P \cdot c + Q \cdot d + \sum_{i=1}^{3} R_i(c, d,\tau), R_i = (R_i^1 \dots R_i^n)^T.$$

3. Introduction of the auxiliary control that satisfies the system

$$\tfrac{d}{d\tau} d(\tau) = \upsilon(\tau),\ \upsilon = (\upsilon_1, \dots, \upsilon_r)^T, d(0) = 0,$$

and constructing of the auxiliary system

$$\frac{d\bar{c}}{d\tau} = \bar{P} \cdot \bar{c} + \bar{Q} \cdot \upsilon, \tag{2.1}$$

where $\bar{P} = \begin{pmatrix} P & Q \\ O_1 & O_2 \end{pmatrix}_{(n+r)\times(n+r)}$, $\bar{Q} = \begin{pmatrix} O_3 \\ E \end{pmatrix}_{(n+r)\times r}$. O_1, O_2, O_3 are null-matrices of the corresponding dimensions, E is the identity matrix.

Further, the linear part of the auxiliary system (2.1) is considered.

Solution to the stabilisation problem for the auxiliary system.

4. Construction of the matrix

$$S = \begin{pmatrix} O_{n\times r} & L_1 & \dots & L_n \\ E_{r\times r} & O_{r\times r} & \dots & O_{r\times r} \end{pmatrix}, \tag{2.2}$$

   where $O_{r\times r}$ is null-matrix with dimension $r \times r$, $L_1 = Q$, $L_i = PL_i - \frac{dL_i}{d\tau}$, $i = 2, \dots, n$.
5. Computation of the matrix $\bar{S} = S^{-1}\left(\bar{P}S - \frac{dS}{d\tau}\right)$.
6. Determining the polynomial coefficients, real parts of the roots of which belong to the left half-plane and satisfy the conditions (see formulae (40) – (42) in [23])
   $\lambda^i_{k_i} \neq \lambda^j_{k_i}$, $i \neq j$; $\lambda^j_{k_i} < -(2n+1)\alpha - 1, j = 1, \dots, k_i, i = 1, \dots ,r.$
7. Computation of the auxiliary control function
   $\upsilon(\tau) = M(\tau)\bar{c}$, $M(\tau) = \delta T^{-1}S^{-1}$, $\bar{c} = (c,d)$, where T is the upper triangle matrix that is constructed based on elements of the matrix $\bar{S}$; δ is a vector-string obtained from $\bar{S}$ and coefficients or the polinom from clause 6.
8. Solution of the Cauchy problem for an auxiliary system with initial conditions $c(0) = x_0$ that is closed by the control $\upsilon(\tau)$. The results of the solution are functions $\bar{c}(\tau) = (c(\tau),d(\tau))$, $\upsilon(\tau)$.
   Solution to the control problem for the original system.
9. Finfing of the switch points by the formula

$$t_k = 1 - e^{-\alpha kh}, k = 0,1, \dots\ , \tag{2.3}$$

   where h is the step of discreteness of the control.
10. Transition to the original independent variable t in the function $\upsilon(\tau)$ according to the formula (1.10) that is giving $\bar{\upsilon}(t) = \upsilon(\tau(t))$.
11. Solution of the Cauchy problem for the system

$$\begin{gathered} \dot{x} = f(x, \bar{u}(t), t),\ \dot{u} = \alpha^{-1}(1-t)^{-1}\bar{\upsilon}(t), \\ x(0) = 0, u(0) = 0,\ \bar{u}(t) = u(t_k), t \in [t_k, t_{k+1}), k = 0,1, \dots\ . \end{gathered} \tag{2.4}$$

In the eighth and eleventh clauses, the Cauchy problem is solving with the help of one of the numerical methods of solution for ODEs [60; 61]. The other clauses may be realised with the help symbolic computation software, e.g. [62].

2.2 Computational cost analysis of the algorithm

In the present section, estimation and analysis of the computational cost of the above discrete control algorithm are considered.

This material was presented at the LIV International conference for postgraduate and undergraduate students on Control Processes and Stability (CPS'23), which was held from April 4-7, 2023, and published in the article [22].

2.2.1 Brief Literature Review

When developing software, it is useful to estimate the computational cost of implemented algorithms.

The computational complexity cost is conducted by a time often. Also, sometimes it may be useful estimate the cost of memory when performing calculations.

The computational costs of some control algorithms are provided in works [44; 54; 55], etc.

Except for the above, we should to notice the publication [63], which is devoted to the analysis of the computational cost for optimal control algorithm synthesis that is obtained from theoretical-game models of the functioning of active systems.

Based on the results of the estimations of the computational cost for algorithms that are provided in [44; 54; 55; 63], etc., it may be followed that a significant part of them have not more than polynomial complexity.

2.2.2 Theoretical computational cost of the algorithm

Lemma 2.1. *The computational complexity of the discrete control algorithm that is constructed in [23] is the following:*

$$O(n^4) + O\big(K \cdot C(n)\big), \tag{2.5}$$

where $O(n^4)$ is the computational complexity of the symbolic computations; $O\big(K \cdot C(n)\big)$ is the computational complexity of the solution of the Cauchy problem with the help of the Runge-Kutta method; $K = \left[\frac{\ln((1-t_m)^{-1})}{\alpha \cdot h}\right] + 1$ is the number of switch-points (1.29)*; $C(n)$ is a function that characterises the number of operations for integrated function components computing.*

Proof of the Lemma 2.1. . The maximum computational complexity are operations of the matrix (2.2) computing and solution of the Cauchy problem (2.4).

For computing the (2.2) $n-1$ multiplications of two matrices with dimensions $(n+r)\times(n+r)$ and $(n+r)\times r$ accordingly are required. The upper bound for the complexity of this operation is $O((n+r)^2 \cdot r)$. Since $r \leqslant n$ from the conditions of the problem, taking into account the necessary multiplications number for defining (2.2), the upper bound for this operaton is $O(n^4)$.

Integrating the Cauchy problem is held on the time-interval $t \in [0,1]$. Swith-points are defined by (2.3) and make up a countable (Problem 1.1, see Chapter 1) or the finite (Problem 1.2, see Chapter 1) set. For estimation of the operation numbers in the worst case scenario, it is necessary to define the largest number of switch-points. For this purpose, it is possible to accept the accuracy of the approximation to 1 of t_m value at the right side of the time-interval [0,1], Then from formula (2.3), we will have the following number of switch-points:

$$K = m + 1 = \left[\frac{\ln((1-t_m)^{-1})}{\alpha \cdot h}\right] + 1,$$

where $|1-t_m| < \varepsilon_2$ is the accuracy of the approximation t_m at the right side of the time-interval [0,1] in accordance with Problem 1.2 from Chapter 1.

2.2.3 Computational cost analyses

Based on the obtained formula (2.5), graphs are plotted that show the nature of the dependency for growth functions of computational complexity and its components from the dimension of the problem and the approximation accuracy at the right side of the time-interval [0,1]. The following values have been accepted: $\varepsilon_{2f} = 10^{-6}$ (float accuracy), $\varepsilon_{2d} = 10^{-16}$ (double accuracy), $\alpha = 0{,}25$, $h = 0{,}01$,

$C(n) = 120 \cdot n$. Values ε_2 have been chosen from standard values for data types float and double, which are applied in different programming languages.

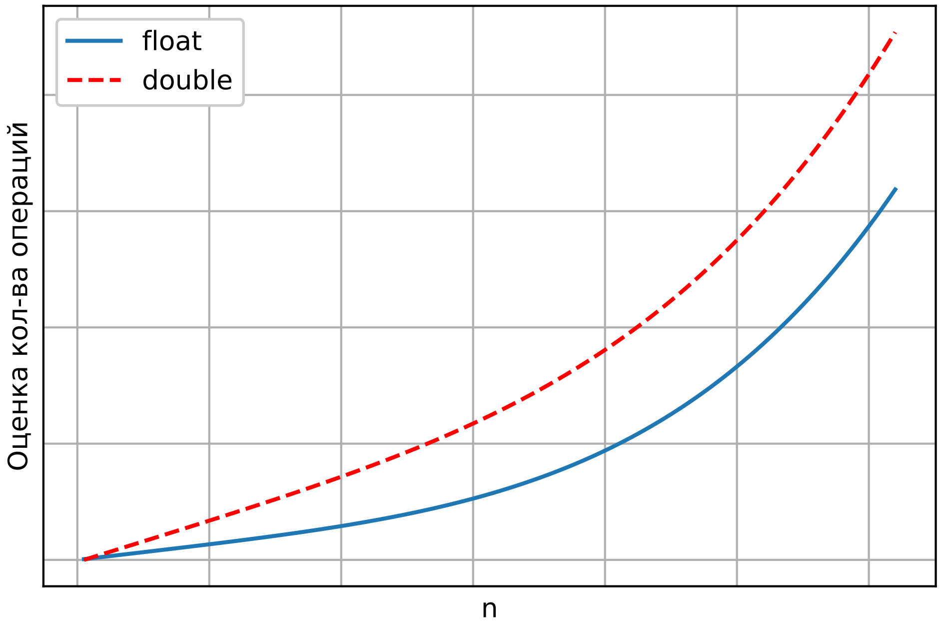

Figure 2.1 — Nature of the dependency for growth functions of computational complexity at different values of the approximation accuracy on the right side of the time-interval [0,1].

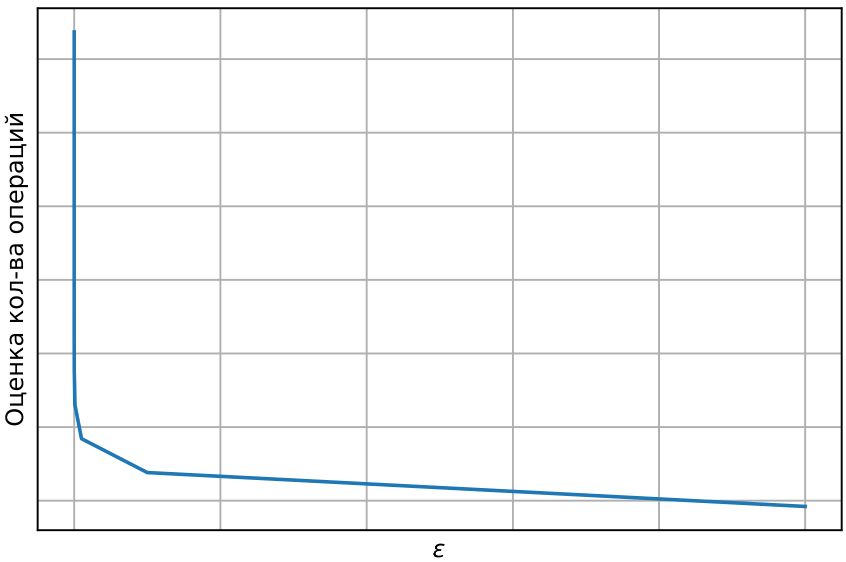

Figure 2.2 — The nature of the dependency on growth functions of computational complexity depends on the accuracy of the problem solution.

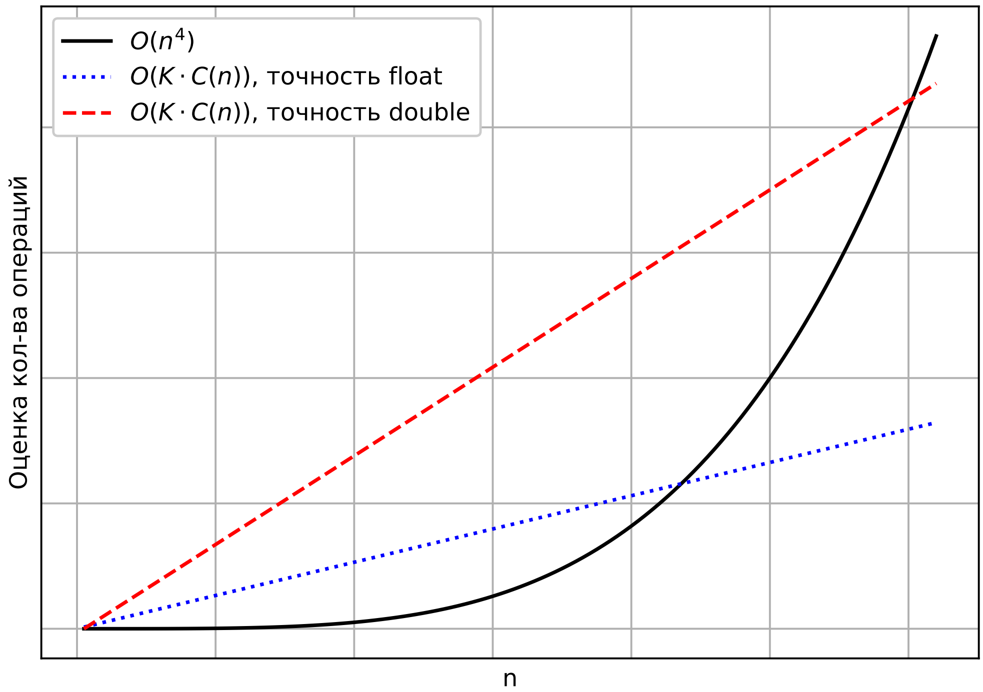

Figure 2.3 — The nature of the dependency on growth functions of computational complexity for symbolic and numerical parts of the algorithm depends on the dimension of the problem.

Analysis of graphs (Figs. 2.1 – 2.3) and obtained formulae allows to the following conclusions:

1. The higher the approximation accuracy at the right side of the time-interval [0,1] then higher computational complexity of the Cauchy problem solution.
2. For small dimensions computational cost of the Cauchy problem solution is higher then complexity of the symbolic calculations.
3. Since any value $N = \left[\sqrt[3]{C \cdot \frac{\ln(\varepsilon_2^{-1})}{\alpha \cdot h}}\right]$, computational cost of the symbolic calculations is higher then computational cost of the Cauchy problem solution.
4. With a sufficiently large dimension of the problem, the computational cost of the solution for the ODE system is possible to not take into account; however, most of the control problems have a small dimension, usually.
5. The computational cost of the studied algorithm depends on the values of α and h.

2.3 Discrete control of the single-link robot-manipulator

Consider the control problem for the robot-manipulator, which moves cargo of variable weight to the given point. The solution to this problem is described in the article [23].

According to [64; 65], let us write the equation system as follows:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a_2(t)\sin x_1 - a_1(t)x_2 + u.$$

Here, x_1 is a deviation angle of manipulator from the vertical axis, x_2 is angular velocity of the manipulator, $a_1(t) = \bar{\alpha}L^{-2}m_1(t)^{-1}$, $m_1(t) = m(t) + M/3$, $a_2(t) = gL^{-1}\big(m(t) + M/2\big)m_1(t)^{-1}$, M is mass of the manipulator, L is length of the manipulator, g is gravitational acceleration, $\bar{\alpha}$ is friction coefficient, $m(t) = m_0 - qt$ is mass of the cargo, $q > 0$, m_0 is initial mass of the cargo, $x = (x_1{,}x_2)^T$ is a state vector, u is a scalar control.

Consider the boundary conditions $x(0) = \bar{x}$, $x(1) = 0$. Analogues of the equation system (1.22) and conditions (1.24) have a form

$$\begin{aligned}
&\frac{dc_1}{d\tau} = \alpha e^{-\alpha\tau}c_2,\\
&\frac{dc_2}{d\tau} = -\alpha e^{-\alpha\tau}a_2(1-e^{-\alpha\tau})\sin c_1 - \alpha e^{-\alpha\tau}a_1(1-e^{-\alpha\tau})c_2 + \alpha e^{-\alpha\tau}d,\\
&\frac{d}{d\tau}d(\tau) = \upsilon,
\end{aligned} \tag{2.6}$$

in which $c_1(\tau) = x_1(t(\tau))$, $c_2(\tau) = x_2(t(\tau))$,

$$c_1(0) = \bar{x}_1,\ c_2(0) = \bar{x}_2,\ d(0) = 0,\ c_i(\tau) \to 0. \tag{2.7}$$

Linear part of the system (2.6):

$$\frac{d\bar{c}}{d\tau} = \bar{P}\bar{c} + \bar{Q}\upsilon,\ \bar{c} = (c_1, c_2, d)^T, \tag{2.8}$$

where $\bar{P}$ and $\bar{Q}$ are constructed by formulae (1.23):

$$\bar{P} = \begin{bmatrix} 0 & \alpha e^{-\alpha\tau} & 0 \\ -a_2(1)\alpha e^{-\alpha\tau} & -a_1(1)\alpha e^{-\alpha\tau} & \alpha e^{-\alpha\tau} \\ 0 & 0 & 0 \end{bmatrix},\ \bar{Q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Consider the matrix (1.30):

$$S = \begin{bmatrix} 0 & 0 & \alpha^2 e^{-2\alpha\tau} \\ 0 & \alpha e^{-\alpha\tau} & \alpha^2 e^{-\alpha\tau} - a_1(1)\alpha^2 e^{-2\alpha\tau} \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.9}$$

It follows from (2.9) that conditions (1.4) and (1.6) have been satisfied.

Matrix $\bar{S}$ has the form

$$\bar{S} = S^{-1}\left(\bar{P}S - \frac{dS}{d\tau}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & \alpha^2\left(-2e^{2\alpha\tau} + a_1(1)e^{\alpha\tau} - a_2(1)\right)e^{-2\alpha\tau} \\ 0 & 1 & 3\alpha - a_1(1)\alpha e^{-\alpha\tau} \end{bmatrix}. \tag{2.10}$$

Let us introduce the notations of elements for the third column of the matrix (2.10):

$$\begin{bmatrix} \varphi_1(\tau) \\ \varphi_2(\tau) \\ \varphi_3(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha^2\left(-2e^{2\alpha\tau} + a_1(1)e^{\alpha\tau} - a_2(1)\right)e^{-2\alpha\tau} \\ 3\alpha - a_1(1)\alpha e^{-\alpha\tau} \end{bmatrix}.$$

Matrix T has the form

$$T = \begin{bmatrix} 1 & -\varphi_3(\tau) & -(\frac{d\varphi_3(\tau)}{d\tau} + \varphi_2(\tau)) \\ 0 & 1 & -\varphi_3(\tau) \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.11}$$

We define the string $\delta = (\delta_1,\delta_2,\delta_3)$ in the form

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}^T = \begin{bmatrix} -\gamma_2 - \varphi_3 \\ -\gamma_1 - 2\frac{d\varphi_3(\tau)}{d\tau} - \varphi_2(\tau) \\ -\gamma_0 - \frac{d^2\varphi_3(\tau)}{d\tau^2} - \frac{d\varphi_2(\tau)}{d\tau} \end{bmatrix}^T, \tag{2.12}$$

where $\gamma_2 = 12$, $\gamma_1 = 47$, $\gamma_0 = 60$.

