Saint-Petersburg University

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Equilibrium behaviour in dynamic models of competition with network interactions

Scientific speciality: 1.2.3. Theoretical informatics, cybernetics

Thesis for candidate of science degree in physics and mathematics

Translation from Russian

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mathematics sciences, associate professor
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Saint-Petersburg
2024

Contents

Introduction

Research Topic Relevance

Network games are a rather young and intensively developing branch of mathematical game theory. A distinctive feature of network games is the assumption that the payoffs each player depend on the structure of interactions of all players. The interaction of players is usually illustrated as a directed or undirected graph with is vertices being identified with players and each edge (undirected interaction) or arc (directed communication) characterizing the interaction, i.e. the influence of the communications on the players connected by it. From a practical perspective, the specifics of network interaction offer a new opportunity to mathematically investigate seemingly ambiguous relations between potential parties to conflict. Thus, for example, there are numerous situations when competing parties can conclude a temporary "truce" without uniting into a coalition — joint research sponsored by competing entities, mutual support of opposing parties, etc. In this respect, it is of great interest to thesis such manifestations, firstly, for their feasibility and conditions of occurrence, secondly, for their duration and dynamics, where the manifestations under consideration can be both short-term and long-term, and thirdly, for the optimal behavior of players at a certain structure of network interaction. Thus, we can conclude that dynamic models of competition with network interaction allow us to explore the peculiarities of the influence of players on each other in terms of their individual relationships.

The network interaction of competing players, including but not limited to, can describe their interaction and feature an exogenous or endogenous nature of the formation of network interactions — respectively, be defined as an external parameter in a game (games on networks), or be part of the strategic behavior of players who independently form network communications among themselves in the game process under a common competitive environment.

The analysis of the exogenous formation of the network interaction between players in dynamics allows us to resolve multiple crucial questions, primarily — from the conceptual point of view of the dynamic game theory, for instance: how does the network interaction (network structure and interaction effects — weights characterizing the strength of influence) affect the players' payoffs and their equilibrium behavior and, if the network interaction affects the players' behavior and the current state of some controlled object, what would it be in equilibrium? The latter issue originates from the optimal control theory, whereunder the stable development of a controlled system requires that the vector describing its states be within the boundaries of some desired region. Thus, the stable development of the system can be achieved by selecting an appropriate network structure [18, 34, 80]. More generally, one can raise the question of identifying for such network structures that would fulfill the desired criterion (where a certain system state or duration of effect would have been achieved — the problem of rapidity), which has become one of the first subject areas in the application of non-cooperative network games in various controllable systems [8].

The analysis of endogenous formation of dynamic network interaction between players can raise the issues of feasibility and conditions of network interaction, duration and sustainability of network interactions, which currently seems to be the most relevant issues of endogenous network interaction. The relevance of the research of these aspects is determined by their being in practical demand, which allows us to successfully apply network games in negotiation processes, joint investments in R&D, reputation management of network participants and other tasks where the network interaction elements are particularly important for the players.

The conceptual features of players' interaction that can be studied using network games allow us to suggest that the methodology may be applicable to all kinds of networks and systems, which have agents with divergent or non-common interests. Transportation, fuel and energy networks, interaction and other types of networks can be referred to as examples of such networks.

Literature review

One of the first research papers devoted to competitive processes, that mentioned the game theory literature is considered to be [52, 59], that analyze the demand and price of goods in a competitive market. Further studies [35, 78, 88], which presented mathematical aspects and applications of game theory, gave rise to the development of the relevant theory as a powerful tool to analyze competitive behavior. Currently, the research on the players' competitive behavior is both relevant and very productive, especially taking into account the aspects of the players' network interaction and network behavior [16, 17, 40, 68]. It is worth noting that the strategic nature of network interaction in a static setting has been investigated in such studies as [48, 54, 63, 70], and the same in a dynamic setting — in [45, 61, 62, 67, 69, 81, 82]. A detailed analysis of current trends and directions in the research on players' network interaction can be found in reviews [16, 36, 62].

Despite the fact that the literature devoted to the research on competitive behavior presents various models of oligopoly with production factor markets, for instance, in studies [5, 6, 19, 75, 79, 84], according to a fair observation made in [25], the literature has so far covered production networks insufficiently. That being said, the analysis of scientific publications has shown that although Cournot's models have become classical examples of competition in mathematical game theory [23, 24, 31, 38, 41], the analysis of competition models based on Cournot's assumptions in various settings remains relevant [1, 9, 11, 64, 65, 83, 86]. It is worth noting that structural interaction and network structure management in competition models directly by the game participants have not been discussed to the necessary extent, although some results have already been made available [49, 63]. It is relevant to continue thesising such aspects since network structures allow us to effectively describe competitive interaction, thus providing the opportunity of expanding the class of solvable game problems due to complementarity and substitutability in the players' behavior [36, 62, 67, 69], as well as the specificity of their mutual influence depending on the presence of network communications. The nature of players' structural interaction is discussed with a sufficient degree of detail and meaningfulness in studies [71, 87, 90], which describe the necessity for the game participants to form structural interactions with each other. Concurrently, the specifics of network interaction expressed via network influence coefficients, as shown in [91], represents a separate focus for research.