Substituting (2.12), (2.11), and (2.9) in (1.45), we obtain:

$$\upsilon(\tau) = M(\tau)\bar{c},\ M(\tau) = \delta T^{-1}S^{-1},\ \bar{c} = (c_1,c_2,d), \tag{2.13}$$

$$M(\tau) = (m_1(\tau),m_2(\tau),m_3(\tau)),$$

$$m_1(\tau) = -a_1(1)a_2(1)\alpha e^{-\alpha\tau} + 3a_2(1)\alpha + a_2(1)\gamma_2 - 8\alpha e^{2\alpha\tau} - 4\gamma_2 e^{2\alpha\tau} - \frac{2\gamma_1 e^{2\alpha\tau}}{\alpha} - \frac{\gamma_0 e^{2\alpha\tau}}{\alpha^2},$$
$$m_2(\tau) = -a_1(1)^2\alpha e^{-\alpha\tau} + 3a_1(1)\alpha + a_1(1)\gamma_2 + a_2(1)\alpha e^{-\alpha\tau} - 7\alpha e^{\alpha\tau} - 3\gamma_2 e^{\alpha\tau} - \frac{\gamma_1 e^{\alpha\tau}}{\alpha},$$
$$m_3(\tau) = -3\alpha + a_1(1)\alpha e^{-\alpha\tau} - \gamma_2.$$

Further, we solve the Cauchy problem for system (2.6), with initial conditions (2.7), which is closed by control (2.13). Finally, we have functions $c_1(\tau)$, $c_2(\tau)$, $d(\tau)$, $\upsilon(\tau)$.

Using formulae (1.10) and (1.29), we obtain the function $\bar{\upsilon}(t) = \bar{\upsilon}(\tau(t))$ and switch-points t_k.

Let us solve the Cauchy problem for the system of equations

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a_2(t)\sin x_1 - a_1(t)x_2 + \bar{u},\\
\dot{u} &= \alpha^{-1}(1-t)^{-1}\upsilon(t)
\end{aligned}$$

with initial conditions $x_1(0) = \bar{x}_1$, $x_2(0) = \bar{x}_2$, $u(0) = 0$.

The following values of parameters were used for numerical simulation: $\bar{x}_1 = -0.5$ radian, $\bar{x}_2 = -0.8$ radian/s, $\bar{\alpha} = 0.1$, $\alpha = 0.25$, $L = 10$ m, $M = 20$ kg, $m_0 = 1$ kg, $h = 0.01$, $q = 0.01$, $t \in [0{,}0.99]$.

The Runge-Kutta method has been applied to solve the Cauchy problem. In the Fig. 2.4 graphs of the phase coordinates $x_1(t)$, $x_2(t)$ and the control $u(t)$ are presented.

The study of the numerical modeling results presented in Fig. 2.4 allows for the following conclusions:

1. graphs of the phase coordinate functions and control illustrate the transition process;
2. the maximum cost of the control resource is in the initial section of the trajectory and belongs to $t \in [0{,}0.2]$; the maximum value of the control action is about 50 radian/s^2;
3. the suggested algorithm may be applied for solving the stabilisation problem at the finite time-interval for a wide class of non-linear non-stationary systems.

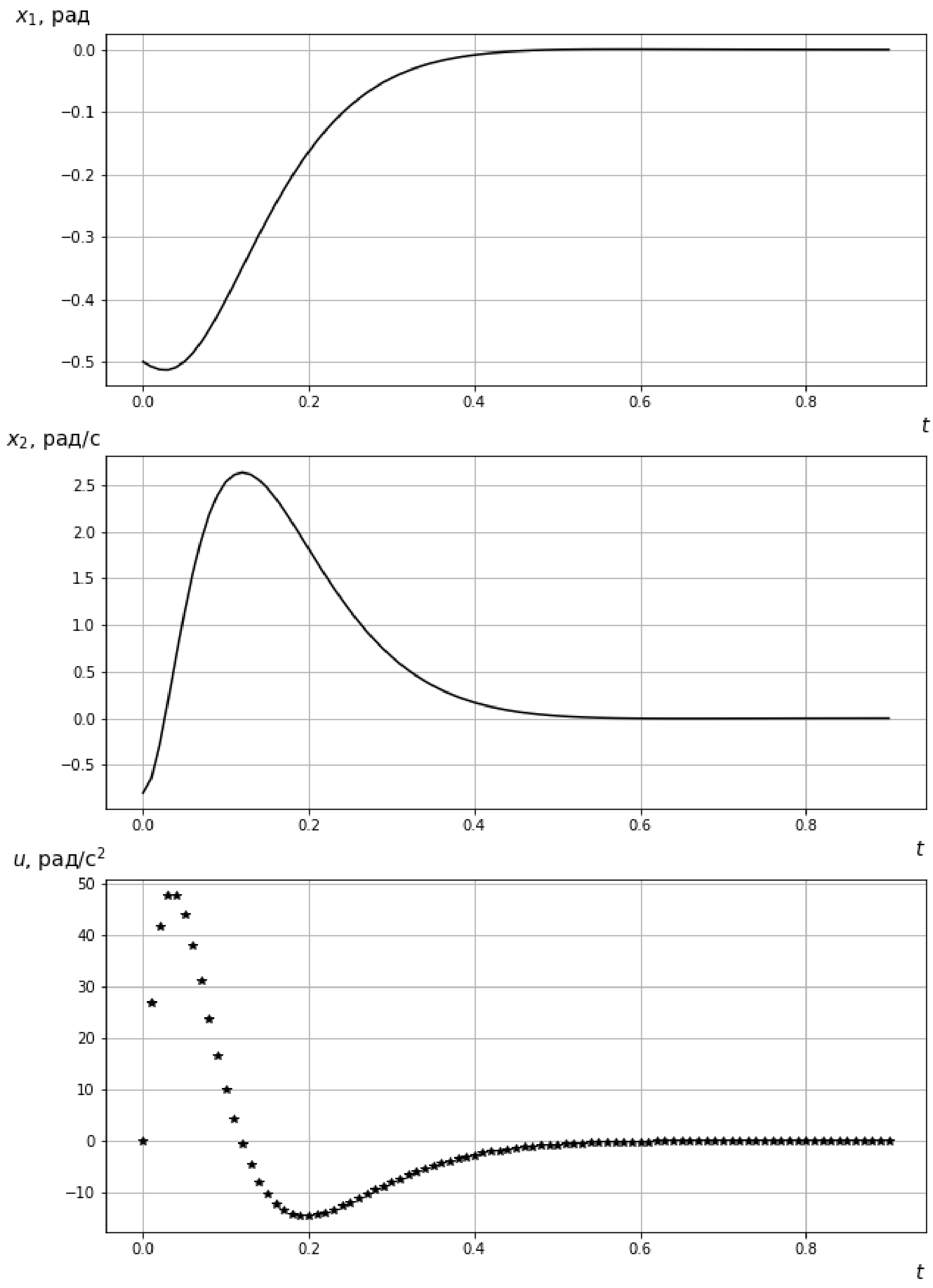

Figure 2.4 — Graphs of the functions $x_1(t)$, $x_2(t)$

2.4 Discrete control of the single-link robot-manipulator with account of pertrubations

Consider the equation system of the control model for the single-link robot-manipulator that is moving cargo with variable weight to the given point, taking into account perturbation actions:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a_2(t)\sin x_1 - a_1(t)x_2 + f(1-t)^2 + u, \end{aligned} \tag{2.14}$$

where $f(1-t)^2$ is the perturbation action, and other parameters are provided in Section 2.3.

The boundary conditions have a form:

$$x(0) = \bar{x}, x(1) = 0. \tag{2.15}$$

Discrete control problem. Find discrete control $u(t_k)$ which is solving the stabilisation problem of the system (2.14) in case of the boundary conditions (2.15), where t_k are switch-points that are defined by the formula:

$$t_k = 1 - e^{-\alpha kh}, k = 0,1, \ldots \tag{2.16}$$

h is the discreteness step of the control, $\alpha > 0$ is an arbitrary constant to be determined, and $t \in [0;1]$.

For solving the problem, we obtain the control function for the system (2.3) with the help of the discrete control algorithm, presented in the first section of this chapter. Let us substitute the obtained control function into the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a_2(t)\sin x_1 - a_1(t)x_2 + f(1-t)^2 + \bar{u}, \\ \dot{u} &= \alpha^{-1}(1-t)^{-1}\upsilon(t). \end{aligned} \tag{2.17}$$

Further, we will solve the Cauchy problem with the initial conditions $x_1(0) = \bar{x}_1$, $x_2(0) = \bar{x}_2$, $u(0) = 0$ and conduct numerical simulations.

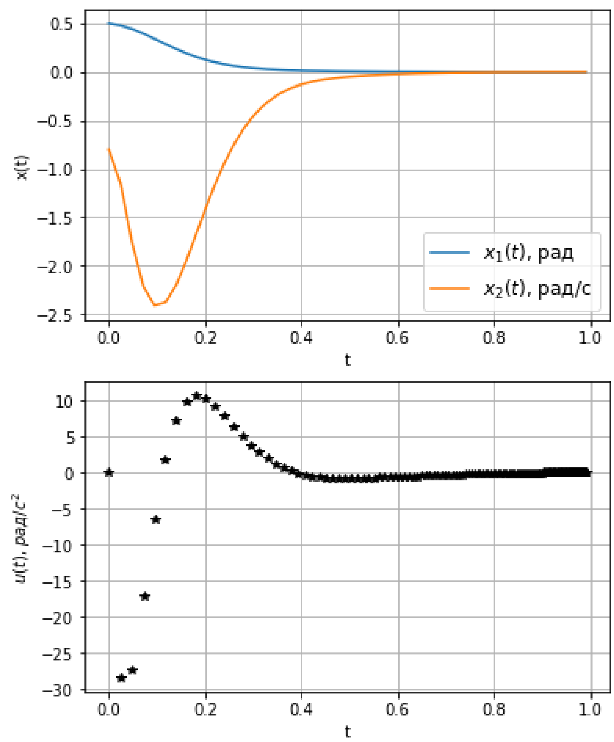

Figure 2.5 — Solution of the Cauchy problem for system (2.17). Values of the parameters: $f = 5$, $\alpha = 0.25$, $h = 0.1$.

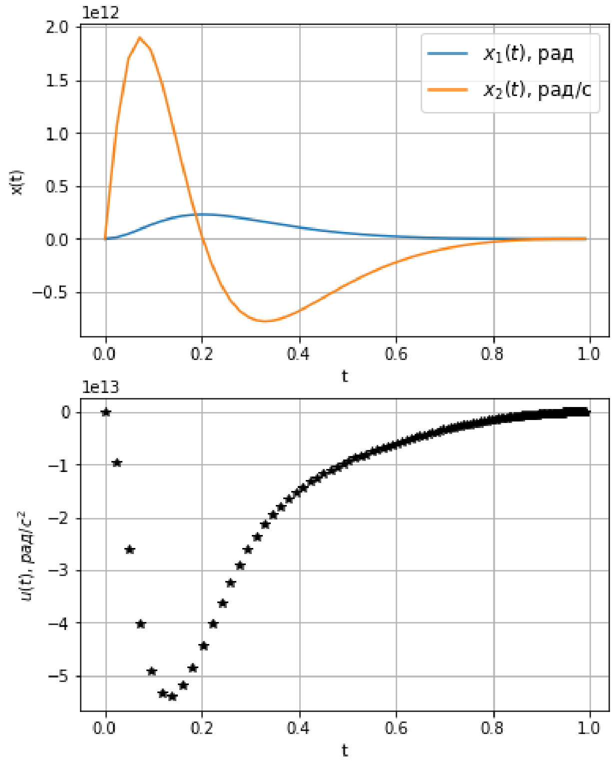

Figure 2.6 — Solution of the Cauchy problem for system (2.17). Values of the parameters: $f = 5 \cdot 10^{13}$, $\alpha = 0.25$, $h = 0.1$.

The following values of the parameters are used for calculations: $\bar{\alpha} = 0.1$, $L = 10$ m, $M = 20$ kg, $m_0 = 1$ kg, $q = 0.01$, $t \in [0{,}0.99]$, $\bar{x}_1 = 0.5$ radians, $\bar{x}_2 = -0.8$ radians/s. To determine the limits of applicability of the constructed algorithm, different values of the perturbation action, discretness step for control h and parameter α have been considered.

The Runge-Kutta method [61] have been applied for solve of Cauchy problem. Graphs of the phase coordinates $x_1(t)$, $x_2(t)$ and control $u(t)$ for different values of parameters α, h and perturbations have been presented in Figs. 2.5—2.8.

Analysis of the simulation results:

1. The target of the control is reached at any value of initial data and does not depend on them.
2. With increasing of the peturbation the time that is necessary for stabilisation of the system increases; however, it happens at any value of the petrubation, see Figs. 2.5, 2.6; this circumstance is related to the decreasing nature of the perturbation.

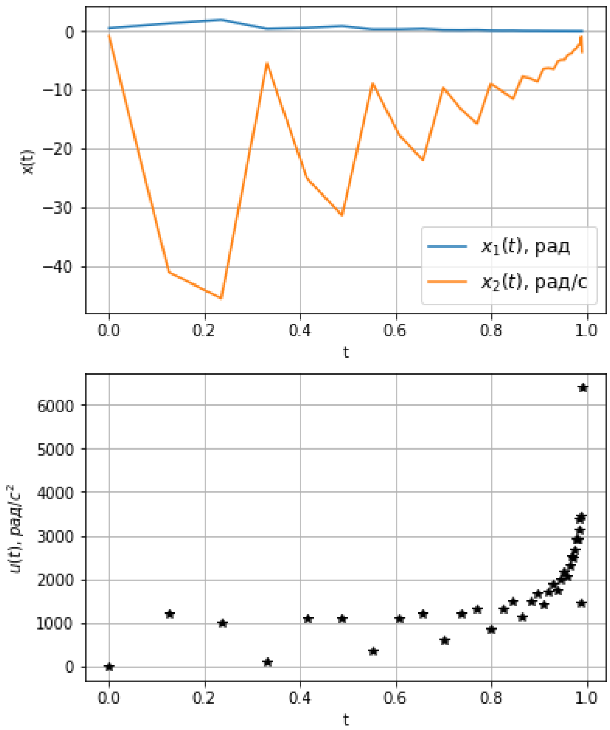

Figure 2.7 — Solution of the Cauchy problem for system (2.17). Values of the parameters: $f = 5$, $\alpha = 0.25$, $h = 0.536$.

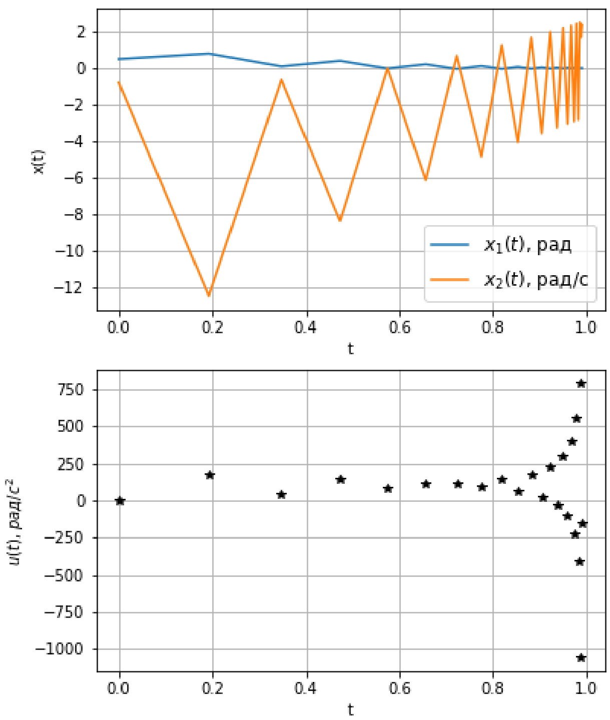

Figure 2.8 — Solution of the Cauchy problem for system (2.17). Values of the parameters: $f = 5$, $\alpha = 2.135$, $h = 0.1$.

3. The value of the discretness control step h is necessary to choose from the range of values $0 < h < 0.536$, see Fig. 2.7; the optimal value of the discretness control step is $h = 0.1$, see Fig. 2.5.
4. The value range of the parameter α belongs to the interval $\alpha \in (0; 2.135)$, see Fig. 2.8, and the optimal value of this constant is 0.25.

2.5 Optimal control of the single-link robot-manipulator with account of pertrubations

In this Section, a solution of the optimal control problem for the single-link robot-manipulator with consideration of perturbations is outlined. The considered system is non-linear.

In the work [66] some approaches to solving the optimal control problems for non-linear problems are considered. For instance, one of them is based on the consideration of the linear-quadratic optimal control problem for the first approximation system of equations and its reduction to the linear programming problem.

In this section, the basis of the applied algorithm is the linear-quadratic problem Statement 2.1 for a linearized system and the solution of the non-stationary differential-matrix Riccati equation. The solution to this equation is possible with Zubov's method of successive approximations [31; 57]. In addition, it is possible to apply the method of integration of the equations system by one of the numerical methods. The problem is solved by numerical integration of the Riccati equation, and analysis of the solution based on one of the theorems from [57] was carried out. A similar approach was developed for the solution of the linear-quadratic problem for linear systems, and it is known as the «establishment method» [67; 68].

Solution to the problem.

The linearized system (2.14) has the form:

$$\dot{x} = A(t) \cdot x + B \cdot u, \tag{2.18}$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Optimal control problem. Find a control that stabilises system (2.18) with quadratic functional of quantity:

$J_1(u) = \int_0^\infty [x^T(t)N_1x(t) + u^2]dt \to \underset{u}{inf}$,

where $x(t)$ is the state vector, u is the scalar control, $N_1 > 0$ is a positive difinite matrix.

We have obtained a non-stationary linear-quadratic problem. The solution to the problem is a function that has a form:

$$u(x,t) = -B^T \cdot P(t) \cdot x(t), \tag{2.19}$$

where $P(t)$ is the solution of the differential-matrix non-stationary Riccati equation.

$$-A(t)^T P(t) - P(t)A(t) + P(t)BB^T P(t) - N_1 = -\dot{P}(t). \tag{2.20}$$

Statement 2.1. *If there exists a restricted solution of the non-stationary Riccati equation* (2.20) *with initial conditions* $P(0) = 0$ *in the form of a symmetric matrix* $P(t)$, *then control* (2.19) *stabilises system* (2.18) *and, in addition, solves the control problem for the original system* (2.14).

Remark 2.1. *In the case of system* (2.18)*, **the Statement 2.1** is a special case of a theorem 2.1, see [57], p. 208.*

Remark 2.2. *The theorem 2.1 (see [57], p. 208) may be considered as "the controllability criteria"for linear-quadratic problems for non-stationary linearly controlled systems.*

Results of the numerical simulation.

Numerical experiments have been carried out at the following values of parameters: $\bar{\alpha} = 0.1$, $L = 10$ m, $M = 20$ kg, $m_0 = 1$ kg, $q = 0.01$, $t \in [0{,}100]$ s, $\bar{x}_1 = 0.5$ radian, $\bar{x}_2 = -0.8$ radian/s, $N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Also, for defining the limits of applicability for the constructed algorithm different values of perturbations have been considered.

Graphs for the symmetric solution matrix elements of the Riccaty equation are represented in Fig. 2.9. Graphs of the phase coordinates $x_1(t)$, $x_2(t)$ taking into account of different cases of the purturbations are shown in Figs. 2.10–2.12. Numerical solutions to the system of ordinary differential equations are obtained with the help of the Runge-Kutta method [61].

Analysis of the numerical experiment results.

1. It follows from the graph (see Fig. 2.9) that the solution matrix of the Riccati equation and its elements are restricted. It means that conditions of the theorem 2.1 (see [57], p. 208) are satisfied. Hence, the control (2.19) stabilises the system (2.18). And, it follows from the graph (see Fig. 2.10) that this control stabilises the original system.
2. The value range of the parameter f that is charaterizing disturbing effect belongs to the span $f \in [0; 2.28)$, see Figs. 2.10–2.12.
3. It follows from graphs (see Figs. 2.10, 2.11) that than larger the purturbating effect value then longer period of time and more significant energy resources of the control are required for stabilisation of the system.
4. It follows from the construction of the problem solution that the target of the control is reached at an unlimited period of time.

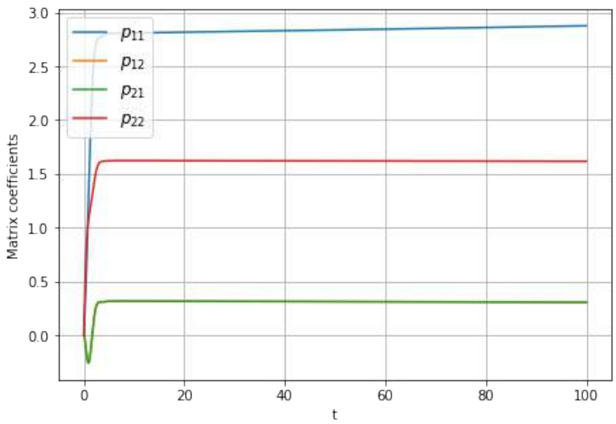

Figure 2.9 — Solution of the Riccati equation (2.20)

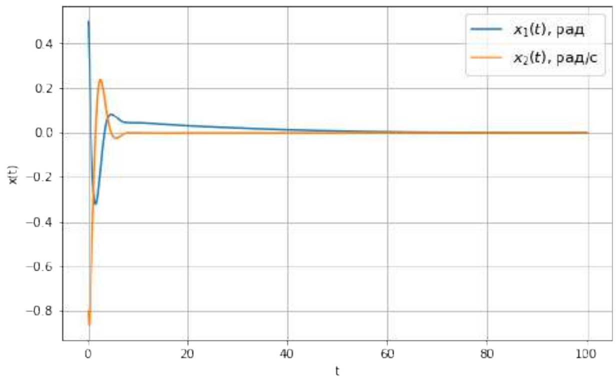

Figure 2.10 — Solution of the Cauchy problem for system (2.14). The value of the parameter f is $f = 0.10$

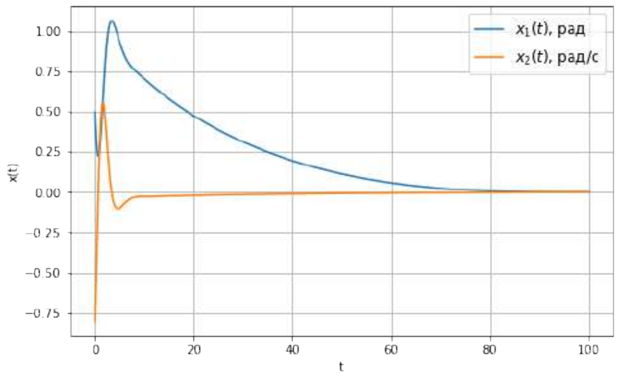

Figure 2.11 — Solution of the Cauchy problem for system (2.14). The value of the parameter f is $f = 1.50$

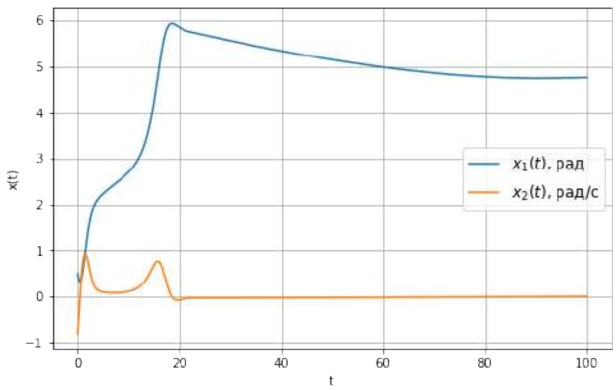

Figure 2.12 — Solution of the Cauchy problem for system (2.14). The value of the parameter f is $f = 2.28$

2.6 Comparison of constructed control algorithms

The results of the numerical experiments for discrete and optimal control problems for system (2.14) with the same model parameters are represented in Figs. 2.13, 2.14.

Analysis of the calculations results:

1. For solving the stabilisation problem in the optimal control case, longer period of time and less energy resources of control are required, see Figs. 2.12, 2.13.
2. It follows from the analysis of graphs (see Figs. 2.5–2.7, 2.10–2.14) and considered models that the discrete control algorithm allows to obtain a solution to the problem at a significantly larger value of the perturbation.

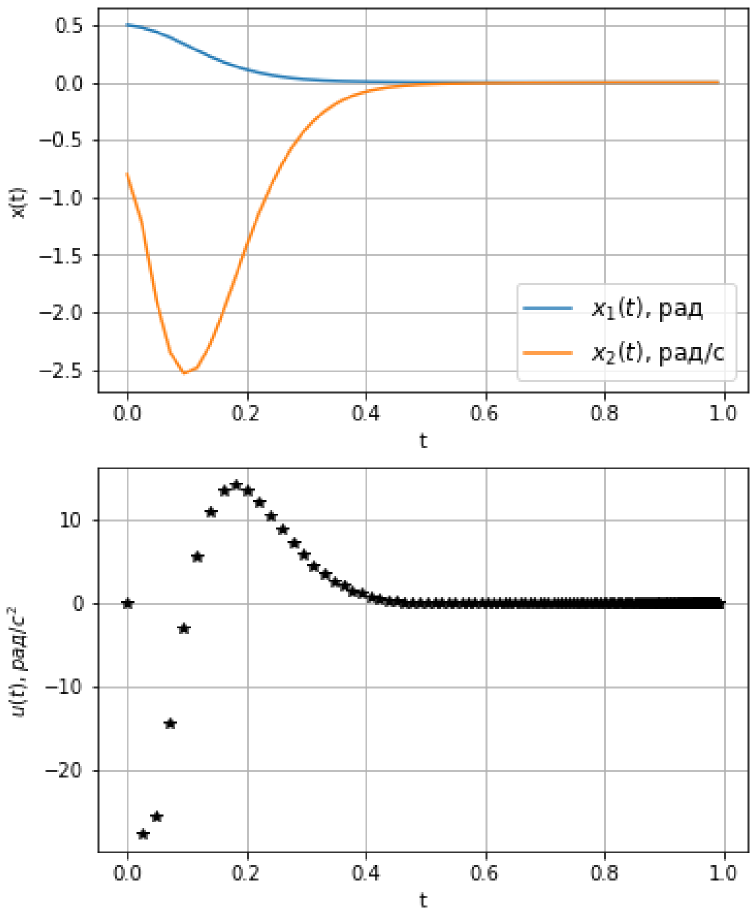

Figure 2.13 — Numerical simulation results for the solution of the discrete control problem for the system (2.14). Parameters are $f = 1.50$, $\alpha = 0.25$, and $h = 0.1$.

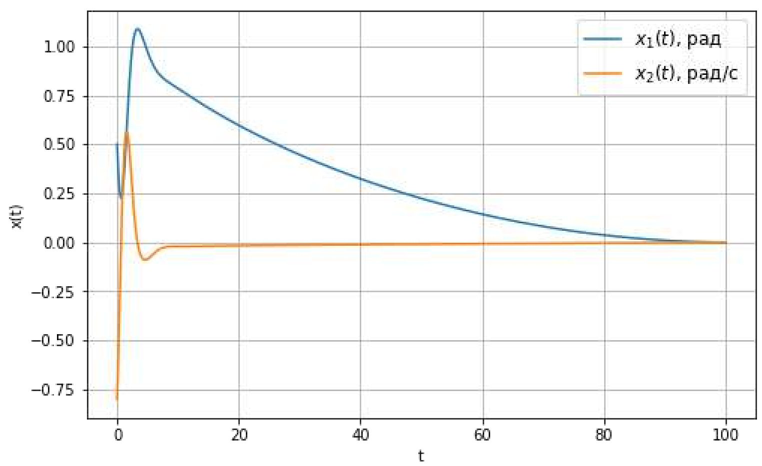

Figure 2.14 — Numerical simulation results for the solution of the optimal control problem for the system (2.14), $f = 1.50$.

2.7 Conclusions for the second chapter

In the present chapter, the construction and analysis of the discrete control algorithm that described in the first chapter are provided.

Based on the presented material, it is possible to draw the following conclusions:

1. The realisation of the considered algorithm is possible with the help of a combination of numerical methods and symbolic calculation software.
2. A constructed discrete control algorithm may be included in the complexity class P [69].
3. For problems with low dimensions, a medium-sized personal computer is enough.
4. The constructed algorithm is more stable to disturbing effects in the system of ODEs in comparison with the optimal control algorithm, but it requires more energy resources for control.

Chapter 3. Solution of the local boundary problem for nonlinear stationary system with account of the computer system verification

3.1 Introduction

Boundary value problems for controlled systems that are concerned with the effectiveness of the mathematical modeling for control processes represent a significant interest. Application of the numerical simulation procedure at different stages of the control system design for technical objects significantly reduces development costs and creation time. Quantity and realibility of the numerical modeling results depend on computer systems serviceability. Due to this circumstance, problem of the control algorithms that allow to verify computer systems during the calculation processes occurs.

In the present work the control function and corresponding phase coordinate functions are found as follows. One of the phase coordinates is given in the form of a polynomial that depends on an independent variable. The verification method is based on the comparison of phase coordinate values received as a result of the calculations with the exact values that are obtained from the given polynomial. If the modulus of the difference between these values is higher than any given number then a decision is made on the use of a reserve computer system.

Similary, the control problem of the computing systems on the control object in the process of control signal generation is solved. The proposed verification method can complement and sometimes replace traditional engineering approaches.

Besides, this algorithm may be applied to the solution of the important and difficult practical problem of choice of the integration step in the solving process of the Cauchy problem for the ordinary differential equations system which is describing the mathematical model of the control object. The difficulty of the solution to this problem is that with large integration steps, the methodical error rate of the calculation scheme grows. And with small integration steps, the calculation error rate increases. With information about the exact value of one of the phase coordinates it is possible to find a balanced integration step for the chosen calculation scheme.

When solving a given problem, the approach applied in the works [23; 34; 70] is used. The main difference of the present result and the one published in [34; 70] is the following: the solution of the original problem tends to be the solution of the boudary problem for non-stationary systems. In the publication [23], an algorithm for solving a boundary problem that is similar to the one considered in [70] is provided. However, it is impossible to use these results because the condition $f(0{,}0{,}t) = 0$ from article [23] is not used in the theorem formulation.

3.2 Problem statement and main theorem

The object of study is a controlled system of ordinary differential equations in the form

$$\dot{x} = f(x{,}u), \tag{3.1}$$

where $x = (x_1, \ldots, x_n)^T$, $x \in R^n$, $u = (u_1, \ldots, u_r)^T$, $u \in R^r$, $r \leqslant n$, $t \in [0{,}1]$.
Here x is a vector of phase coordinates, and, u is a vector of control.
Let the conditions be fulfilled

$$f \in C^{2n}(R^n \times R^r{,}R^n), f = (f_1, \ldots, f_n)^T, \tag{3.2}$$

$$f(0{,}0) = 0; \tag{3.3}$$

$$\frac{\partial f_1}{\partial u_1}(0{,}0) \neq 0. \tag{3.4}$$

Introduce the following matrices:
$A_0 = \{a_{ij}\}$, $a_{ij} = \frac{\partial f_i}{\partial x_j}(0{,}0) - \frac{\partial f_i}{\partial u_1}(0{,}0) \cdot \frac{\partial f_1}{\partial x_j}(0{,}0) \cdot \left(\frac{\partial f_1}{\partial u_1}(0{,}0)\right)^{-1}$,
$B_0 = \{b_{ij}\}$, $\{b_{ij}\} = \frac{\partial f_i}{\partial u_j}(0{,}0) - \frac{\partial f_i}{\partial u_1}(0{,}0) \cdot \frac{\partial f_1}{\partial u_j}(0{,}0) \cdot \left(\frac{\partial f_1}{\partial u_1}(0{,}0)\right)^{-1}$, $i = 2, \ldots, n$, $j = 2, \ldots, r$,
Let $S_0 = (B_0{,}A_0B_0, \ldots{,}A_0^{n-1}B_0)$.
Suppose that matrix S_0 satisfies the following condition:

$$rank\ S_0 = n - 1. \tag{3.5}$$

The following restrictions are imposed on the control u:

$$\|u\| \leqslant N, N > 0, N = const \tag{3.6}$$

Problem 1. Find pair of functions $x(t) = (x_1(t), \ldots, x_n(t))^T$, $x(t) \in C^1[0,1]$; R^n, $u(t) \in C^1[0,1]$; R^r which is satisfying to the system (3.1) and conditions:

$$x(0) = x_0, x(1) = 0, x_0 = (x_1^0, \ldots, x_n^0)^T, \tag{3.7}$$

$$x_1(t) = x_1^0 \cdot (1 - t), x_1^0 \in R^1. \tag{3.8}$$

This pair of functions we will call the solution of the problem (3.1), (3.7), and (3.8).

Solution to the problem may be found based on the proof of the following theorem.

Theorem 3.1. *Let for system* (3.1) *conditions* (3.2) - (3.5) *are sutisfied. Then there exists such* $\varepsilon > 0$ *that* $\forall x_0 \in R^n : \|x_0\| < \varepsilon$ *solution of the problem* (3.1), (3.7), (3.8) *exists which may be obtained afrer solving of the stabilisation problem for linear non-stationary system of a special form and the following solution of the Cauchy problem for an auxiliary system of the ordinary differential equations.*

The proof of the theorem may be divided into several stages.

3.3 Formulation of the auxiliary problems and auxiliary systems construction

From conditions (3.2) - (3.4) and the implicit function theorem, it follows that there exists $\varepsilon_1 > 0$ such, that function $u_1(t, x_1^0, x_2, \ldots, x_n, u_2, \ldots, u_r)$ exists for all $x_1^0, x_i, i = 2, \ldots, n, u_j, j = 2, \ldots, r : |x_1^0| < \varepsilon_1, |x_i| < \varepsilon_1, |u_i| < \varepsilon_1$ and satisfies the equation

$$x_1^0 = f_1(x_1^0 \cdot (1 - t), x_2, \ldots, x_n, u_1(x_1(t), x_1^0, x_2, \ldots, x_n, u_2, \ldots, u_r), u_2, \ldots, u_r),$$
$$t \in [0,1], \tag{3.9}$$

and the condition

$$u_1(t,0,\ldots,0) \equiv 0. \tag{3.10}$$

After substituting of function $u_1(t,x_1^0,x_2,\ldots,x_n,u_2,\ldots,u_r)$ and $x_1(t)$ from formula (3.8) into the right side of all equations of the system (3.1), except the first, we will obtain the system

$$\begin{gathered}\dot{x}_i = f_i(x_1(t),x_2,\ldots,x_n,u_1(t,x_1^0,x_2,\ldots,x_n,u_2,\ldots,u_r),u_2,\ldots,u_r),\\ i=2,\ldots,n.\end{gathered} \tag{3.11}$$

Consider the problem.

Find pair of functions $\bar{x}(t)$, $\bar{u}(t)$, where $\bar{x}(t) = (x_2(t),\ldots,x_n(t))^T$, $\bar{u}(t) = (u_2(t),\ldots,u_r(t))^T$, which are satisfying the system (3.11) and the conditions

$$\bar{x}(0)=\bar{x}_0, \bar{x}(1)=\bar{0}, \bar{x}_0=(x_2^0,\ldots,x_n^0)^T, \bar{0}=(0,\ldots,0)_{n-1\times 1}. \tag{3.12}$$

Pair of functions $\bar{x}(t)$, $\bar{u}(t)$ satisfying the system (3.11) and the conditions (3.12) we will call a solution of the problem (3.11), (3.12).

Remark 3.3.1. *If to substitute solution of the problem* (3.11), (3.12) *in the function* $u_1(t,x_1^0,x_2,\ldots,x_n,u_2,\ldots,u_r)$, *then we will have a set of functions* $x_1(t),\bar{x}(t)$ *and* $u_1(t),\bar{u}(t)$ *which is solution of the Problem 3.1.*

Introduce notations

$$\begin{gathered}\bar{f}_i(t,x_1^0,x_2,\ldots,x_n,u_1,\ldots,u_r) = \\ f_i(x_1(t),x_2,\ldots,x_n,u_1(t,x_1^0,x_2,\ldots,x_n,u_2,\ldots,u_r),u_2,\ldots,u_r),\ i=2,\ldots,n.\end{gathered} \tag{3.13}$$

It follows from the conditions (3.3) and (3.10)

$$\bar{f}_i(t,0,\bar{0},\bar{\bar{0}}) = 0,\ \bar{\bar{0}} = (0,\ldots,0)_{r-1\times 1},\ i=2,\ldots,n. \tag{3.14}$$

Let us replace the independent variable t on τ by a formula:

$$t(\tau) = 1 - e^{-\alpha\tau},\ \tau \in [0,\infty), \tag{3.15}$$

where $\alpha > 0$ is a real number to be determined.

As a result, a vector form of the system (3.11) is as follows

$$\frac{d\bar{c}}{d\tau} = \alpha e^{-\alpha\tau}\bar{f}(t(\tau),x_1^0,\bar{c},\bar{d}), \bar{f} = (\bar{f}_2,\ldots,\bar{f}_n)^T, \tag{3.16}$$

$$\bar{c}(\tau) = \bar{x}(t(\tau)),\ \ \bar{c} = (c_2,\ldots,c_n)^T, \bar{d}(\tau) = \bar{u}(t(\tau)),\ \ \bar{d} = (d_2,\ldots,d_r)^T. \tag{3.17}$$

Consider the problem.

Find a pair of functions $\bar{c}(\tau) \in C^1([0,\infty))$, $\bar{d}(\tau) \in C^1([0,\infty))$ satisfying the system (3.16) and the conditions

$$\bar{c}(0) = \bar{x}_0, \bar{c}(\tau) \to \bar{0} \text{ при } \tau \to \infty \tag{3.18}$$

This pair of functions we will call the solution of problem (3.16), (3.18).

Remark 3.3.2. *It is easy to see that after passing to limit $\tau \to \infty$ and to original independent variable t by formulae* (3.15), (3.17) *in the solution of problem* (3.16)*,* (3.18) *we will obtain solution of the problem* (3.11)*,* (3.12)*.*

Let us introduce the following notations for solving the problem (3.16), (3.18):

$\tilde{\bar{c}} = \theta_i\bar{c}$, $\tilde{\bar{d}} = \theta_i\bar{d}$, $\tilde{t} = 1 - \theta_i e^{-\alpha\tau}$, $\theta_i \in [0,1]$, $i = 2,\ldots,n$;

$|k| = \sum_{j=2}^{n} k_j$; $|m| = \sum_{j=2}^{r} m_j$, $k! = k_2!\ldots k_n!$, $m! = m_2!\ldots m_r!$.