It should be noted, that a significant part of network game studies is often devoted to games with externalities, predominantly arising in the consumer context only [55, 56, 58]. Along with this, the players' efforts determined by the complementarity of their actions are often considered as investment efforts and mainly directed towards some ephemeral concepts – knowledge, opinions, impressions, etc. [46, 73, 76, 77, 85]. Recently, the area of application of network games devoted to the optimization of systems with a network structure, in which players need to share the publicly available resources, has also been very popular; a detailed description of the problems of this area of application can be found, for instance, in studies [36, 37, 57].

Numerous researchers have chosen network games as the focus of their studies [2, 7, 12, 25, 42, 43, 44]. Such popularity is primarily explained by the value of applying theoretical results to actually existing network structures, for instance, to organized groups of people, market or political relations, and even social or wireless networks [13, 50], etc. However, the authors of the cited studies have noted that there are quite a lot of issues with insufficient coverage at the moment, which still remain largely open in theory, highlighting the conditions, criteria and principles of network interaction between competing players as the most relevant and insufficiently covered aspects.

To investigate dynamic competition models with network interaction, this thesis will adress and analyze the Nash equilibrium. Despite the fact that the Nash equilibrium has certain drawbacks, which are described in detail in [21], with its intrinsic merits it embodies the fundamental concept of solving non-zero-sum games — according to [78, 88]. Therefore, we can conclude that the Nash equilibrium, although well covered, remains relevant in the literature various on competition models under conditions of simultaneous and independent behavior of players.

Research Purpose and Objectives

The purpose of the thesis is to find and analyze the equilibrium behavior of competing firms under conditions of their dynamic network interaction. Dynamic competition models with network interaction are considered as the object of the thesis, and the equilibrium behavior of firms the subject of the thesis taking into account the structure of their network interaction.
The purpose is achieved by accomplishing the following tasks:

1. Describe the dynamic equilibrium behavior of firms under exogenous formation of network interaction. For this purpose, it is necessary to build an economic and mathematical model of dynamic competition of firms, to find and characterize the Nash equilibrium for this model in two classes of firms' strategies — open-loop and feedback.

2. Investigate the effect of network parameters on the Nash equilibrium in the model with exogenously formed interaction. It is necessary to analyze the effect of the network structure and network coefficients, which characterize the effect obtained by firms from other firms' investments on their strategic behavior, the dynamics of their competitiveness and profit, as well as on the market price of goods.

3. Describe the firms' dynamic equilibrium behavior under endogenous formation of network interaction. For this purpose, the firms' feasible behavior should be extended by the network component, i.e. the firms' capability of forming network interactions with competitors. Analyze the variants of the firms' network interaction formation rule. Find the Nash equilibria for two variants of the firms' network interaction — with the formation of a constant and variable network structure. Benchmark the obtained results and draw conclusions.

4. Determine the conditions for the formation of network interactions between firms in the Nash equilibrium. Determine and analyze the conditions under which firms are interested in network interaction — for variants of formation of unilateral and bilateral network communications.

5. Adapt the models under consideration to the practical interaction of competing firms in the market. Suggest and justify assumptions in the models under consideration to adapt them to the real conditions of economic interaction between firms, including the choice of business partners and duration of interaction in the Nash equilibrium in each of the variants of the firms' network interaction under consideration.

Scientific Novelty

The dynamic investment and network modification in the Cournot oligopoly with exogenously formed network interaction has been constructed. For the model built, the conditions ensuring the uniqueness of open-loop and feedback Nash equilibria have been found, for the equilibria obtained a benchmarking has been conducted. The model has investigated the effect of network parameters characterizing the firms' mutual influence, as well as the influence of the network structure on the equilibrium behavior of firms, their profits and the dynamics of competitiveness. Consideration has been given to the unit price in the market as an external effect (consumer externalities) arising from the equilibrium behavior of competing firms.

Based on dynamic investment and network modification in the Cournot oligopoly with exogenously formed network interaction, the variant of endogenously formed network interaction has been discussed — on the models where firms can enter into short-term or long-term network interaction. The network interaction duration-dependent prospects have been benchmarked, while long-term interaction has been investigated in the settings with unique-time and regular network interaction costs. The open-loop Nash equilibrium has been found for each model, and the pairwise network interaction condition has also been determined for two variants of network communications — unilateral and bilateral.

The thesis has demonstrated that the conceptual approach to dynamic models with endogenous network interaction can be applied to the problems of the economic sector: the conditions for selecting business partners under various variants of firms' investment behavior — constant and variable investment volumes, which can be interpreted as cautious and risky investment behavior in an unstable market. The structural generality of all conditions of interaction in the open-loop Nash equilibrium has been shown, the comparison of considered investment behavior variants under common input parameters has been exemplified.

All the main results were obtained by the author personally and are new.