If we represent the right-hand side of the system (3.11) as a Taylor formula in the neighbourhood of point $(1,\ x_1^0,\ \bar{0},\ \bar{\bar{0}})$, taking into account the replacement of function f with $\bar{f}$, then the system will take the form

$$\frac{dc_i}{d\tau} = \alpha e^{-\alpha\tau}\Bigg(\bar{f}_i(1,x_1^0,\bar{0},\bar{\bar{0}}) + \sum_{j=2}^{n}\frac{\partial \bar{f}_i}{\partial x_j}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j + \sum_{j=2}^{r}\frac{\partial \bar{f}_i}{\partial u_j}(1,x_1^0,\bar{0},\bar{\bar{0}})d_j -$$

$$- e^{-\alpha\tau}\frac{\partial \bar{f}_i}{\partial t}(1,x_1^0,\bar{0},\bar{\bar{0}})\Bigg) +$$

$$+ \frac{1}{2}\alpha e^{-\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_k}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j c_k + \sum_{j=2}^{r}\sum_{k=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial u_j \partial u_k}(1,x_1^0,\bar{0},\bar{\bar{0}})d_j d_k +$$

$$+ 2\sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial u_k}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j d_k - 2\alpha e^{-\alpha\tau}\sum_{j=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial t}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j -$$

$$- 2\alpha e^{-\alpha\tau}\sum_{j=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial u_j \partial t}(1,x_1^0,\bar{0},\bar{\bar{0}})d_j + e^{-2\alpha\tau}\frac{\partial^2 \bar{f}_i}{\partial t^2}(1,x_1^0,\bar{0},\bar{\bar{0}})\Bigg) + \ldots +$$

$$+ \alpha e^{-\alpha\tau}\sum_{|k|+|m|+l=2n-2,}\frac{1}{k!m!l!}\frac{\partial \bar{f}_i^{|k|+|m|+l}}{\partial {x_2}^{k_2}\ldots\partial {x_n}^{k_n}\partial {u_2}^{m_2}\ldots\partial {u_r}^{m_r}\partial t^l}(1,x_1^0,\bar{0},\bar{\bar{0}}) \times$$

$$\times {c_2}^{k_2}\ldots {c_n}^{k_n}\bar{d}_2^{m_2}\ldots {d_r}^{m_r}(-1)^l e^{-l\alpha\tau} +$$

$$+ \alpha e^{-\alpha\tau} \times \sum_{|k|+|m|+l=2n-1,}\frac{1}{k!m!l!}\frac{\partial \bar{f}_i^{|k|+|m|+l}}{\partial x_2^{k_2}\ldots\partial x_n^{k_n}\partial u_2^{m_2}\ldots\partial u_r^{m_r}\partial t^l}(\tilde{t}(\tau),x_1^0,\tilde{\bar{c}},\tilde{\bar{d}}) \times$$

$$\times {c_2}^{k_2}\ldots {c_n}^{k_n}{d_2}^{m_2}\ldots {d_r}^{m_r}(-1)^l e^{-l\alpha\tau},\ i = 2,\ldots,n. \tag{3.19}$$

Further considerations we will conduct on the condition of the restrictions for function $\bar{c}(\tau)$:

$$\|\bar{c}(\tau)\| < C, C > 0. \tag{3.20}$$

Let us to perform $2n-1$ shift-transformations of the function $c_i \to c_i^{(2n-1)}$. The main goal of these transformations is the norm of right-side summands, which do not contain components of vectors $\bar{c}^{(2n-1)}$ and $\bar{d}$, should satisfy the following estimation $O(e^{-2n\alpha\tau}|x_1^0|)$, $\tau \to \infty$, $x_1^0 \to 0$. At the first stage, we are replacing $c_i(\tau)$ with $c_i^{(1)}(\tau)$ by a formula:

$$c_i = c_i^{(1)}(\tau) - e^{-\alpha\tau}\bar{f}_i(1,x_1^0,\bar{0},\bar{\bar{0}}),\ i = 2,\ldots,n. \tag{3.21}$$

Let $D^{|k|+|m|+l}\bar{f}_i = \frac{\partial \bar{f}_i^{|k|+|m|+l}}{\partial x_2^{k_2}\ldots\partial x_n^{k_n}\partial u_2^{m_2}\ldots\partial u_r^{m_r}\partial t^l}$.

For compact recording, we introduce the notation:

$$F_i(x_1^0) = \bar{f}_i(1, x_1^0, \bar{0}, \bar{\bar{0}}),\ i = 2, \ldots, n. \tag{3.22}$$

Obviously,

$$F_i(0) = 0,\ i = 2, \ldots, n. \tag{3.23}$$

We obtain the following system after substituting (3.21) into the left and right sides of (3.19) and taking into account the introduced notations:

$$\frac{dc_i^{(1)}}{d\tau} = -\alpha e^{-2\alpha\tau}\sum_{j=2}^{n}\frac{\partial \bar{f}_i}{\partial x_j}(1, x_1^0, \bar{0}, \bar{\bar{0}})F_j(x_1^0)$$

$$+\alpha e^{-3\alpha\tau}\big(\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})F_j(x_1^0)F_k(x_1^0) + \sum_{j=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial t}(1, x_1^0, \bar{0}, \bar{\bar{0}})F_j(x_1^0)\big)$$

$$+\alpha e^{-\alpha\tau}\left(\sum_{j=2}^{n}\frac{\partial \bar{f}_i}{\partial x_j}(1, x_1^0, \bar{0}, \bar{\bar{0}})c_j^{(1)} + \sum_{j=2}^{r}\frac{\partial \bar{f}_i}{\partial u_j}(1, x_1^0, \bar{0}, \bar{\bar{0}})d_j - e^{-\alpha\tau}\frac{\partial \bar{f}_i}{\partial t}(1, x_1^0, \bar{0}, \bar{\bar{0}})\right)$$

$$-\frac{1}{2}\alpha e^{-2\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})c_j^{(1)}F_k(x_1^0)$$

$$+\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})F_j(x_1^0)c_k^{(1)} + 2\sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial u_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})F_j(x_1^0)d_k\Bigg)$$

$$+\frac{1}{2}\alpha e^{-\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})c_j^{(1)}c_k^{(1)} + \sum_{j=2}^{r}\sum_{k=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial u_j \partial u_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})d_j d_k$$

$$+ 2\sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial u_k}(1, x_1^0, \bar{0}, \bar{\bar{0}})c_j^{(1)}d_k - 2\alpha e^{-\alpha\tau}\sum_{j=2}^{n}\frac{\partial^2 \bar{f}_i}{\partial x_j \partial t}(1, x_1^0, \bar{0}, \bar{\bar{0}})c_j^{(1)}$$

$$- 2\alpha e^{-\alpha\tau}\sum_{j=2}^{r}\frac{\partial^2 \bar{f}_i}{\partial u_j \partial t}(1, x_1^0, \bar{0}, \bar{\bar{0}})d_j + e^{-2\alpha\tau}\frac{\partial^2 \bar{f}_i}{\partial t^2}(1, x_1^0, \bar{0}, \bar{\bar{0}})\Bigg) + \ldots +$$

$$+ \alpha e^{-\alpha\tau}\sum_{|k|+|m|+l=2n-2,}\frac{1}{k!m!l!}D^{|k|+|m|+l}\bar{f}_i(1, x_1^0, \bar{0}, \bar{\bar{0}})\times$$

$$(c_2^{(1)} - e^{-\alpha\tau}F_2(x_1^0))^{k_2}\ldots(c_n^{(1)} - e^{-\alpha\tau}F_n(x_1^0))^{k_n}d_2^{\,m_2}\ldots d_r^{\,m_r}(-1)^l e^{-l\alpha\tau}$$

$$+ \alpha e^{-\alpha\tau}\sum_{|k|+|m|+l=2n-1,}\frac{1}{k!m!l!}D^{|k|+|m|+l}\bar{f}_i(\tilde{t}(\tau), x_1^0, \tilde{\bar{c}}, \tilde{\bar{\bar{d}}})\times$$

$$(c_2^{(1)} - e^{-\alpha\tau}F_2(x_1^0))^{k_2}\ldots(c_n^{(1)} - e^{-\alpha\tau}F_n(x_1^0))^{k_n}d_2^{\,m_2}\ldots d_r^{\,m_r}(-1)^l e^{-l\alpha\tau},$$

$$i = 2, \ldots, n. \tag{3.24}$$

Initial conditions will take the form:

$$c_i^{(1)}(0) = x_i^0 + F_i(x_1^0),\ i = 2, \ldots, n. \tag{3.25}$$

At the next stage, we are replacing variables using the formula:

$$c_i^{(1)} = c_i^{(2)}(\tau) + e^{-2\alpha\tau}\Bigg(\sum_{j=2}^{n}\frac{\partial \bar f_i}{\partial x_j}(1, x_1^0,\bar 0,\bar{\bar 0})\Bigg)F_i(x_1^0) =$$
$$= c_i^{(2)}(\tau) + e^{-2\alpha\tau}\varphi_i^{(2)}(x_1^0),\ i = 2,\ldots,n, \tag{3.26}$$

where $\varphi_i^{(2)}(x_1^0) = \sum_{j=2}^{n}\frac{\partial \bar f_i}{\partial x_j}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0)$,

It follows from (3.23) that

$$\varphi_i^{(2)}(0) = 0. \tag{3.27}$$

After substituting (3.26) in the left-hand and right-hand sides of the system (3.24), we obtain

$$\begin{aligned}
\frac{dc_i^{(2)}}{d\tau} &= \alpha e^{-3\alpha\tau}\Bigg(\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0)F_k(x_1^0)\\
&+\sum_{j=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial t}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0) + \sum_{j=2}^{n}\frac{\partial \bar f_i}{\partial x_j}(1, x_1^0,\bar 0,\bar{\bar 0})\varphi_j^{(2)}(x_1^0)\Bigg)\\
&-\frac{1}{2}\alpha e^{-4\alpha\tau}\times\Bigg(\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar 0)\varphi_j^{(2)}(x_1^0)F_k(x_1^0)\\
&+\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0)\varphi_k^{(2)}(x_1^0)\\
&+2\sum_{j=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial t}(1, x_1^0,\bar 0,\bar 0)\varphi_j^{(2)}(x_1^0)\\
&+\alpha e^{-\alpha\tau}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar 0)\varphi_j^{(2)}(x_1^0)\varphi_k^{(2)}(x_1^0)\Bigg)\\
&+\alpha e^{-\alpha\tau}\Bigg(\sum_{j=2}^{n}\frac{\partial \bar f_i}{\partial x_j}(1, x_1^0,\bar 0,\bar{\bar 0})c_j^{(2)} + \sum_{j=2}^{r}\frac{\partial \bar f_i}{\partial u_j}(1, x_1^0,\bar 0,\bar{\bar 0})d_j - e^{-\alpha\tau}\frac{\partial \bar f_i}{\partial t}(1, x_1^0,\bar 0,\bar{\bar 0})\Bigg)\\
&-\alpha e^{-2\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar{\bar 0})c_j^{(2)}F_k(x_1^0)\\
&+\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0)c_k^{(2)} + \sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2 \bar f_i}{\partial x_j\partial u_k}(1, x_1^0,\bar 0,\bar{\bar 0})F_j(x_1^0)d_k\Bigg)\\
&+\frac{1}{2}\alpha e^{-3\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar{\bar 0})c_j^{(2)}\varphi_k^{(2)}(x_1^0)\\
&+\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2 \bar f_i}{\partial x_j\partial x_k}(1, x_1^0,\bar 0,\bar 0)\varphi_j^{(2)}(x_1^0)c_k^{(2)} +\\
&\sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2 \bar f_i}{\partial x_j\partial u_k}(1, x_1^0,\bar 0,\bar{\bar 0})\varphi_j^{(2)}(x_1^0)d_k\Bigg)
\end{aligned}\tag{3.28}$$

$$+\tfrac{1}{2}\alpha e^{-\alpha\tau}\Bigg(\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{\partial^2\bar{f}_i}{\partial x_j\partial x_k}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j^{(2)}c_k^{(2)}+\sum_{j=2}^{r}\sum_{k=2}^{r}\frac{\partial^2\bar{f}_i}{\partial u_j\partial u_k}(1,x_1^0,\bar{0},\bar{\bar{0}})d_jd_k$$
$$+2\sum_{j=2}^{n}\sum_{k=2}^{r}\frac{\partial^2\bar{f}_i}{\partial x_j\partial u_k}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j^{(2)}d_k-2\alpha e^{-\alpha\tau}\sum_{j=2}^{n}\frac{\partial^2\bar{f}_i}{\partial x_j\partial t}(1,x_1^0,\bar{0},\bar{\bar{0}})c_j^{(2)}$$
$$-2\alpha e^{-\alpha\tau}\sum_{j=2}^{r}\frac{\partial^2\bar{f}_i}{\partial u_j\partial t}(1,x_1^0,\bar{0},\bar{\bar{0}})d_j+e^{-2\alpha\tau}\frac{\partial^2\bar{f}_i}{\partial t^2}(1,x_1^0,\bar{0},\bar{\bar{0}})\Bigg)+\ldots+$$
$$+\alpha e^{-\alpha\tau}\sum_{|k|+|m|+l=2n-2,}\frac{1}{k!m!l!}D^{|k|+|m|+l}\bar{f}_i(1,x_1^0,\bar{0},\bar{\bar{0}})\times$$
$$(c_2^{(2)}-e^{-\alpha\tau}F_2(x_1^0)+e^{-2\alpha\tau}\varphi_2^{(2)}(x_1^0))^{k_2}\ldots(c_n^{(2)}-e^{-\alpha\tau}F_n(x_1^0)+e^{-2\alpha\tau}\varphi_n^{(2)}(x_1^0))^{k_n}\times$$
$$d_2{}^{m_2}\ldots d_r{}^{m_r}(-1)^le^{-l\alpha\tau}+\ \alpha e^{-\alpha\tau}\sum_{|k|+|m|+l=2n-1,}\frac{1}{k!m!l!}D^{|k|+|m|+l}\bar{f}_i(\tilde{t}(\tau),x_1^0,\tilde{\bar{c}},\tilde{\bar{d}})\times$$
$$(c_2^{(2)}-e^{-\alpha\tau}F_2(x_1^0)+e^{-2\alpha\tau}\varphi_2^{(2)}(x_1^0))^{k_2}\ldots(c_n^{(2)}-e^{-\alpha\tau}F_n(x_1^0)+e^{-2\alpha\tau}\varphi_n^{(2)}(x_1^0))^{k_n}\times$$
$$d_2{}^{m_2}\ldots d_r{}^{m_r}(-1)^le^{-l\alpha\tau},\ i=2,\ldots,n.$$

Initial conditions will take the form

$$c_i^{(2)}(0)=x_i^0+F_i(x_1^0)-\varphi_i^{(2)}(x_1^0),\ i=2,\ldots,n. \tag{3.29}$$

Obviously, the summands of the right side of the system (3.28), which are not consisting components of vectors $\bar{c}^{(2)}$ and $\bar{d}$ satisfy the estimation $O(e^{-3\alpha\tau}|x_1^0|)$ at $\tau\to\infty$, $x_1^0\to 0$ in the domain (3.20), (3.6).

Based on (3.21), (3.24), (3.26), (3.28) and inductive transition, the shift-transformasion on the $k-$th stage will take a form

$$c_i^{(k-1)}=c_i^{(k)}(\tau)+e^{-k\alpha\tau}\varphi_i^{(k)}(x_1^0),\ \varphi_i^{(k)}(0)=0,\ i=2,\ldots,n. \tag{3.30}$$

If transformation (3.30) to apply $2n-1$ times and in the right-hand side of obtained system to join linear by components of vector $\bar{c}^{(2n-1)}$ summands with factors $e^{-i\alpha\tau}$ $i=1,\ldots,n$ and also, summands which are linear by components of vector $\bar{d}$ with factors $e^{-i\alpha\tau}$ $i=1,\ldots,n$, then in accordance with formulae (3.21), (3.24), (3.25), (3.26), (3.28), and (3.29) we will obtain system and initial data which can be written in vector form:

$$\begin{gathered}\frac{d\bar{c}^{(2n-1)}}{d\tau}=P(x_1^0)\cdot\bar{c}^{(2n-1)}+Q(x_1^0)\cdot\bar{d}+R_1(\tau,x_1^0,\bar{c}^{(2n-1)},\bar{d})+\\+R_2(\tau,x_1^0,\bar{c}^{(2n-1)},\bar{d})+R_3(\tau,x_1^0,\bar{c}^{(2n-1)},\bar{d})+R_4(\tau,x_1^0,\bar{c}^{(2n-1)},\bar{d});\\R_1=(R_2^1,\ldots,R_n^1)^T,\ R_2=(R_2^2,\ldots,R_n^2)^T,\ R_3=(R_2^3,\ldots,R_n^3)^T,\\R_4=(R_2^4,\ldots,R_n^4)^T.\end{gathered} \tag{3.31}$$

From condition (3.13) and differentiation as complex functions of functions $f_i, i = 2, \dots, n$ by components of vectors $\bar{x}$, $\bar{u}$ and variable t we obtain equality:

$$\begin{aligned} P(x_1^0) &= \alpha e^{-\alpha\tau}(A(x_1^0) + \alpha e^{-\alpha\tau}P_2(x_1^0) + \ldots + \alpha e^{-(n-1)\alpha\tau}P_n(x_1^0)), \\ Q(x_1^0) &= \alpha e^{-\alpha\tau}(B(x_1^0) + \alpha e^{-\alpha\tau}Q_2(x_1^0) + \ldots + \alpha e^{-(n-1)\alpha\tau}Q_n(x_1^0)), \end{aligned} \tag{3.32}$$

$$\begin{aligned} c_i^{(2n-1)}(0) &= x_i^0 + F_i(x_1^0) - \varphi_i^{(2)}(x_1^0) - \ldots - \varphi_i^{(2n-1)}(x_1^0),, \; i = 2, \dots, n, \\ \varphi_i^{(k)}(x_1^0) &= (\varphi_2^{(k)}(x_1^0), \dots, \varphi_n^{(2n-1)}(x_1^0)), \, k = 2, \dots, 2n-1, \; \varphi_i^{(k)}(0) = 0. \end{aligned} \tag{3.33}$$

It is easy to see that

$$P_i(x_1^0) \to 0, Q_i(x_1^0) \to 0 \text{ при } x_1^0 \to 0, \; i = 2, \dots, n. \tag{3.34}$$

Besides

$$A(0) = A_0, B(0) = B_0. \tag{3.35}$$

Functions R_i^1 contain summands of the right-hand side of the system (3.31) which are linear depending from components $c^{(2n-1)}$ with factors $e^{-i\alpha\tau}$, where $i \geqslant n+1$. R_i^2 includes summands of the right-hand side of the system (3.31) that are linearly dependent from components $\bar{d}$ with factors $e^{-i\alpha\tau}$, $i \geqslant n+1$. R_i^3 contains summands of the right-hand side of the system (3.31), which are linear depending on the components of vectors $\bar{c}^{(2n-1)}$ and $\bar{d}$. R_i^4 contains from the summands which are not contain degrees of the components of vectors $\bar{c}^{(2n-1)}$ and $\bar{d}$.