Theoretical and Practical Significance of Research

The dynamic competition models with network interaction investigated herein complement the existing models of game-theoretic analysis, which have not been considered so far in terms of structural interaction of players. Despite the fact that the thesis has investigated the market competition models, we would like to note that the results obtained are of universal applicable nature — with regard to game modeling of competitive processes that occur beyond the economy. Indeed, given the specifics of the subject area of the problem under consideration, the thesis results can be relevantly applied to such processes as the competition for the leading position of a party (political science), the formation of public opinion (sociology and marketing), the distribution of server resources capabilities (computing systems), the construction and operation of roads (transportation systems and logistics), as well as to various tasks of ecology, psychology, jurisprudence, sociology and other sciences, where competing parties or individuals with their own goals and individual influence collide. Thus, this thesis contributes to the development of the network game theory due to such game components as dynamics and multicomponent behavior of players in their network structural interaction.

In the exogenous formation of network interaction, the game process participants can implement their equilibrium behavior depending on the information structure of the model. The thesis results allow, through the choice of network structures and coefficients of players' exogenous interaction, to adjust their competitiveness, the state of the controlled system, as well as the external factors arising from the dynamic process of competition. This can be effectively applied, for instance, to state antimonopoly programs, as well as in other programs for stabilization, maintenance and development of the market economy.

For dynamic processes of competition with endogenously formed network interaction, the results have been obtained that allow players to implement their equilibrium behavior, relying on the minimum amount of information required to make decisions (time and costs of network interaction), while choosing their network environment (direct neighbors within the network), relying on the formal condition of feasibility of direct network interaction — regardless of the interaction variant, which can either be unilateral or bilateral. This allows building «stable» network structures — secured against existing network interactions being broken or new ones being established, thus avoiding waste of funds — as noted in [58], as well as influencing their own competitiveness and winning in the game process.

The thesis results also allow us comparing not only the prospects of the short--term or long-term network interaction variants, but also network interaction in combinations with the specifics of other components of strategic behavior — with their variable or constant value. Thus, the network interaction conditions derived from the thesis will hopefully compensate to some extent for the currently lacking coverage of the matters of feasibility, duration and conditions of interaction in a competitive environment.

Based on the above, we can conclude that the thesis has attempted not only to methodologically supplement the network game theory by covering some insufficiently studied issues of the theory, but also to conceptually suggest a universal approach that can be applied in economic analysis and management tasks for effective planning of firm's activities, as well as, on a larger scale — in the tasks of stabilization, maintenance, management and development of the market economy.

The thesis results have been used in the research project under the fellowship of the Russian Science Foundation No. 22-11-00051 "Development of Methods for Management of Multi-Agent Systems under Conflict Conditions".

Structure of the Thesis and the Main Scientific Results

The thesis structure includes an introduction, three Chapters divided into sections and subsections, a description of the main results and conclusions — in each chapter, conclusion and a list of references consisting of 91 sources. The thesis consists of 116 pages of typewritten text and includes 25 Tables and 1 Figure.

Chapter 1 describes the construction and analysis of a dynamic competition model with double-component (production and investment) behavior of firms with exogenous formation of their network interaction (Section 1.1). Two classes of behavior of firms have been considered — according to the open-loop information structure and feedback information structure. For each class of firms' strategies, a unique Nash equilibrium has been obtained (Section 1.2). The results have been benchmarked in the Nash equilibrium for each of the considered classes of firms' strategies by means of numerical simulations (Section 1.3). Section 1.4 separately investigates the function and significance of network elements that characterize the specifics of firms' mutual influence — through unit costs. It also considers the network structure influence in the Nash equilibrium on the firms' competitiveness dynamics and the externalities arising from competition — in particular, the unit price in a common competitive market. The Chapter is summarized with a presentation of the main results and conclusions.

Chapter 2 continues to investigate the dynamic competition model from Chapter 1, but with its extension: now firms implement behaviors that are divided into network, production, and investment behaviors. We have initially discussed the variants of endogenous interaction between firms — unilateral interactions represented by arcs and bilateral communications represented by edges in network structures, described the formal component of the endogenous interaction variants, and, respectively, specified some components of the model under consideration (Section 2.1). Section 2.2 seeks for the open-loop Nash equilibrium under bilateral interaction of firms. The thesis continues with a search for the open-loop Nash equilibrium for models with constant network interaction between firms under two variations of network communication costs (unique-time and variable), as well as the conditions of network interaction between firms (Section 2.3). Section 2.4 shows how the results obtained earlier can be applied to unilateral interaction of firms in network structures, and how the Nash equilibrium is modified. Section 2.5, through numerical simulations, benchmarks the open-loop Nash equilibria for all models considered, and evaluates the advantages and disadvantages of two variants of network interaction durations — short-run (firms rearrange the network structure at each decision point) and long-term (the model is implemented on the network built by firms at the initial time point). The Chapter is summarized with a presentation of the main results and conclusions.

Chapter 3 presents assumptions aimed at adapting the investigated models of competition to real economic processes and demonstrates the application of the previously obtained results, both conceptual and methodological, to the real process of interaction between firms. For this purpose, we introduce a set of special assumptions that firms adhere to in real conditions of competitive production and conduct a benchmarking of short-term (Section 3.1) and long-term (Section 3.2) network interaction in the Nash equilibrium under risky and cautious investment behavior of firms — in Section 3.3. We also investigate a model in which firms implement unique-time investments (Section 3.4). The Chapter is summarized with a presentation of the main results and conclusions.

Research Methodology and Methods

This thesis has employed the tools which represent common research techniques and approaches in applied mathematics: dynamic game theory (Nash equilibrium), operations research (Bellman recurrence relations, Lagrange multiplier method), optimal control theory (Pontryagin's maximum principle), mathematical modeling, benchmarking, numerical simulations in Wolfram Mathematica.

Degree of Credibility and Evaluation of Results

The main results of this thesis have been discussed and reported at the following scientific events: All-Russian Conference on Natural Sciences and Humanities with International Participation "SPSU Science – 2022", St. Petersburg; Fifty-Second (LII) Scientific and Educational-Methodological Conference of ITMO University, Section "Mathematical Modeling", St. Petersburg; XII Congress of Young Scientists of ITMO, Section "Artificial Intelligence and Behavioral Economics", St. Petersburg; 16th International Conference on Game Theory and Management (GTM2023), St. Petersburg; 22nd International Conference on Mathematical Optimization Theory and Operations Research "MOTOR 2023", Ekaterinburg; Workshop on Dynamic Games and Applications, Tashkent, Uzbekistan; All-Russian Conference on Natural Sciences and Humanities with International Participation "SPSU Science – 2023", St. Petersburg; Scientific Seminar of the Applied Mathematical Research Institute under the Karelian Scientific Center of the Russian Academy of Sciences, Petrozavodsk; Scientific Seminar of the Department of Mathematical Game Theory and Statistical Decisions of St. Petersburg State University, St. Petersburg.

The validity and credibility of the thesis research results is ensured by the correctness of problem statements, proof points and conclusions, the rigor of mathematical evidence and the receipt of positive feedback from members of the editorial boards of scientific outlets where the main results were published.

Publications

The main thesis results have been published in three scientific papers [27, 28, 29], included in the list of peer-reviewed scientific publications recommended by the HAC (Higher Attestation Commission) of the Russian Federation and included in the core of RSCI (Russian Science Citation Index), with the thesis [27] also indexed in the Scopus and Web of Science international scientific databases. Received the certificate of registration of the computer program [26] with registration number RU 2023685627 have been obtained.

Main scientific results

1. An network modification of the Cournot oligopoly is constructed, for which the two-component Nash equilibrium behavior of firms with dynamic exogenous network formation is obtained. The equilibrium is presented and characterized for a open-loop strategies. In addition, a feedback Nash equilibrium has been obtained and the «proximity» of the two equilibria found has been established [27].

2. The influence of the network structure and the associated coefficients of the model on the behavior of firms in equilibrium, and how the structure of interaction of firms affects changes in their unit costs, competitiveness in the market, profits, as well as the price of a unit of goods in the market are analysed [27].

3. A feasible behavior of each firm is complemented by a component that characterizes the attitude toward network interaction with its competitors and is responsible for its network behavior. A functional structure of the Nash equilibrium behavior of firms with dynamic endogenous network formation obtained [28].

4. A Nash equilibrium is obtained for two types of network interaction — with the formation of a constant and a variable network structure. At the same time, the costs associated with the networking of firms are also considered in two types — one-time and regular. A comparative analysis of the obtained results is carried out [28].

5. Assumptions are proposed and justified that serve to adapt the studied game theoretic models to the practical interaction of competing firms in the market. The conditions for choosing business partners and options for the duration of interaction between firms in the Nash equilibrium are considered. A comparative analysis of the Nash equilibrium is given for two types of the investment behavior of firms common in real conditions - risky (variable) and cautious (constant), taking into account the duration of their interaction [29].

6. A functional expression of the equilibrium behavior of firms under their one-time investment in their production is obtained. The relation between the

changes of the upper limits of the allowable costs of network interaction and the duration of the investments of the firms is shown [29].

7. For each model of endogenous network formation studied in the thesis, the equilibrium network behavior of competitors are obtained, the fulfillment of which makes firms interested in network interaction with their competitors. At the same time, two types for the formation of network interaction are considered, represented by undirected or directed links between firms. The thesis notes that in network structures, which are formed when firms implement their equilibrium network behavior, it is unprofitable for any firm to unilaterally break any of its existing connections, as well as to strive to create a new one, for which the condition of equilibrium network behavior is not fulfilled [28, 29].

Main results to be Defended

1. Open-loop and feedback Nash equilibria and their uniqueness in the dynamic competition model with exogenously formed network interaction.

2. Open-loop Nash equilibria for dynamic competition models with endogenously formed short-term or long-term network interaction.

3. Open-loop Nash equilibria for dynamic competition models under risky or cautious investment behavior of firms with their endogenously formed short-term or long-term interaction.

4. Conditions of Nash equilibrium network behavior under open-loop information structure in dynamic competition models with endogenously formed network interaction.

Acknowledgements

The influence of certain people on us — our formation and progressive development — sometimes cannot be overemphasized. And as the catchphrase says "the teacher continues in his student", in my works I can feel the people who influenced my choice of science for a professional career and allowed me to fall in love with mathematics. I couldn't be more grateful to them for this. Feeling grateful and cherishing priceless moments of our collaborative work over the years, I would like to dedicate my thesis research to my first Teachers: Candidate of Physical and Mathematical Sciences, Associate Professor Nadezhda Aleksandrovna Chuesheva, and to memory of my school teachers — Lyubov Ilyinichna Kozlova, Svetlana Stepanovna Gvozdeva.