It follows from conditions (3.21), (3.26), and (3.30) that there exist constants $C_1 > 0$ and $\varepsilon_1 > 0$ such that, for all $\bar{c}^{(2n-1)}$ and x_1^0 which belong to the domain

$$||\bar{c}^{(2n-1)}|| < C_1, \; |x_1^0| < \varepsilon_1. \tag{3.36}$$

The corresponding function $\bar{c}(\tau)$ will belong to the domain (3.20).

It follows from conditions (3.2), (3.3), and the construction of functions R_1, R_2, R_3, R_4 that estimations

$$\|R_1(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d})\| \leqslant e^{-(n+1)\alpha\tau} L_1 \|\bar{c}^{(2n-1)}\|, L_1 > 0, \tag{3.37}$$

$$\|R_2(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d})\| \leqslant e^{-(n+1)\alpha\tau} L_2 \|\bar{d}\|, L_2 > 0, \tag{3.38}$$

$$\|R_3(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d})\| \leqslant e^{-\alpha\tau} L_3(\|\bar{c}^{(2n-1)}\|^2 + \|\bar{d}\|^2), L_3 > 0, \tag{3.39}$$

$$\|R_4(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d})\| \leqslant e^{-2n\alpha\tau} L_4(x_1^0), L_4(x_1^0) \to 0 \text{ при } x_1^0 \to 0. \tag{3.40}$$

are fair in the domain (3.6), (3.36).

Constants $L_i, i = 1 \dots 4$ depend from domain (3.36).

Introduce the auxiliary control function $\bar{\upsilon}$ which is related to the original $\bar{d}$ by the following system of differential equations

$$\frac{d}{d\tau}\bar{d}(\tau) = \bar{\upsilon}, \bar{\upsilon} = (\upsilon_1, \dots, \upsilon_{r-1})^T. \tag{3.41}$$

Let

$$\bar{d}(0) = \bar{\bar{0}}. \tag{3.42}$$

Then the equations system (3.31), (3.41), and initial conditions (3.33), (3.42) will take a form

$$\begin{gathered}\frac{d}{d\tau}\tilde{c}^{(2n-1)} = \bar{P}(x_1^0) \cdot \tilde{c}^{(2n-1)} + \bar{Q}(x_1^0) \cdot \bar{\upsilon} + \bar{R}_1(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d}) + \\ + \bar{R}_2(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d}) + \bar{R}_3(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d}) + \bar{R}_4(\tau, x_1^0, \bar{c}^{(2n-1)}, \bar{d}),\end{gathered} \tag{3.43}$$

$$\bar{P}(x_1^0) = \begin{pmatrix} P(x_1^0) & Q(x_1^0) \\ O_1 & O_2 \end{pmatrix}_{n+r-2 \times n+r-2}, \ \bar{Q} = \begin{pmatrix} O_3 \\ E \end{pmatrix}_{n+r-2 \times r-1},$$

where $\tilde{c}^{(2n-1)} = (\bar{c}^{(2n-1)}, d)^T_{n+r-2\times 1}$, $\bar{R}_1 = (R_2^1, \dots, R_n^1, 0, \dots, 0)^T_{n+r-2\times 1}$,
$\bar{R}_2 = (R_2^2, \dots, R_n^2, 0, \dots, 0)^T_{n+r-2\times 1}$, $\bar{R}_3 = (R_2^3, \dots, R_n^3, 0, \dots, 0)^T_{n+r-2\times 1}$,
$\bar{R}_4 = (R_2^4, \dots, R_n^4, 0, \dots, 0)^T_{n+r-2\times 1}$, O_1, O_2, O_3 are null-matrices with corresponding dimensions, and E is an identity matrix,

$$\tilde{c}^{(2n-1)}(0) = \tilde{c}_0^{(2n-1)}, \tilde{c}_0^{(2n-1)} = (\bar{c}_0^{(2n-1)}, 0, \dots, 0)^T_{n+r-2\times 1}. \tag{3.44}$$

Further proof will be based on the following lemma.

3.4 Formulation and proof of the auxiliary lemma

Consider the linear part of the system (3.43):

$$\frac{d}{d\tau}\tilde{c}^{(2n-1)} = \bar{P}\cdot\tilde{c}^{(2n-1)} + \bar{Q}\cdot\bar{\upsilon}. \qquad (3.45)$$

Lemma 3.1. *Let conditions* (3.2), (3.5) *be satisfied. Then there exists a positive number* ε_2*:* $0 < \varepsilon_2 < \varepsilon_1$ *such that for all* x_1^0*:* $|x_1^0| < \varepsilon_2$ *there exists an auxiliary control function* $\bar{\upsilon}(\tau)$ *in the form*

$$\bar{\upsilon}(\tau) = M(\tau)\tilde{c}^{(2n-1)},\ \|M(\tau)\| = O(e^{(n-1)\alpha\tau})\ \text{при}\ \tau\to\infty, \qquad (3.46)$$

which obtain exponential decreasing of the fundamental matrix of the system (3.45) *which is closed by an auxiliary control function* (3.46).

P r o o f o f t h e l e m m a. Let us denote j-th column of the matrix $\bar{Q}$ as $L_1^j,\ j = 1,\ldots,r-1$. Consider matrix

$$S_1 = \{L_1^1, L_2^1, \ldots, L_{k_1}^1, L_1^2, L_2^2, \ldots, L_{k_2}^2, \ldots, L_1^{r-1}, L_2^{r-1}, \ldots, L_{k_{r-1}}^{r-1}\},$$
$$L_i^j = \bar{P}L_i^{j-1} - \frac{dL_i^{j-1}}{d\tau},\ j = 1,\ldots,r-1,\ i = 2,\ldots,k_j,$$

where $k_j, j = 1,\ldots,r-1$, is the maximum column number of the matrix $L_1^j,\ldots,L_{k_j}^j$, $j = 1,\ldots,r-1$, such that vectors $L_1^1, L_2^1, \ldots, L_{k_1}^1, L_1^2, L_2^2, \ldots, L_{k_2}^2, \ldots, L_1^{r-1}, L_2^{r-1}, \ldots, L_{k_{r-1}}^{r-1}$ are linear independent.

With accuracy before permutation of columns, the structure of the matrix S_1 is

$$\begin{pmatrix} O_{n-1\times r-1} & L_1 & \ldots & L_{n-1} \\ E_{r-1\times r-1} & O_{r-1\times r-1} & \ldots & O_{r-1\times r-1} \end{pmatrix},$$

where $O_{r-1\times r-1}$ is null-matrix with dimension $r-1\times r-1$;

$$L_1 = Q,\ L_i = PL_i - \tfrac{dL_{i-1}}{d\tau},\ i = 2,\ldots,n-1.$$

Let us show that the rank of the matrix S_1 equals to $n+r-2$.

Let $S_2 = \{L_1,\ldots,L_{n-1}\}$.

Introduce a matrix $S_3 = \{\bar{L}_1,\ldots,\bar{L}_{n-1}\}$. $\bar{L}_1 = \alpha e^{-\alpha\tau}B(x_1^0)$, $\bar{L}_i = \alpha e^{-\alpha\tau}A(x_1^0)L_i - \frac{dL_{i-1}}{d\tau}$, $i = 2,\ldots,n-1$.

Conditions (3.34) guarantee $\bar{\varepsilon} < \varepsilon_1$ existation of $\bar{\varepsilon} < \varepsilon_1$ such that $\forall x_1^0 : |x_1^0| < \bar{\varepsilon}$, $rank\ S_2 = rank\ S_3$.

Reasoning by contradiction, taking into account conditions (3.5), (3.35), it is easy to see that there exists $\tilde{\varepsilon}$: $0 < \tilde{\varepsilon} < \bar{\varepsilon}$ such that $\forall x_1^0 : |x_1^0| < \tilde{\varepsilon}$ and

$$rank\ S_3 = rank\ S_0 = n - 1. \tag{3.47}$$

From equality (3.47) and structure of the matrix S_1 follows

$$\text{rank } S_1 = n + r - 2. \tag{3.48}$$

It follows from the column construction of the matrix S_2 that its elements decreases not higher than $e^{-(n-1)\alpha\tau}$, as $\tau \to \infty$. Hence, elements of the matrix S_2^{-1} increase not faster than $e^{(n-1)\alpha\tau}$, as $\tau \to \infty$. As a result, we obtain an estimation:

$$\|S_1^{-1}\| = O(e^{(n-1)\alpha\tau}), \tau \to \infty. \tag{3.49}$$

Let us replace variables, taking into account (3.48):

$$\tilde{c}^{(2n-1)} = S_1(\tau)y, y = (y_1, \dots, y_{n+r-2})^T. \tag{3.50}$$

Finally, the system (3.45) takes the form:

$$\frac{dy}{d\tau} = S_1^{-1}\left(\bar{P}S_1 - \frac{dS_1}{d\tau}\right)y + S_1^{-1}\bar{Q}\bar{\upsilon}. \tag{3.51}$$

According to work [57], the matrix of the right-hand side of (3.51) takes a form

$$\begin{gathered} S_1^{-1}\left(\bar{P}S_1 - \frac{dS_1}{d\tau}\right) = \\ \{e_2, \dots, e_{k_1}, \varphi_{k_1}(\tau), \dots, e_{k_1+\ldots+k_{r-1}+2}, \dots, e_{k_1+\ldots+k_{r-1}}, \varphi_{k_{r-1}}(\tau)\}. \end{gathered} \tag{3.52}$$

In (3.52) $e_i = (0, \dots, 1, \dots, 0)^T_{n+r-2\times 1}$ is a matrix column in which 1 is on the i-th place.

Components of the vector $\varphi_{k_j}(\tau)$ take the form:

$$\varphi_{k_j}(\tau) = (-\varphi^1_{k_1}(\tau), \dots, -\varphi^{k_1}_{k_1}(\tau), \dots, -\varphi^1_{k_j}(\tau), \dots, -\varphi^{k_j}_{k_j}(\tau), 0, \dots, 0)^T_{n+r-2\times 1},$$

where $-\varphi^i_{k_j}(\tau)$ are coefficients of decomposition of the vector $L^j_{k_j+1}$ by vectors L^1_i, $i = 1,\ldots,k_1$; L^2_i, $i = 1,\ldots,k_2$; L^j_i, $i = 1,\ldots,k_j$, $j = 1,\ldots,r-1$, $\sum_{j=1}^{r-1} k_j = n + r - 2$.

$$L^j_{k_j+1} = -\sum_{i=1}^{k_1} \varphi^i_{k_1}(\tau)L^1_i - \ldots - \sum_{i=1}^{k_j} \varphi^i_{k_j}(\tau)L^j_i,$$
$$S_1^{-1}Q = \{e_1,\ldots,e_{k_j+1},\ldots,e_{\gamma+1}\},\ \gamma = \sum_{i=1}^{r-2} k_i. \tag{3.53}$$

Consider a stabilisation problem for a system of ordinary differential equations

$$\frac{dy_{k_j}}{d\tau} = \{e_2^{k_j},\ldots,e_{k_j}^{k_j},\bar{\varphi}_{k_j}\}y_{k_j} + e_1^{k_j}d_j,\ j = 1,\ldots,r-1, \tag{3.54}$$

where $y_{k_j} = (y^1_{k_j},\ldots,y^{k_j}_{k_j})^T_{k_j\times 1}$, $e_1^{k_j} = (0,\ldots,1,\ldots,0)^T_{k_j\times 1}$, 1 is in the i-th position, $\bar{\varphi}^i_{k_j} = (-\varphi^1_{k_j},\ldots,-\ \varphi^{k_j}_{k_j})^T_{k_i\times 1}$.

Let $y^{k_j}_{k_j} = \psi$.

Equalities

$$y^{k_j}_{k_j} = \psi,\ y^{k_j-1}_{k_j} = \psi^{(1)} + \varphi^{k_j}_{k_j}\psi,$$
$$y^{k_j-2}_{k_j} = \psi^{(2)} + \varphi^{k_j}_{k_j}\psi^{(1)} + \left(\frac{d\varphi^{k_j}_{k_j}}{d\tau} + \varphi^{k_j-1}_{k_j}\right)\psi, \tag{3.55}$$
$$y^1_{k_j} = \psi^{(k_j-1)} + r_{k_j-2}(\tau)\psi^{(k_j-2)} + \ldots + r_1(\tau)\psi^{(1)} + r_0(\tau)\psi.$$

follow from the structure of the right-hand side matrix of the system (3.54).

Let us differentiate the last equality from (3.55) and substitute the resulting expression in the first equation of the system (3.54). As a result, we obtain the following system:

$$\psi^{(k_j)} + \varepsilon_{k_j-1}(\tau)\psi^{(k_j-1)} + \ldots + \varepsilon_0(\tau)\psi = \upsilon_j,\ j = 1,\ldots,r-1. \tag{3.56}$$

Remark 3.3. *It follows from columns construting of the matrix* S_1 *and formulae* (3.53) *that functions* $\varphi^{k_j}_{k_j},\ldots,\varphi^2_{k_j},\varphi^1_{k_j}$, *them derivatives and also functions* $r_{k_j-2}(\tau),\ldots,\ r_1(\tau), r_0(\tau)$ *are limited.*

Let

$$\upsilon_j = \sum_{i=1}^{k_j}(\varepsilon_{k_j-i}(\tau) - \gamma_{k_j-i})\psi^{(k_j-i)},\ j = 1,\ldots,r-1, \tag{3.57}$$

and coefficients γ_{k_j-i} are chosen so that the roots of the characteristic equation

$$\lambda^{k_i}+\gamma_{k_i-1}\lambda^{k_i-1}+\ldots+\gamma_0=0,\, i=1,\ldots,r-1$$

satisfy the conditions

$$\lambda^i_{k_i}\neq\lambda^j_{k_i},\, i\neq j;\; \lambda^j_{k_i}<-(2n+1)\alpha-1, j=1,\ldots,k_i,\, i=1,\ldots,r-1. \qquad (3.58)$$

Returning to the original variables, we obtain

$$\upsilon_j=\delta_{k_j}T^{-1}_{k_j}S^{-1}_{1k_j}\tilde{c}^{(2n-1)},\quad j=1,\ldots,r-1, \qquad (3.59)$$

where $\delta_{k_j}=(\varepsilon_{k_j-1}(\tau)-\gamma_{k_j-1},\ldots,\varepsilon_0(\tau)-\gamma_0)$, T_{k_j} is a matrix from inequalities (3.55) such that $y_{k_j}=T_{k_j}\bar{\psi}$, $\bar{\psi}=(\psi^{k_j-1},\ldots,\psi)^T$, $S^{-1}_{1k_j}$ is a matrix, consisting of the corresponding k_j strings of the matrix S^{-1}_1.

The resulting auxiliary function (3.59) may be written in the form (3.46), where $M(\tau)=\delta_k T^{-1}_k S^{-1}_{1k}=(\delta_{k_1}T^{-1}_{k_1}S^{-1}_{1k_1},\ldots,\delta_{k_r}T^{-1}_{k_r}S^{-1}_{1k_r})^T$.

Let $\Psi(\tau)$ be a fundamental matrix of the system (3.56) that is closed by an auxiliary control function (3.57). It follows from (3.58) that $\Psi(\tau)$ is the fundamental matrix of the exponentially stable linear system of ODEs with constant coefficients. Hence,

$$\|\Psi(\tau)\Psi(t)^{-1}\|\leqslant\bar{M}e^{-\lambda(\tau-t)},\, \bar{M}>0,\; \lambda>0. \qquad (3.60)$$

Consider the system (3.45) closed by the auxiliary control function (3.59):

$$\frac{d\tilde{c}^{(2n-1)}}{d\tau}=D(\tau)\tilde{c}^{(2n-1)},\, D(\tau)=\bar{P}(\tau)+\bar{Q}(\tau)M(\tau). \qquad (3.61)$$

Let $\Phi(\tau)$ ($\Phi(0)=E$) be the fundamental matrix of the system (3.61). E is an identity matrix. Introduce a block diagonal matrix $T(\tau)$. Matrices T_{k_j}, $j=1,\ldots,r-1$ are on its diagonal. Then, equality

$$\Phi(\tau)=S_1(\tau)T(\tau)\Psi(\tau)\Psi^{-1}(0)T^{-1}(0)S^{-1}_1(0) \qquad (3.62)$$

follows from formulae (3.50) and (3.55).

Further, taking into account Remark 3.3 and formulae (3.49), (3.50), (3.55), (3.60), and (3.62), we obtain estimations

$$\begin{gathered}\|\Phi(\tau)\| \leqslant \bar{K}e^{-\lambda\tau},\ \lambda > 0,\ \bar{K} > 0,\\ \|\Phi(\tau)\Phi^{-1}(t)\| \leqslant \bar{K}_1 e^{-\lambda(\tau-t)}e^{(n-2)\alpha t}, \tau \geqslant t,\ \bar{K}_1 > 0,\\ \|M(\tau)\| = O(e^{(n-1)\alpha\tau}),\ \tau \to \infty.\end{gathered} \tag{3.63}$$

It is possible to take the value $\varepsilon_2 = \tilde{\varepsilon}$ as ε_2 from the formulation of the lemma.

The lemma is proved.

3.5 Proof of the theorem

Consider the system (3.43) that is closed by the auxiliary control function (3.46):

$$\begin{aligned}\frac{d\tilde{c}^{(2n-1)}}{d\tau} = D(\tau)\tilde{c}^{(2n-1)} + \bar{R}_1(\tau, x_1^0, \tilde{c}^{(2n-1)}) + \bar{R}_2(\tau, x_1^0, \tilde{c}^{(2n-1)}) + \\ + \bar{R}_3(\tau, x_1^0, \tilde{c}^{(2n-1)}) + \bar{R}_4(\tau, x_1^0, \tilde{c}^{(2n-1)}).\end{aligned} \tag{3.64}$$

Let us replace variables in the system (3.64) using formulae:

$$\tilde{c}^{(2n-1)} = z(\tau)e^{-(n-1)\alpha\tau},\ \tilde{c}^{(2n-1)}(0) = z(0). \tag{3.65}$$

As a result, we obtain

$$\begin{gathered}\frac{dz}{d\tau} = C(\tau)z + e^{(n-1)\alpha\tau}(\bar{R}_1(\tau, x_1^0, ze^{-(n-1)\alpha\tau}) + \\ + \bar{R}_2(\tau, x_1^0, ze^{-(n-1)\alpha\tau}) + \bar{R}_3(\tau, x_1^0, ze^{-(n-1)\alpha\tau}) + \\ + \bar{R}_4(\tau, x_1^0, ze^{-(n-1)\alpha\tau})),\ C(\tau) = D(\tau) + (n-1)\alpha E.\end{gathered} \tag{3.66}$$

Let us show that all solutions of the system (3.66) with initial conditions (3.65) that are beginning in the small neighbourhood of zero decrease exponentially.

Let $\Phi_1(\tau)$, $\Phi_1(0) = E$ is a fundamental matrix of the system $\frac{dz}{d\tau} = C(\tau)z$. Then according to (3.63), (3.65):

$$\begin{aligned}\|\Phi_1(\tau)\| \leqslant \bar{K}e^{-\beta\tau},\ \|\Phi_1(\tau)\Phi_1^{-1}(t)\| \leqslant K_1 e^{-\beta(\tau-t)}e^{(n-2)\alpha t},\\ \beta = \lambda - (n-1)\alpha,\ \tau \geqslant t.\end{aligned} \tag{3.67}$$

Let us choose a value of the coefficient α so that the condition

$$\beta > 0 \tag{3.68}$$

is satisfied.