During the thesis research, I would like to express my gratitude to my thesis advisor, Doctor of Physical and Mathematical Sciences, Professor Artem Aleksandrovich Sedakov, who has always been responsive, impartial, patient and friendly in our joint efforts. Thank you for your attention and support of my scientific interests. Your comments and recommendations on formatting research results are always valuable.

Chapter 1.
Dynamic model with exogenous network formation

The dynamic competition model, which is the subject of this chapter, is based on the classical Cournot[1] oligopoly model presented to the scientific community in [59]. Since its appearance, the model has been subjected to various criticisms by economists, however, having proved its suitability in modeling of economic processes, and as noted in [4], Cournot oligopoly later became a paradigmatic economic and mathematical model. Some of its formal provisions are summarized below.

Suppose $\mathcal{N} = \{1, \ldots, n\}, n \geq 2$, is a finite set of players, which are firms that produce homogeneous and indivisible goods. The produced goods are fully sold in the common market. Each firm $i \in \mathcal{N}$ decides on the volume of goods to produce, i.e. $u_i \in \mathbb{U} = [0, +\infty)$. Then $\sum_{j=1}^{n} u_j$ is the total and non-negative volume of goods in the market in the situation where $u = (u_1, \ldots, u_n) \in \mathbb{U}^n$. Suppose, that the market demand Q is known and defined by a decreasing linear function: $Q\left(P\right) = p - \theta \cdot P$, where $p, \theta \in \mathbb{R}_+$, and $P = P\left(u\right)$ is the unit price such that demand matches supply of the goods available in the market: $\sum_{j=1}^{n} u_j = p - \theta \cdot P$. For simplicity, it is often assumed that $\theta = 1$. The inverse demand function, reflecting the marginal value of a unit of the goods at a given volume is $P\left(u\right) = p - \sum_{j=1}^{n} u_j$, where p is the maximum possible price of a unit (assumed to be fixed and constant). Fixed costs are most often not included in the model, while unit costs $c \in \mathcal{C} = [0; p]$ are assumed to be fixed and equal for all firms. The profit of each firm i is determined by the value

$$F_i\left(u\right) = \left(p - c - \sum_{j=1}^{n} u_j\right) \cdot u_i. \tag{1.1}$$

To maximize their profits, each firm pursues a strategy that enters in the Nash equilibrium $u^{\mathrm{N}} = \left(u_1^{\mathrm{N}}, \ldots, u_n^{\mathrm{N}}\right) \in \mathbb{U}^n$ such that for each firm $i \in \mathcal{N}$ and any of its admissible strategies $u_i \in \mathbb{U}$ the inequality

$$F_i\Big(u_1^{\mathrm{N}}, \ldots, u_{i-1}^{\mathrm{N}}, u_i, u_{i+1}^{\mathrm{N}}, \ldots, u_n^{\mathrm{N}}\Big) \leqslant F_i\Big(u_1^{\mathrm{N}}, \ldots, u_{i-1}^{\mathrm{N}}, u_i^{\mathrm{N}}, u_{i+1}^{\mathrm{N}}, \ldots, u_n^{\mathrm{N}}\Big)$$

is true. For each firm $i \in \mathcal{N}$ it is easy to determine its strategy that enters in the

[1] Antoine Augustin Cournot (28 august 1801 — 30 march 1877, Paris) — French economist, philosopher and mathematician, author of several economic and mathematical models that have become classic for game-theoretic analysis.

Nash equilibrium and is represented by the following volume[2]: $u_i^{\mathrm{N}} = (p - c)/(n+1)$, where $u_i^{\mathrm{N}} > 0$ at $p > c$.

Apart from the presented provisions, the described model features multiple assumptions known both from the thesis by A. O. Cournot and from other classical scientific literature, e.g. [10, 31, 41, 46], which will be adhered to.

In addition to the model discussed above, we will suppose that a firm is able to influence the state of unit costs (hereinafter within the chapter referred to as the costs) of all firms and discuss such influence in dynamics. We will assume that each firm i is capable of exerting an additional effect that affects the costs of other firms by the nature of their networking with the firm i. We will proceed to the detailed description and formalization of the mentioned capabilities of firms in the model under consideration.

1.1. Description and formalization of dynamic investment and network modification in the Cournot oligopoly

In addition to the classical Cournot model, consider oligopoly as a dynamic game in discrete time with periods given by the set $\mathcal{T} = \{0, 1, \ldots, T\}$, $T \geqslant 2$. In real conditions, the unit costs of firms are dynamic. We will consider the costs of firms $i \in \mathcal{N}$, denoted by $c_i(t) \in \mathcal{C}$, at $t \in \mathcal{T}$ as a value that tends to change over time. By managing its costs, a firm can ensure that it increases the competitiveness and profitability of its business or position. This seems to be a potentially important application of dynamic game theory. Let the set $c(t) = (c_1(t), \ldots, c_n(t))'$ denote the state of firms' costs at time $t \in \mathcal{T}$ for given initial costs $c(0) \equiv c_0 = (c_{10}, \ldots, c_{n0})'$, where the symbol «$\,'\,$» here and in the future will denote the transposition operation. The equation of the cost dynamics of firms is written in vector form as

$$c(t+1) = f(t, \mathbf{g}(t), c(t), y_1(t), \ldots, y_n(t)), \quad t \in \mathcal{T} \setminus \{T\}.$$

The investment efforts of firm i at time t, in monetary terms, are represented by $y_i(t)$ which belongs to $\mathbb{Y}_i$, a subset of the non-negative real numbers. The value of the expression $\frac{\varepsilon_i(t)}{2} y_i^2(t)$ determines the monetary value of the investments. Here, $\varepsilon_i(t)$ is the current value of a given parameter. The $f(t, \cdot)$ is a continuously differentiable

[2] The solution is known as the Cournot-Nash equilibrium. It was first obtained by A. O. Cournot without a conceptual justification of the equilibrium. Later, J. Nash determined the justification.

function. Additionally, we will discuss the structure of the network interaction of firms, denoted by $\mathbf{g}(t)$.

Have firms invest in their production technologies. The specifics of the impact of one firm's investment on others can be illustrated using a graph. To do this, we identify a set of firms with vertices of a graph $(\mathcal{N}, \mathbf{g}(t))$, in which $\mathbf{g}(t) \subseteq \mathcal{N} \times \mathcal{N}$ — the set of connections represented by the edges of the graph, and define its structure at time $t \in \mathcal{T} \backslash \{T\}$. To simplify writing, the graph structure at time t is denoted by $\mathbf{g}(t)$. When evaluating the relationship between the firms $i, j \in \mathcal{N}$, $j \neq i$, denoted by (i, j), we assume that $(i, j) = (j, i)$ and $(i, j) \in \mathbf{g}(t)$ if and only if $g_{ij}(t) = g_{ji}(t) = 1$, where $g_{ij}(t)$ and $g_{ji}(t)$ are elements of a binary adjacency matrix for the network $\mathbf{g}(t)$ without self loops. Next, for convenience, we denote by $\mathbf{g} = \{\mathbf{g}(t)\}_{t=0}^{T-1}$ the sequence of network structures specified in the model.

For the firm $i \in \mathcal{N}$, the rule for changing its costs over time is defined by a recurrent equation with a given initial condition

$$c_i(t+1) = \delta c_i(t) - \sum_{j=1}^{n} \mu_{ij}(t, \mathbf{g}(t)) y_j(t), \quad t \in \mathcal{T} \setminus \{T\}, \quad c_i(0) = c_{i0}, \tag{1.2}$$

where

$$\mu_{ij}(t, \mathbf{g}(t)) = \begin{cases} \alpha_i(t), & j = i, \\ \beta_{ij}(t)\, g_{ij}(t) + \gamma_{ij}(t)(1 - g_{ij}(t)), & j \neq i. \end{cases}$$

In the future, in order to simplify the notation, where an explicit clarification of the dependence on the network $\mathbf{g}(t)$ is not necessary, we write $\mu_{ij}(t)$ instead of $\mu_{ij}(t, \mathbf{g}(t))$.

The parameter $\delta \geqslant 1$ characterizes the rate of change of a firm's costs over time due to the possible obsolescence of the production technologies it uses in the absence of additional investments in their modernization. The parameter $\alpha_i(t) > 0$ reflects the effect of the firm's own investments i at the current time, and $\beta_{ij}(t) \geqslant 0$ and $\gamma_{ij}(t) \geqslant 0$ — current effects of investments of neighboring firms in the network $\mathbf{g}(t)$, i.e. from $j \in \mathcal{N}_i(\mathbf{g}(t)) := \{r \,|\, (i,r) \in \mathbf{g}(t)\}$, and also from firms $j \notin \mathcal{N}_i(\mathbf{g}(t)) \cup \{i\}$ that are not neighbors of firm i in the network $\mathbf{g}(t)$.

$$u = (u(0), \ldots, u(T-1)), \quad u(t) = (u_1(t), \ldots, u_n(t)),$$

$$y = (y(0), \ldots, y(T-1)), \quad y(t) = (y_1(t), \ldots, y_n(t)).$$

The profit of firm i at time $t \in \mathcal{T} \setminus \{T\}$, based on equation (1.1), is written as:

$$F_i(t, c_i(t), u(t), y_i(t)) = \left(p - \sum_{j=1}^{n} u_j(t)\right) u_i(t) - c_i(t)u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t), \qquad (1.3)$$

as the difference between its current revenue, formed according to the classical Cournot oligopoly model with a linear inverse demand function, and its current costs, which include production costs and investments in production. At the terminal period, the profit of firm i is determined by the residual value of its production according to the function $\Phi_i(T, c_i(T)) = \eta_i - \eta c_i(T)$, where $\eta > 0$ is the liquidity ratio of production and $\eta_i > 0$ is the maximum market value of production, additionally assuming $\eta_i > \eta p$. Then the total profit of firm i for all periods in the model takes the form

$$J_i(c_0, u, y) = \sum_{t=0}^{T-1} \rho^t F_i(t, c_i(t), u(t), y_i(t)) + \rho^T \Phi_i(T, c_i(T)), \qquad (1.4)$$

where $\rho \in (0, 1]$ is the factor rate common to all firms and constant over time. In this setting, the dynamic model of competitive production with investment is a linear-quadratic discrete-time game with an n-dimensional state variable and two-dimensional actions players. Note that the costs to firms of maintaining network connections do not affect their behavior, since network interaction is exogenous and not controlled by the players. For this reason, such costs are not considered in this model.