The solution of the system (3.64) with the initial conditions (3.33), (3.44), and (3.65) may be written as follows:

$$\begin{gathered} z(\tau) = \Phi_1(\tau)\Phi_1^{-1}(\tau_1)z(\tau_1) + \int\limits_{\tau_1}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{(n-1)\alpha t}\times \\ (\bar{R}_1(t, x_1^0, ze^{-(n-1)\alpha t}) + \bar{R}_2(t, x_1^0, ze^{-(n-1)\alpha t})+ \\ \bar{R}_3(t, x_1^0, ze^{-(n-1)\alpha t}) + \bar{R}_4(t, x_1^0, ze^{-(n-1)\alpha t}))dt,\ \tau \in [\tau_1,\infty); \end{gathered} \tag{3.69}$$

$$\begin{gathered} z(\tau) = \Phi_1(\tau)\tilde{c}^{(2n-1)}(0) + \int\limits_{0}^{\tau} \Phi_1(\tau)\Phi_1^{-1}(t)e^{(n-1)\alpha t}\times \\ (\bar{R}_1(t, x_1^0, ze^{-(n-1)\alpha t}) + \bar{R}_2(t, x_1^0, ze^{-(n-1)\alpha t})+ \\ \bar{R}_3(t, x_1^0, ze^{-(n-1)\alpha t}) + \bar{R}_4(t, x_1^0, ze^{-(n-1)\alpha t}))dt, \tau \in [0,\tau_1]. \end{gathered} \tag{3.70}$$

Using (3.69), (3.70), and taking into account (3.37) - (3.40), (3.67), (3.68) in the domain (3.6), and (3.36), we obtain estimations:

$$\begin{gathered} \| z(\tau) \| \leqslant \bar{K}e^{-\beta\tau} \| \Phi_1^{-1}(\tau_1)z(\tau_1) \| + \\ \int\limits_{\tau_1}^{\tau} K_1 e^{-\beta(\tau-t)}(Le^{-\alpha t}\|z\| + L_4(x_1^0)e^{-n\alpha t})dt,\ \tau \in [\tau_1,\infty), \end{gathered} \tag{3.71}$$

$$\begin{gathered} \| z(\tau) \| \leqslant \bar{K}e^{-\beta\tau} \| \tilde{c}^{(2n-1)}(0) \| + \\ \int\limits_{0}^{\tau} K_1 e^{-\beta(\tau-t)}(Le^{-\alpha t}\|z\| + L_4(x_1^0)e^{-n\alpha t})dt,\ \tau \in [0, \tau_1], \end{gathered} \tag{3.72}$$

where $L > 0$ is a constant that depends on the domain (3.6), (3.36).

Applying the well-known result (see [58]) in the domain (3.6), (3.36) to formulae (3.71), (3.72), we will obtain inequalities

$$\|z(\tau)\| \leqslant \bar{K}e^{-\mu\tau}\|\Phi_1^{-1}(\tau_1)z(\tau_1)\| + \int\limits_{\tau_1}^{\tau} K_1 e^{-\mu(\tau-t)}L_4(x_1^0)e^{-n\alpha t}dt, \qquad \tau \in [\tau_1,\infty), \tag{3.73}$$

where $\mu = \beta - K_1 L e^{-\alpha\tau_1}$,

$$\|z(\tau)\| \leqslant \bar{K}e^{-\mu_1\tau}\|\tilde{c}^{(2n-1)}(0)\| + \int\limits_{0}^{\tau} K_1 e^{-\mu_1(\tau-t)}L_4(x_1^0)e^{-n\alpha t})dt, \qquad \tau \in [0,\tau_1], \tag{3.74}$$

$\mu_1 = \beta - K_1 L$.

Let us choose a value $\tau_1 > 0$ so that the condition $\mu > 0$ is fulfilled.

We obtain the following formula after computing the integrals in the right-hand parts of (3.73), (3.74):

$$\begin{gathered} \|z(\tau)\| \leqslant \bar{K}e^{-\mu\tau}\|\Phi_1^{-1}(\tau_1)\|\|z(\tau_1)\| + K_2 e^{-\alpha\tau}L_4(x_1^0), \tau \in [\tau_1,\infty), \\ \|z(\tau)\| \leqslant K_3\|\tilde{c}^{(2n-1)}(0)\| + K_4 L_4(x_1^0), \tau \in [0,\tau_1]. \end{gathered} \tag{3.75}$$

Here, coefficients $\bar{K}$, K_1, K_2, K_3 and K_4 are strictly positive.

Estimations (3.75) may be written as a single inequality:

$$\|z(\tau)\| \leqslant K_5 e^{-\alpha\tau}\|\tilde{c}^{(2n-1)}(0)\| \tag{3.76}$$

It follows from (3.65), (3.76) that for all $\tilde{c}^{(2n-1)}(0)$, which belong to the domain

$$\|\tilde{c}^{(2n-1)}(0)\| < \frac{C_1}{K_5}, \tag{3.77}$$

a solution of the system (3.64) does not leave domain (3.36) and exponentially decreases.

Using condition (3.33), we will find $\varepsilon > 0$ such that $\forall x_0$, satisfying the inequality $\|x_0\| < \varepsilon$, the condition (3.77) is satisfied.

A substitution of the obtained function $z(\tau)$ into the formula (3.65) gives the known function $\tilde{c}^{(2n-1)}(\tau) = (\bar{c}^{(2n-1)}(\tau), d(\tau))$. After substituting $\tilde{c}^{(2n-1)}$ into the formula (3.46), we obtain the known function $\bar{\upsilon}(\tau)$. Further, using $\tilde{c}^{(2n-1)}$, with help of formulae (3.30), (3.26), and (3.21), we will have the pair function $\bar{c}(\tau)$, $\bar{d}(\tau)$ which is a solution of the problem (3.16), (3.18).

The found functions $\bar{x}(t) = \bar{c}(\tau(t))$, $\bar{u}(t) = \bar{d}(\tau(t))$ and $\tilde{\upsilon}(t) = \bar{\upsilon}(\tau(t))$ satisfy the system

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(x_1(t), \bar{x}, u_1(x_1(t), \bar{x}, \bar{u}), \bar{u}), \\ \dot{\bar{u}} &= \alpha^{-1}(1-t)^{-1}\tilde{\upsilon}(t) \end{aligned} \tag{3.78}$$

and the initial conditions

$$\bar{x}(0) = \bar{x}_0, \bar{u}(0) = \bar{\bar{0}}. \tag{3.79}$$

Remark 3.4. *It follows from the theorem proof that after minor changes to the condition* (3.5) *and the proof, its statement will be correct, if we substitute a function* $x_i(t) = p_i(t)$, $i = 1, \ldots, n$ *instead of the known function* $x_0(t)$ *specified by the formula* (3.8). *Here,* $p_i(t)$ *is a polynomial of degree* $n \geqslant 1$ *satisfying the boundary condition* (3.7).

The theorem has been proved.

3.6 Algorithm description

The algorithm for the original problem solution consists of the following stages:

1. finding the function u_1 from the condition (3.9); as a result, we obtain the function
   $u_1(x_1(t), x_2, \ldots, x_n, u_2, \ldots, u_r)$;
2. construction of the system (3.11);
3. construction the auxiliary system (3.43);
4. solution of the stabilisation problem of the system (3.45); as a result, we obtain the auxiliary control function of the form (3.46) in symbolic form;
5. solution of the Cauchy problem for the system (3.43) with initial conditions (3.44) closed by the auxiliary control, obtained at stage 4; as a result, we obtain the known functions $\bar{c}^{(2n-1)}(\tau)$,$\bar{d}(\tau)$;
6. substitution of the functions $\bar{c}^{(2n-1)}(\tau)$,$\bar{d}(\tau)$ into the formula (3.46) gives the known function $\upsilon(\tau)$;

7. transition to the independent variable t by the formula (3.15) in the function $\bar{\upsilon}(\tau)$ into the function $\bar{\upsilon}(\tau)$; as a result, we will obtain $\tilde{\upsilon}(t) = \bar{\upsilon}(\tau(t))$;
8. solution of the Cauchy problem (3.78) with initial data (3.79); finally, we obtain the known functions $\bar{x}(t)$, $\bar{u}(t)$;
9. substitution of the found solutions $\bar{x}(t)$, $\bar{u}(t)$ into the function $u_1(x_1(t), \bar{x}(t), \bar{u}(t))$ and its calculation.

Implementation of the stages from the first to the fourth and also the sixth, seventh, and ninth may be carried out by analytical methods with the help of symbolic calculation software. E.g., it is possible to apply the Python Sympy library [62] or the computer algebra software Wolfram Mathematica [71].

The fifth and eighth stages are being performed by numerical solution methods for systems of ODEs. For example, one of the Runge-Kutta methods may be applied.

The computational cost of the algorithm is

$$O(n^4) + O\left(\frac{C_2(n) \cdot \tau_f}{h}\right) + O\left(\frac{C_2(n)}{h}\right), \tag{3.80}$$

where $O(n^4)$ is a complexity of symbolic operations; $O(\frac{C_2(n) \cdot \tau_f}{h})$ is a complexity cost of the fifth stage of the algorithm; $O(\frac{C_2(n)}{h})$ is a complexity cost of the eighth stage; $C_2(n)$ a function characterising the number of necessary operations for the calculation of the integrated function; τ_f is a finite value of the time interval; h is an integration step of the Cauchy problem solution. To determine the computational cost, the Runge-Kutta method of the 4th degree with a constant integration step has been taken.

3.7 Solution of the interorbital flight problem

Consider the transition problem of the material point moving along a circular orbit with constant speed into the central gravitational field to the given point, lying in the plane of the orbit by a reactive force.

In accordance with [72], the equations system relating to the moving by circular orbit has the following form:

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 = \nu_1(x_1, x_4) + u_1, \\ &\dot{x}_3 = x_4, \\ &\dot{x}_4 = \nu_2(x_1, x_2, x_4) + \nu_3(x_1)u_2 \ , \end{aligned} \tag{3.81}$$

where $x_1 = r - r_0$, $x_2 = \dot{r}$, $x_3 = \psi - \alpha_0 t$, $x_4 = \dot{\psi} - \alpha_0$, $u_1 = a_\gamma \dot{m}/m$, $u_2 = a_\psi \dot{m}/m$; r_0 is a radius of the circular ofbit, $\dot{r}$ is a radial velocity; ψ is a polar angle, $\dot{\psi}$ is an angular velocity; a_γ, a_ψ is a projection of the relative speed vector of the particle on radius and transverse directions, respectively, $m, \dot{m}$ is a mass and speed of the mass-changing; α_0 is an angular velocity of the moving by the given cicular orbit, $\nu_1 = -\frac{\nu}{(x_1+r_0)^2} + (x_1 + r_0)(x_4 + \alpha_0)^2$, $\nu_2 = -2\frac{x_2(x_4+\alpha_0)}{(x_1+r_0)}$, $\nu_3 = \frac{1}{x_1+r_0}$,

where $\nu = G \cdot M$; $G = 6.6743 \times 10^{-11}$ м^3 kg^{-1} c^{-2} is a gravitational factor, $M = 5.972{\times}10^{24}$ kg is an Earth mass, $\alpha_0 = \sqrt{\frac{\nu}{r_0^3}}$, $r_0 = 7{\cdot}10^6$ m, $x = (x_1, x_2, x_3, x_4)^T$, $u = (0, u_1, 0, u_2)^T$.

Problem 3.2. Find functions $x(t)$, $u(t)$ satisfying the system (3.81) and the conditions:

$$x_1(0) = 100, x_2(0) = 0.2, x_3(0) = -\alpha_0 \cdot 10^{-6}, x_4(0) = 10^{-5}, x(1) = 0 \tag{3.82}$$

$$x_4(t) = (3x_4^0 + 2x_3^0) \cdot t^2 - (4x_4^0 + 6x_3^0) \cdot t + x_4^0. \tag{3.83}$$

Initial conditions and values of parameters are given according to [37].

The solution to the control problem has been conducted with the help of the algorithm described in the previous section, using the value $\alpha = 0.25$.

Let us find the control u_1. Consider the system consisting of the first two equations of (3.81):

$$\begin{gathered} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{\nu}{(x_1+r_0)^2} + ((x_4^0(1-t))^2 + 2\alpha_0 x_4^0(1-t) + \alpha_0^2)x_1 + \\ + r_0(x_4^0(1-t))^2 + 2\alpha_0 r_0 x_4^0(1-t) + \alpha_0^2 r_0 + u_1, \end{gathered}$$

We do the transition to the new independent variable using the formula (3.15):

$$
\begin{aligned}
&\frac{d}{d\tau}c_1 = \alpha e^{-\alpha\tau}c_2,\\
&\frac{d}{d\tau}c_2 = \alpha e^{-\alpha\tau}\Bigg(-\frac{\nu}{(c_1+r_0)^2} + ((x_4^0 e^{-\alpha\tau})^2 + 2\alpha_0 x_4^0 e^{-\alpha\tau} + \alpha_0^2)c_1 + \\
&\qquad\qquad + r_0(x_4^0 e^{-\alpha\tau})^2 + 2\alpha_0 r_0 x_4^0 e^{-\alpha\tau} + \alpha_0^2 r_0 + d_1\Bigg),
\end{aligned} \tag{3.84}
$$

where $c_1(\tau) = x_1(t(\tau))$, $c_2(\tau) = x_2(t(\tau))$,

$$c_1(0) = \bar{x}_1,\ c_2(0) = \bar{x}_2,\ d_1(0) = 0,\ c_i(\tau) \to 0.$$

Let us perform shift transformations and add the auxiliary control function $\upsilon(\tau)$ to the formula (3.84). As a result, we have a system:

$$
\begin{gathered}
\tfrac{d}{d\tau}c_1 = \alpha e^{-\alpha\tau}c_2,\\
\tfrac{d}{d\tau}c_2 = \alpha e^{-\alpha\tau}\Bigg(-\tfrac{\nu}{(c_1+r_0)^2} + ((x_4^0 e^{-\alpha\tau})^2 + 2\alpha_0 x_4^0 e^{-\alpha\tau} + \alpha_0^2)c_1 + + r_0(x_4^0 e^{-\alpha\tau})^2 + \\
2\alpha_0 r_0 x_4^0 e^{-\alpha\tau} + \alpha_0^2 r_0 + d_1\Bigg), \tfrac{d}{d\tau}d_1 = \upsilon(\tau).
\end{gathered}
$$

Finally, we obtain

$$
\begin{aligned}
u_1 = &\left[2.16\cdot 10^{-8}\cdot t - 3.51\cdot 10^{-6} - \frac{4.0}{(1-t)^2} - \frac{78.0}{\alpha(1-t)^2} - \right.\\
&\left. - \frac{380}{\alpha^2(1-t)^2}\right]\cdot x_1 - \frac{3(\alpha+13)}{\alpha(1-t)}\cdot x_2
\end{aligned} \tag{3.85}
$$

We obtain a control u_2 from the fourth equation of the system (3.81), taking into account the given function (3.83):

$$u_2 = \frac{((6\cdot x_4^0 + 4\cdot x_3^0)\cdot t - (4\cdot x_4^0 + 6\cdot x_3^0) - \nu_2(x_1, x_2, x_4))}{\nu_3(x_1)} \tag{3.86}$$

We find a numerical solution of the Cauchy problem for the given system after substitition of control functions (3.85) and (3.86) into the formula (3.81).

The calculation results are presented in Figs. 3.1—3.4. All computations and graphs were accomplished in the Jupiter Notebook software.

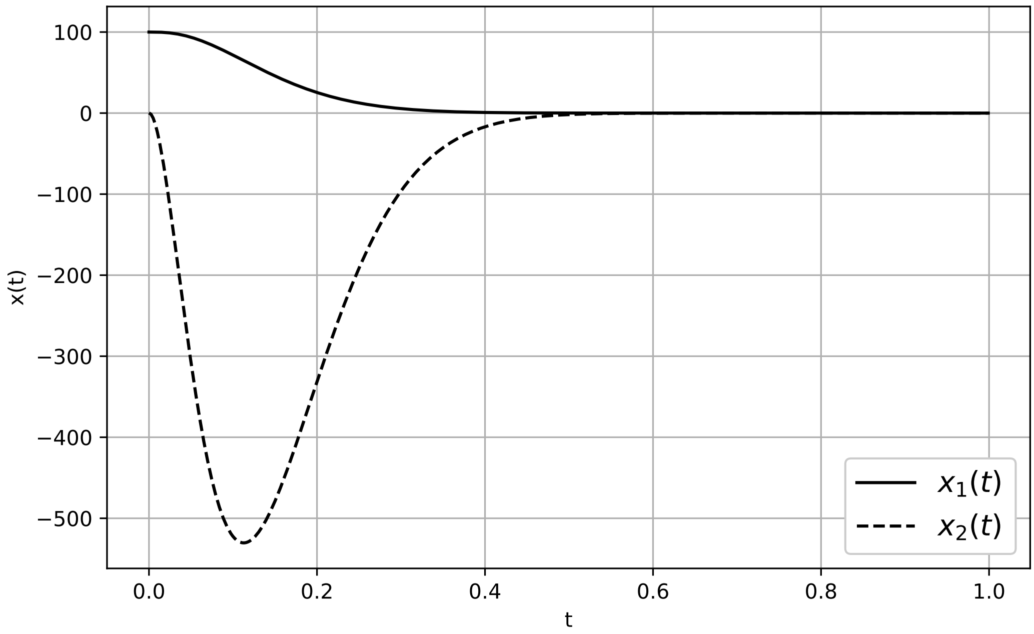

Figure 3.1 — Graphs of the functions $x_1(t)$, $x_2(t)$

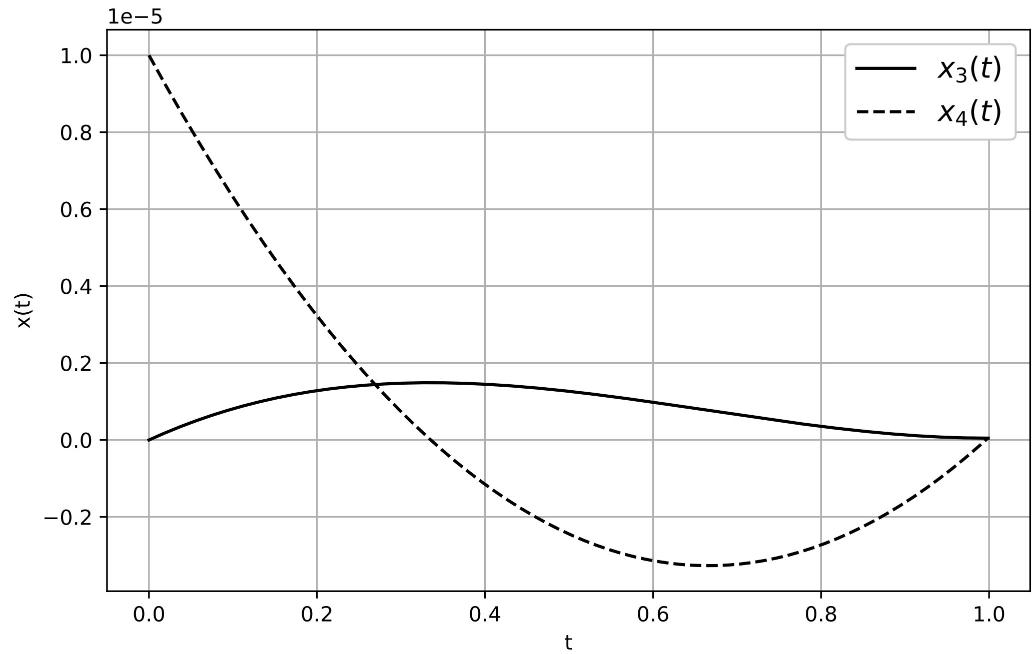

Figure 3.2 — Graphs of the functions $x_3(t)$, $x_4(t)$

Analysis of the numerical simulation results allows us to draw the following conclusions:

1. It follows from the analysis of graphs (Figs. 3.1—3.4) that the obtained functions of the state vector correspond to the conditions (3.82), (3.83), and (3.7).

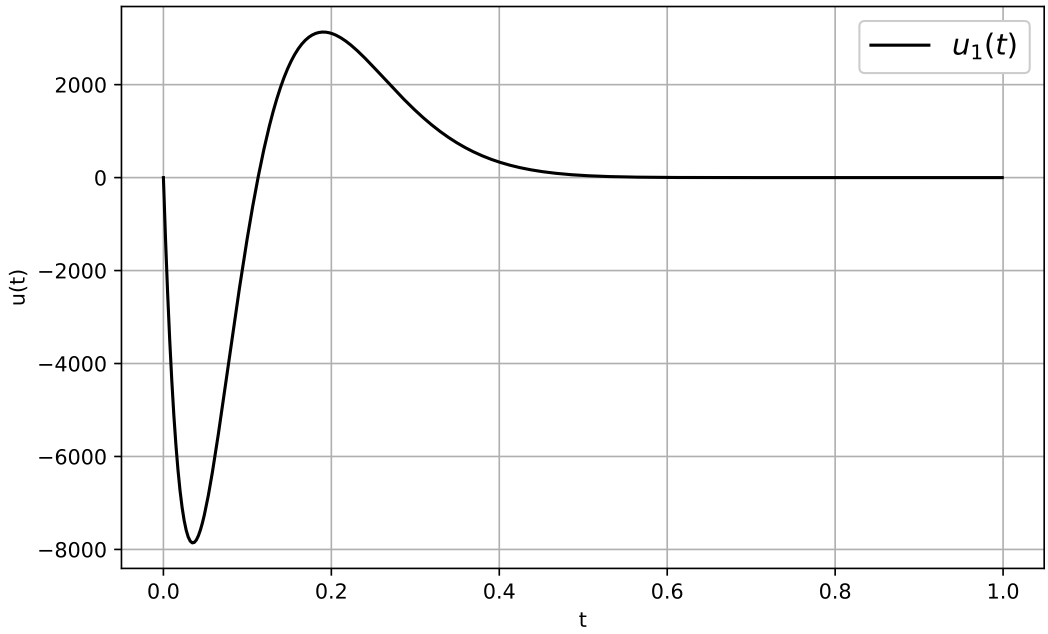

Figure 3.3 — Graph of the function $u_1(t)$

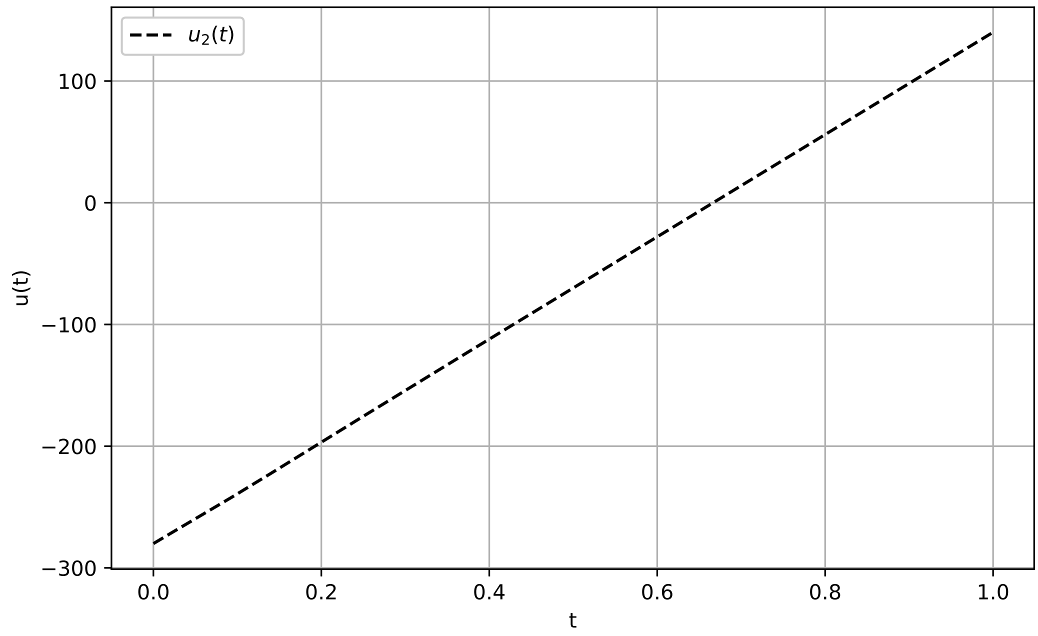

Figure 3.4 — Graph of the function $u_2(t)$

2. It follows from the graphs that the norm of the control vector is $||u||_\infty = 7858.86$.
3. The efficiency of the constructed algorithm is shown.

3.8 Conclusions for the third chapter

The proposed method may be applied to solve the concrete problems of computer system verification.

It follows from the formula (3.80) that the computational cost of the algorithm depends on the integration step of the solution of the Cauchy problem for auxiliary and original equation systems.

Solutions for problems of small dimension are possible on the personal computer with middle parameters, and also in the Google Colab software.

For the considered example, the polynomial (3.83) with a degree higher than in the condition (3.8) has been given, which allows us to conclude the generalisation possibility of the Problem 3.1 solution.

Chapter 4. Solution of the control problem for Josephson junction arrays

4.1 Introduction

A Josephson junction is a possible way to construct a quantum bit (qubit) [73]. A Josephson qubit can be presented as a non-linear resonator. There are three types of superconducting qubits that are distinguished by how the non-linear resonator is constructed, namely *phase*, *flux*, and *charge* qubits. Phase and flux qubits are sensitive to the phase value of the Josephson current [74; 75].

To perform quantum computations, it is necessary to create a chain or an array of qubits. The dynamics of Josephson junction arrays were studied in detail in a significant number of works, e.g., [76—78]. The dynamics of an identical non-linear oscillator network was studied in [79].

Control of one Josephson junction was investigated in [80].

The Josephson junction array may be presented as a multidimensional controlled system with periodic non-linearities. The dynamics of such systems were considered in [81].

In [78], the authors show how the phases depend on the global variables of an electrical circuit: the phases are not stable and grow up with time.

In the current work, an approach to stabilisation of the Josephson junction array phases is suggested. It is based on the solution of the optimal control problem for the Josephson junction array in terms of global variables.

The results of this chapter were published in the paper [24].

4.2 Models description

4.2.1 Identical Josephson junction array model

The identical junctions can be described by the system of ordinary differential equations in dimensionless global variables [77; 78]:

$$\begin{gathered}\dot{x}_i = I - \sin x_i - \varepsilon x_{N+2}, \dot{x}_{N+1} = x_{N+2}, \\ \dot{x}_{N+2} = I - \gamma x_{N+2} - \omega_0^2 x_{N+1} - \frac{1}{N}\sum_{i=1}^{N} \sin x_i\end{gathered} \tag{4.1}$$

where x_i is the phase of the i-th junction; N is the total number of junctions; x_{N+1} is the load capacitor charge; I is the external current; ε, γ and ω_0^2 are dimensionless parameters of the parallel RLC-load. Initial conditions for the ODE system are $x_0 = (0,\ldots,0,0.5,0)^T$.

Initial conditions for the ODE's system are

$$x_0 = (0,\ldots,0,0.5,0)^T. \tag{4.2}$$

An equivalent circuit for the identical Josephson junction array with an RLC-load connected in parallel is presented in Fig. 4.1.

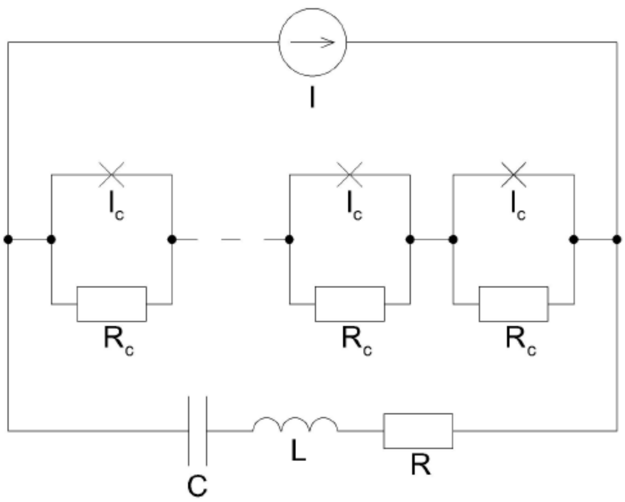

Figure 4.1 — Equivalent circuit for the identical Josephson junction array with the common RLC load [82]

The numerical simulation for the solution of equation (4.1) given in Fig. 4.2. The simulation was made for a 200-junction array, following [78].

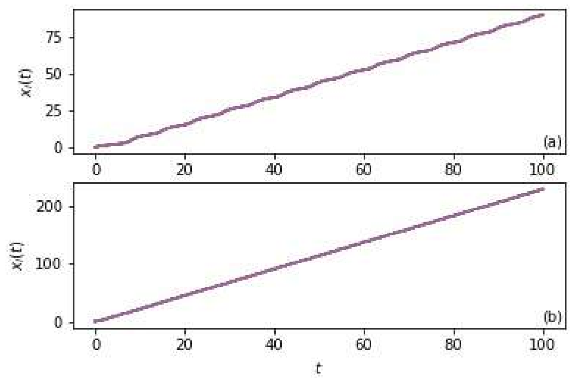

Figure 4.2 — Identical Josephson junction array simulation results. The values of the parameters are (a) $I = 1.2$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$; (b) $I = 2.5$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$.

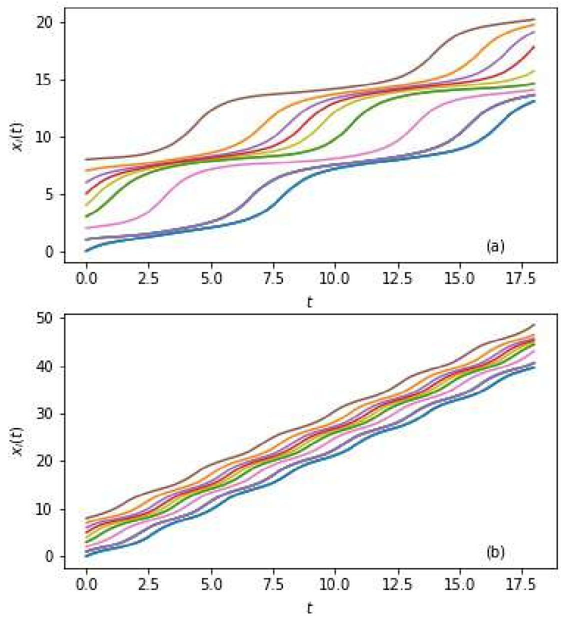

Figure 4.3 — Identical Josephson junctions array simulation results for different initial values of the phases. 15 phases of 200 are shown. The values of the parameters are (a) $I = 1.2$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$; (b) $I = 2.5$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$.

From simulation, it follows that identical junctions are in a synchronous state.

Also, the simulation for different initial values of phases was carried out. The initial values of phases are in the range $[0; 10]$. The simulation results are presented in Fig. 4.3. As follows from the graphs, Josephson junctions are in an asynchronous state.

4.2.2 Non-identical Josephson junction array model

The ODE system describing non-identical junctions is [77; 78]:

$$\dot{x}_i = I - (1 + \xi_i)\sin x_i - \varepsilon x_{N+2},\ \dot{x}_{N+1} = x_{N+2},$$
$$\dot{x}_{N+2} = I - \gamma x_{N+2} - \omega_0^2 x_{N+1} - \frac{1}{N}\sum_{i=1}^{N}(1 + \xi_i)\sin x_i \tag{4.3}$$

where the values have the same meaning as in (4.1) and ξ_i are the parameters characterizing the difference of the critical currents from the nominal value.

An equivalent circuit for the non-identical Josephson junction array with a parallel RLC-load is presented in Fig. 4.4.

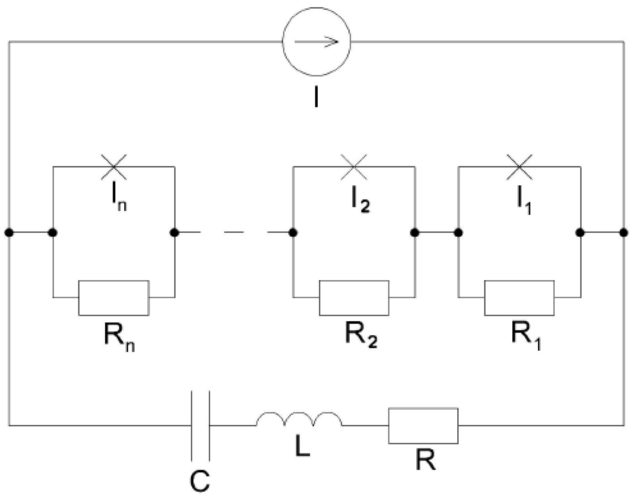

Figure 4.4 — Equivalent circuit for the non-identical Josephson junction array with the common RLC load [82]

The simulation of (4.3) is presented in Fig. 4.5. The simulation was done for an array with 200 junctions, according to [78].

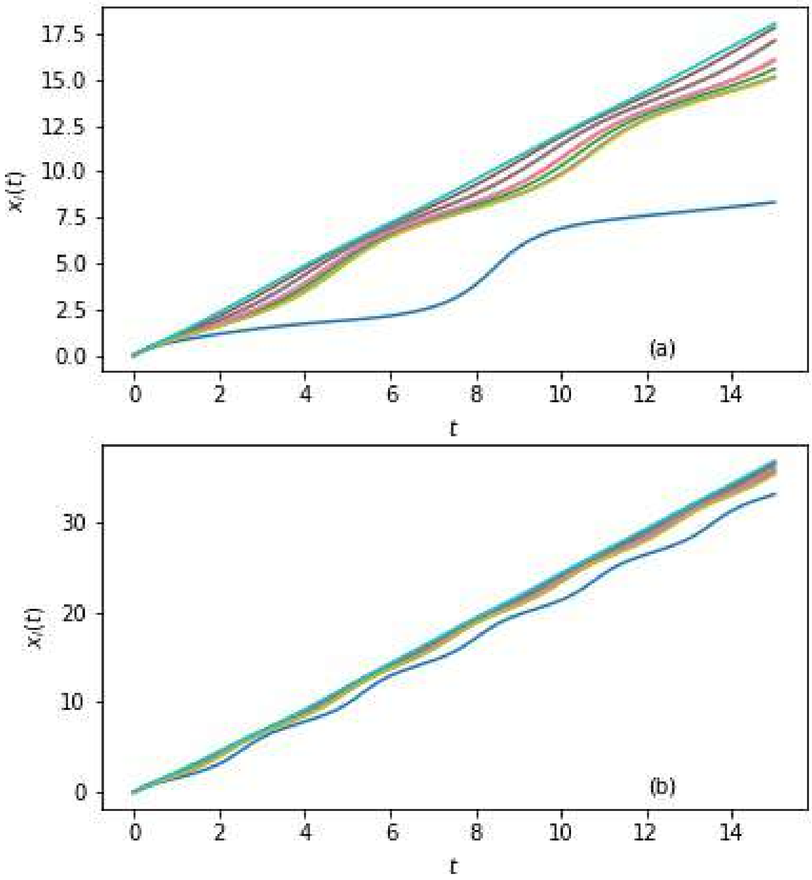

Figure 4.5 — Non-identical Josephson junction array simulation results. 10 phases of 200 are shown. The values of the parameters are (a) $I = 1.2$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, $\gamma = 1$, $\xi_i \in [-1; 1]$; (b) $I = 2.5$, $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, $\gamma = 1$, $\xi_i \in [-1; 1]$.

4.3 Problem statement and its solution

Let us consider systems (4.1) and (4.3) in general form and expand them with a control function:

$$\dot{x} = f(x) + Bu + \bar{I} = F(x,u) + \bar{I}, \tag{4.4}$$

where $x = (x_1, \ldots, x_n)^T$, $x \in R^n$,
$u = (u_1, \ldots, u_r)^T$, $u \in R^r$, $r \leqslant n$,
$f = (f_1, \ldots, f_n)^T$, $f \in C^\infty(R^n \times R^r; R^n)$;
$\bar{I} = (I, \ldots, I, 0, I)^T$, $F(0,0) = 0$,

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.1. Find a pair of the functions $x(t)$, $u(t)$, which satisfy the system (4.4) and given initial conditions:

$$x_0 = (x_0^1, \dots, x_0^n). \tag{4.5}$$

Here, $u(t)$ is the optimal control for the linearized system (4.4).

Problem solution. Let us consider ODE systems (1) and (2). These systems contain significant non-linearity. Thus, for the start, we make a Taylor expansion and keep only linear terms. The linearized systems have a general form:

$$\dot{x} = Ax + Bu, \tag{4.6}$$

where A is the matrix of the first approximation summands, which has the form

$$A = \begin{pmatrix} -(1+\xi_1) & \dots & 0 & 0 & -\varepsilon \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -(1+\xi_N) & 0 & -\varepsilon \\ 0 & \dots & 0 & 0 & 1 \\ -\frac{1+\xi_1}{N} & \dots & -\frac{1+\xi_N}{N} & -\omega_0^2 & -\gamma \end{pmatrix} \tag{4.7}$$

The case $\xi_i = 0$, $i = 1,...,N$ corresponds to an identical junction array.

A test of the controllability conditions for these systems shows that

$$rank S = n, \tag{4.8}$$

where $S = (B, AB, \dots, A^{n-1}B)$, n is a dimension of the system.

Now we study a stationary Linear-Quadratic problem (LQ-problem) for (4.6), following [64].

LQ-problem. Find a pair of the functions $x(t)$, $u(t)$, which satisfy the system (4.6) over the infinite time interval and minimise the quality criterion

$$J(u) = \int_0^\infty [x^T(s)N_2x(s) + u^T(s)N_3u(s)]ds, N_3 > 0, N_2 > 0, \tag{4.9}$$

where matrices N_2 of dimension $[n \times n]$ and N_3 of dimension $[r \times r]$ are positively defined.

The possibility of the LQ-problem solution follows from condition (4.8).