Let us describe the behavior of firms in dynamics. At the initial period, with the known network structure $\mathbf{g}(0)$ and initial costs c_0, firms simultaneously and independently choose their feasible behavior — pairs $(u_i(0), y_i(0))$, $i \in \mathcal{N}$, — each of them decides how much output to produce and how much investment to implement at the current time. This decision gives the firm i the profit $F_i(0, c_{i0}, u(0), y_i(0))$ — according to (1.3). Furthermore, the costs of firm i change according to the rule (1.2) and become equal to $c_i(1)$, $i \in \mathcal{N}$. At an intermediate nonterminal time $t \in \mathcal{T} \setminus \{T\}$ with the known network structure $\mathbf{g}(t)$, firms simultaneously and independently choose their feasible behavior — for firm $i \in \mathcal{N}$ the current volume of production and investment $(u_i(t), y_i(t))$, which will lead the firm i to the profit $F_i(t, c_i(t), u(t), y_i(t))$ and the next period cost $c_i(t+1)$. At the moment $t = T$, the firm i receives the residual value determined by the function $\Phi_i(T, c_i(T))$, $i \in \mathcal{N}$, and the total profit is calculated according to (1.4). After the described actions the game ends.

In order to represent the dynamic nature of the interaction of firms in the form of a game in a normal form, according to [47], we denote by s_i a strategy of firm $i \in \mathcal{N}$, which prescribes to it an choice of feasible behavior depending on the current information, and we denote the set of strategies of this firm by $\mathcal{S}_i$. Due to the unambiguity of the choice of actions prescribed by the strategies, we define the payoff function of the firm i as $\mathcal{J}_i(s) = J_i(c_0, u, y)$ for the set of strategies chosen by the firms $s = (s_1, \ldots, s_n)$. Thus, the dynamic investment and network modification in the Cournot oligopoly with the exogenous network formation can be represented by a game in normal form:

$$\Gamma^{\mathrm{ex}} = \big\langle \mathcal{N}, \{\mathcal{S}_i\}_{i \in \mathcal{N}}, \{\mathcal{J}_i\}_{i \in \mathcal{N}} \big\rangle.$$

1.2. Nash equilibrium for two information structures

In the sections of game theoretic analysis devoted to competitive models, the Nash equilibrium embodies the fundamental concept of solving non-zero-sum games according to [78, 88]. In Γ^{ex}, a Nash equilibrium is set of strategies $s^{\mathrm{N}} = \big(s_1^{\mathrm{N}}, \ldots, s_n^{\mathrm{N}}\big) \in \prod_{j \in \mathcal{N}} \mathcal{S}_j$, where $\prod_{j \in \mathcal{N}} \mathcal{S}_j = \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ — the Cartesian product of sets of firm strategies, and for any firm $i \in \mathcal{N}$ the condition is satisfied

$$s_i^{\mathrm{N}} = \arg \max_{s_i \in \mathcal{S}_i} \mathcal{J}_i\big(s_{-i}^{\mathrm{N}} \,|\, s_i\big),$$

where the set of strategies $\big(s_{-i}^{\mathrm{N}} \,|\, s_i\big)$ differs from s^{N} only in that the firm i uses $s_i \in \mathcal{S}_i$ instead of the strategy s_{-i}^{N}, that is $\big(s_{-i}^{\mathrm{N}} \,|\, s_i\big) = \big(s_1^{\mathrm{N}}, \ldots, s_{i-1}^{\mathrm{N}}, s_i, s_{i+1}^{\mathrm{N}}, \ldots, s_n^{\mathrm{N}}\big)$.

In order to find a Nash equilibrium in Γ^{ex}, it is important to understand what the information structure is, that is, the type and amount of information available to firms to choose their strategies. In the current chapter, we consider two variants of the information structure: open-loop and feedback, we denote corresponding strategies by s_i^{OL} and s_i^{FB}, respectively, following [60, 89]. For each of the considered variants of the information structure, the Nash equilibrium profiles are presented.

1.2.1. Open-loop Nash equilibrium

The open-loop information structure in Γ^{ex} assumes the choice of actions by the participants of the game, based on the knowledge of the current moment and the initial state of the costs of all firms c_0. A feasible strategy of a firm corresponding to the described information structure is similarly called — open-loop, and should prescribe to it feasible behavior, taking into account the current intermediate period and the state of c_0. More formally, our can define the open-loop strategy of the firm $i \in \mathcal{N}$ as a rule $s_i^{\text{OL}}(t, c_0) : \mathcal{T} \setminus \{T\} \mapsto \mathbb{U}_i \times \mathbb{Y}_i$, which clearly matches the feasible behavior of $s_i^{\text{OL}}(t, c_0) = \left(s^{\text{OL}}\right)$ at any intermediate time and initial values of the unit costs of firms $\left(s_{i1}^{\text{OL}}(t, c_0), s_{i2}^{\text{OL}}(t, c_0)\right) = (u_i(t), y_i(t))$.