The required control function $u(t)$ that satisfies (4.9) is defined as [64]:

$$u(x) = -N_3^{-1} B^T P x, \tag{4.10}$$

where matrix $P > 0$ solves the algebraic Riccati equation

$$A^T P + PA + N_2 - PBN_3^{-1} B^T P = 0, \tag{4.11}$$

The Riccati equation (4.11) is solved with MATLAB Control Toolbox. We substitute the solution of the Ricatti equation P into (4.10) to obtain the control functions. At the last step, we substitute the obtained control functions into the initial non-linear systems.

4.4 Numerical simulation results and analysis

Simulation was done for controlled systems of identical and non-identical Josephson junction arrays.

Initial conditions (4.5) for ODE systems are $x_0 = (0, \dots, 0, 0.5, 0)^T$. We considered two values of the external current, namely $I = 1.2$ and $I = 2.5$. The values of parameters ξ_i for the non-identical Josephson junction array vary in the range [−1,1]. Parameters of the RLC-load are $\varepsilon = 0.5$, and $\gamma = 1$, $\omega_0^2 = 1.2$ for both cases. The parameters are taken from [77; 78]..We have chosen matrices N_2 and N_3 to be identity matrices of the corresponding sizes. Simulation results are presented in Figs. 4.6, 4.7, 4.8.

In case of different initial phase values for identical Josephson junction array, these values are chosen in the range $[0, 10]$. To determine the required external current value, numerical experiments were carried out. Results are presented in Fig. 4.7.

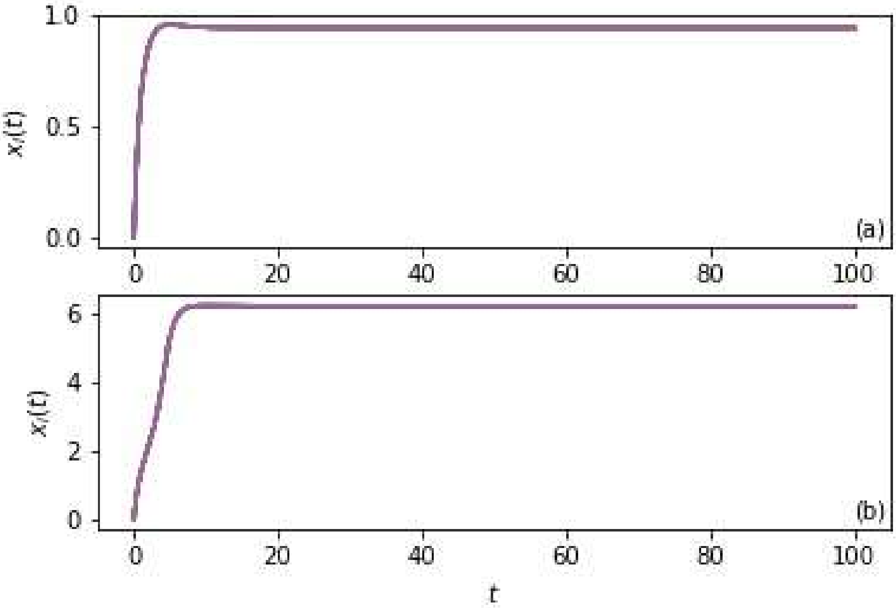

Figure 4.6 — The simulation results for the solution of the control problem for an Identical Josephson junctions array. The values of the parameters are $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, $\gamma = 1$. (a) $I = 1.2$. (b) $I = 2.5$.

Analysis of the simulation results

1. The solution to the control problem ensures stabilisation of the phase values for identical and non-identical junction arrays.
2. The dependence of the phase values on external current is detected.
3. In the case of identical junctions with different initial values of the phases the synchronisation effect is detected (see Fig. 4.8). When external current $I < 1.738$ phases tend to two different values. Phases tend to be in a one synchronous-state with an external current $I \geqslant 1.738$.

A numerical simulation was performed in Jupyter Notebook.

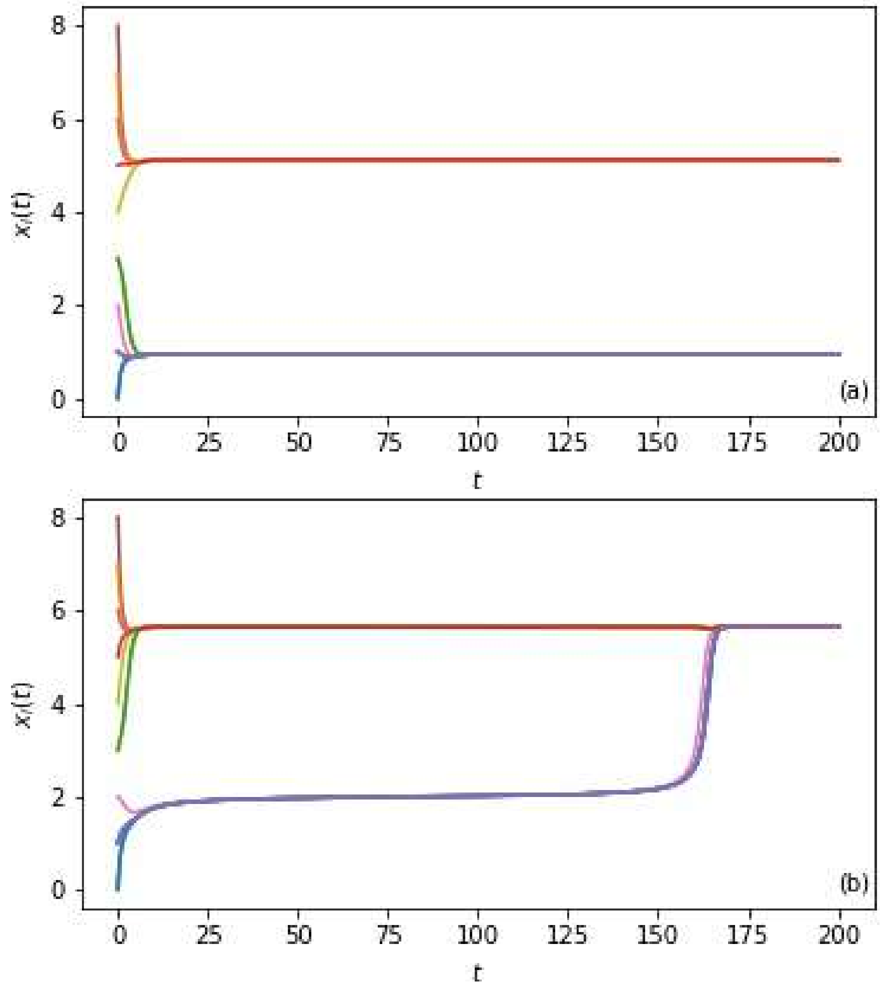

Figure 4.7 — The simulation results for the solution of the control problem for an identical Josephson junction array with different initial values of the phases. The first 15 junctions of 200 are shown. The values of the parameters are $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$. (a) $I = 1.2$(b) $I = 1.738$

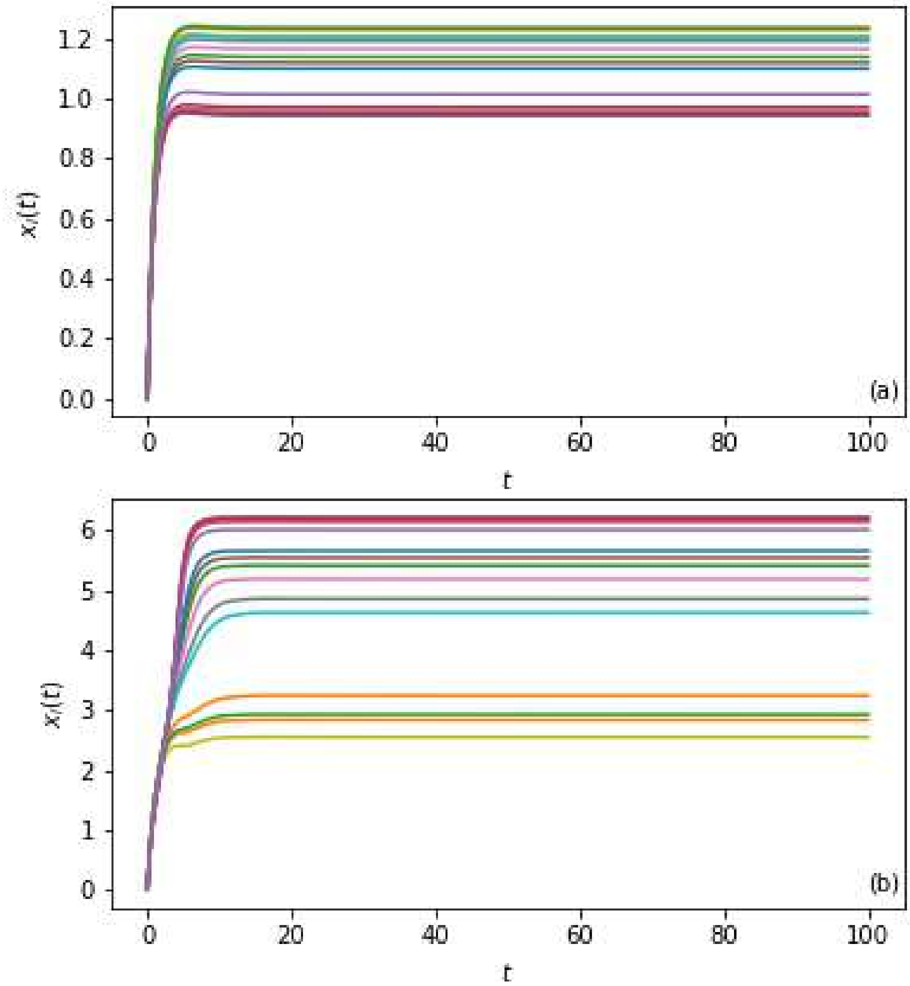

Figure 4.8 — The simulation results for the solution of the control problem for non-identical Josephson junctions array. The first 15 junctions of 200 are shown. The values of the parameters are $\varepsilon = 0.5$, $\omega_0^2 = 1.2$, and $\gamma = 1$, $\xi_i \in [-1; 1]$. (a) $I = 1.2$. (b) $I = 2.5$.

Conclusion

In the presented work, the following control algorithms for non-linear systems have been constructed and studied:

1. the algorithm of discrete control construction for non-linear non-starionary systems;
2. the algorithm of control construction for non-linear starionary systems, taking into account computer system verification.

The efficiency of the constructed algorithms has been shown in the concrete examples, and their complexity cost have been determined. These algorithms allow for the construction of control functions for a wide class of non-linear systems. For the discrete control algorithm, a library of functions has been developed that simplifies the simulation process.

Besides, the dynamics of systems describing identical and non-identical Josephson junction arrays, taking into account the optimal control method, have been studied. Josephson junctions are applied in the development of different technical systems. The solution to this control problem may be useful for solving the problems.

In conclusion, we discuss the further development of the presented research.

Except for the suggested discrete control method, it is possible to consider piecewise control problems for non-linear systems, taking into account incomplete information or the controllability of the system. Also, it is possible to consider discrete control problems for systems with a delay.

The verification problem of computer systems has been considered in the case of one known component of the state vector. Further, may be performed a generalization on the larger number of given components of the state vector. And also, it is possible to consider the more complex functions of the given components of the state vector. Besides, it is interesting to obtain a solution to the discrete control problem, taking into account computer system verification.

It is possible to solve the identical and non-identical Josephson junction arrays optimal control problem taking into account the incomplete measurability of the system. And also, it is interesting to obtain estimations of the maximal number of junctions for which the control problem solutions take place.

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Appendix A

Program code for solving of the discrete control problem of a robot-manipulator

```
import sympy as sp
import math as mt
import numpy as np
import matplotlib.pyplot as plt
from sympy import *
import DiscrControlLib as ds


#Function of the graph plotting
def PlotFig(y,t,m,label_y):
  plt.figure(figsize=(16, 5))
  plt.subplot(121)
  st = label_y
  label=['$x_1(t)$, рад','$x_2(t)$, рад/с']
  if m > 1:
    for n in range(0,m):
      r = [y[i][n] for i in range(0,len(y))]
      plt.plot(t,r,label=label[n])
      plt.legend(loc='best', fontsize=12)

  else:
    plt.plot(t,y,label=st)
  plt.xlabel('t')
  plt.ylabel(st)
  plt.grid()
  plt.subplot(122)
  sg = '* k'
  s = len(t)
```

```
  plt.plot(t[:s],y[:s,2], sg)
  plt.grid()
  plt.xlabel('t')
  plt.ylabel('$u(t)$, $рад/c^2$')
  plt.show()


#variables

u = sp.var('u:5')
c = sp.var('c:5')
d = sp.var('d:5')
a = var('a')
sp.var('alpha')
t, tau = symbols('t tau')
y = sp.var('y:3')

#Values of the parameters
a = 0.25
x_0 = 0.05
a1 = 0.1
q = 0.01
L = 10
M = 20
m_0 = 1
g = 9.81

m = m_0 - q*t
m_1 = m + M/3
a_1 = a1/(L**2*m_1)
a_2 = g*(m + M/2)/(L*m_1)

"""###Matrices of an auxiliary system constructing"""

#Vector-function
```

```
F = sp.Matrix([y1,- a_2*sp.sin(y0) - a_1*y1 + y2, 0])

P = ds.Matrix_P(F, y, t, 1)

Q = sp.Matrix([0,0,1])


"""###Constructing of the matrix S,
test of the Kalman's type conditions"""

S, R = ds.Test_controllability(P, Q, tau)


"""### Constructing of the matrix $S^{-1}(PS-\frac{dS}{d\tau})$"""

U = sp.simplify(S.inv()*(P*S - sp.diff(S,tau)))


"""### Coefficients of the stable polynom"""

gamma = ds.Koeffs_Sym(3,a)


"""### Constructing of the matrix $T$"""

phi = U.col(-1)


T = sp.Matrix([[1, -phi[-1], -(sp.diff(phi[-1],tau)+phi[-2])],
[0, 1, -phi[-1]],[0, 0, 1]])


"""Constructing of the vector $\delta$"""

delta = sp.simplify(sp.Matrix([-gamma[0]-phi[-1],
```

```
-gamma[1]-2*sp.diff(phi[-1],tau)-phi[-2],
-gamma[2] - sp.diff(phi[-1],tau,2)- sp.diff(phi[-2],tau)]))


"""### Constructing of the control function"""

M_u = sp.simplify(delta.T*T.inv()*S.inv())
M_u

"""### Return to initial variables"""

TAU = - sp.log(1-t)/alpha
TAU

M_t = M_u.subs(tau, TAU)
M_t

"""### Substitution of the control function in the initial system
and solving of the Cauchy problem"""

M_t1 = M_t.subs(alpha, a)
M_t1

u_t = ds.Sym_to_Num(M_t1,t)
m = len(u_t)
for i in range(m):
  print(u_t[i](t))

at1 = sp.lambdify(t, a_1, "numpy")

at2 = sp.lambdify(t, a_2, "numpy")

per = 5*10**13
def f(t,y):
    f = np.zeros((m),'float')
```

```
    u = 0
    #h = 1-mt.exp(-a*t)
    for i in range(m):
        u += u_t[i](t)*y[i]

    f[0] = y[1]
    f[1] = - at2(t)*np.sin(y[0]) - at1(t)*y[1] + y[2] + per*(1-t)**2
    f[2] = u/(a*(1-t))
    return f

t0 = 0.
tEnd = 0.99
y0 = np.array([0.5, -0.8, 0.])
tau1 = 0.1

k = len(y0)

t_u = []
y_u = []

for i in range(k):
  t_f, y_f = ds.rungediscr_23(f,t0,y0[i],tEnd,tau1,a)
  t_u.append(t_f)
  y_u.append(y_f)
  PlotFig(y_f,t_f,2,'x(t)')
```

Appendix B

Program code for solving of the optimal control problem of a robot-manipulator

```
import numpy as np
import matplotlib.pyplot as plt
import math as mt
import sympy as sp
import OptRicControlLib as opt

def PlotFig(y,t,m,label_y):
  plt.figure(figsize=(8, 5))
  #plt.subplot(121)
  st = label_y
  if m == 2:
    label=['$x_1(t)$, рад','$x_2(t)$, рад/с']
  else:
    label = ["$p_{11}$","$p_{12}$","$p_{21}$","$p_{22}$"]
  if m > 1:
    for n in range(0,m):
      r = [y[i][n] for i in range(0,len(y))]
      plt.plot(t,r,label=label[n])
      plt.legend(loc='best', fontsize=12)

  else:
    plt.plot(t,y,label=st)
  plt.xlabel('t')
  plt.ylabel(st)
  plt.grid()
  plt.show()

#Required symbolic variables
t, tau = sp.var('t tau')
```

```
sp.var('alpha')
y = sp.var('y:2')
x = sp.var('x:4')

#Parameters of the model
a = 1/2
x_0 = 0.05
a1 = 0.1
q = 0.01
L = 10
M = 20
m_0 = 1
g = 9.81
m = m_0 - q*t
m_1 = m + M/3
a_1 = a1/(L**2*(m_0 - q*t + M/3))
a_2 = g*(m_0 - q*t + M/2)/(L*(m_0 - q*t + M/3))

#Vector-function
F = sp.Matrix([y1,- a_2*sp.sin(y0) - a_1*y1])

Y_f = sp.Matrix([y0,y1])


"""###Matrices of an auxiliary system constructing""

P = sp.Matrix([[0,1],[-a_2,-a_1]])

Q = sp.Matrix([0,1])

"""###The Kalman's type conditions test"""

S, R = opt.Test_control_one_col(P, Q, tau)

#Matrices of the system
```

```
N_0 = 1
N_1 = sp.eye(2)
N_1[1,1] += 1
A = P
B = Q

#Solution of the Riccati equation
t0 = 0.
tEnd = 100
p0 = np.array([0, 0, 0, 0])
tau1 = 0.02

t_p, y_p = opt.Riccati_solve(A,B,N_0,N_1,t0,tEnd,tau1)

label = ["$p_{11}$","$p_{12}$","$p_{21}$","$p_{22}$"]

opt.PlotFigRic(y_p,t_p,4,label,figsize=(8,5))

"""###Constructing of the control function"""

X = sp.Matrix([[x0,x1],[x2,x3]])
u = -N_0*B.T*X
u = sp.lambdify(t, u, "numpy")

"""###Substitution of the control function in the initial system
and solving of the Cauchy problem"""

at1 = sp.lambdify(t, a_1, "numpy")

at2 = sp.lambdify(t, a_2, "numpy")

def f(t,y,y_p,k,h):
    f = np.zeros((2),'float')
    f[0] = y[1]
    f[1] = - at2(t)*np.sin(y[0]) - at1(t)*y[1] -
```

```
    y_p[k][2]*y[0] - y_p[k][3]*y[1] + h*(1 - t/tEnd)**2
    return f

h = [0.10,1.5,2.28]

t0 = 0.
y0 = np.array([0.5, -0.8])
tau1 = 0.02

t_u = []
y_u = []
for i in range(len(h)):
  t_f, y_f = opt.runKut_23_U(f,t0,y0,tEnd,tau1,y_p,h[i])
  t_u.append(t_f)
  y_u.append(y_f)
  PlotFig(y_f,t_f,2,'x(t)')
```