Next, we will introduce some notations for convenience: $\mathbf{e} \in \mathbb{R}^n$ is a vector consisting of units, $\mathbf{e}_i \in \mathbb{R}^n$ is a unit vector with 1 in position i, I is identics matrix $(n \times n)$ and $\mu_i(t) = (\mu_{1i}(t), \ldots, \mu_{ni}(t))'$ for $i \in \mathcal{N}$. These notations are used in the following theorem, which characterizes the unique open-loop Nash equilibrium in the model Γ^{ex}. The thesis only describes interior Nash equilibria, which are those where firms' behavior is the interior point of the set of feasible actions.

Theorem 1.1. *Let $\ell_{i1}(t) \in \mathbb{R}^n$, $\ell_{i2}(t) \in \mathbb{R}$ satisfy the recurrence relations:*

$$\ell_{i1}(t) = \begin{cases} \delta^2 M^{-1}(t+1)\ell_{i1}(t+1) - \rho^t \dfrac{\mathbf{e} - (n+1)\mathbf{e}_i}{n+1}, & t \neq T, \\ 0, & t = T, \end{cases} \tag{1.5}$$

$$\ell_{i2}(t) = \begin{cases} \delta\left[\ell_{i1}'(t+1)M^{-1}(t+1)m(t+1) + \ell_{i2}(t+1)\right] - \rho^t \dfrac{p}{n+1}, & t \neq T, \\ -\rho^T \eta, & t = T, \end{cases} \tag{1.6}$$

for all firm $i \in \mathcal{N}$, where matrices $M(t)$ and vectors $m(t)$ are set according to the rules:

$$M(t) = I - \sum_{j \in \mathcal{N}} \frac{\alpha_j(t-1)\mu_j(t-1)}{\rho^{t-1}\varepsilon_j(t-1)} \ell_{j1}'(t) \;\; u \;\; m(t) = \sum_{j \in \mathcal{N}} \frac{\alpha_j(t-1)\mu_j(t-1)}{\rho^{t-1}\varepsilon_j(t-1)} \ell_{j2}(t).$$

If the matrices $M(t)$ invertible for all $t \neq 0$, then in model Γ^{ex}, a set of strategies $s^{\text{OLN}} = (s_1^{\text{OLN}}, \ldots, s_n^{\text{OLN}})$ is the unique open-loop Nash equilibrium whose compo-

nents are $s_i^{\mathrm{OLN}}(t, c_0)$ *for the* $i \in \mathcal{N}$, $t \in \mathcal{T} \setminus \{T\}$ *and have the form:*

$$u_i^{\mathrm{OLN}}(t, c_0) = \frac{p + (\mathbf{e} - (n+1)\mathbf{e}_i)' c^{\mathrm{OLN}}(t)}{n+1}, \tag{1.7}$$

$$y_i^{\mathrm{OLN}}(t, c_0) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)} \Big(\ell_{i1}'(t+1) M^{-1}(t+1)(\delta c^{\mathrm{OLN}}(t) + m(t+1)) + \ell_{i2}(t+1)\Big), \tag{1.8}$$

where the current equilibrium profit of cost $c^{\mathrm{OLN}}(t)$ *is recursively found from equation*

$$c^{\mathrm{OLN}}(t) = M^{-1}(t)(\delta c^{\mathrm{OLN}}(t-1) + m(t)), \quad t \in \mathcal{T} \setminus \{0\}, \quad c^{\mathrm{OLN}}(0) = c_0. \tag{1.9}$$

Proof. To determine the open-loop Nash equilibrium, we will use the Pontryagin maximum principle [41, 47], for which we will introduce the Hamiltonian function for the firm $i \in \mathcal{N}$ and $t \in \mathcal{T} \setminus \{T\}$:

$$\begin{aligned}\mathcal{H}_i(t, c(t), u(t), y(t), \psi_i(t+1)) = \rho^t \Big[\Big(p - c_i(t) - \sum_{j \in \mathcal{N}} u_j(t)\Big) u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t)\Big] + \\ + \sum_{j \in \mathcal{N}} \psi_{ij}(t+1) \Big[\delta c_j(t) - \sum_{r \in \mathcal{N}} \mu_{jr}(t) y_r(t)\Big],\end{aligned}$$

where $\psi_i(t) = (\psi_{i1}(t), \ldots, \psi_{in}(t))'$ is vector of costate variables. According to [47] if the strategy set $s^{\mathrm{OLN}}(t, c_0)$ is a Nash equilibrium, then there exist non-zero costate variables $\psi_i(t)$, $t \in \mathcal{T} \setminus \{0\}$, $i \in \mathcal{N}$ which satisfy the relations:

$$s_{i1}^{\mathrm{OLN}}(t, c_0) = \frac{p - c_i(t) - \sum_{j \in \mathcal{N} \setminus \{i\}} s_{j1}^{\mathrm{OLN}}(t, c_0)}{2},$$

$$s_{i2}^{\mathrm{OLN}}(t, c_0) = -\frac{\sum_{j \in \mathcal{N}} \psi_{ij}(t+1) \mu_{ji}(t)}{\rho^t \varepsilon_i(t)},$$

$$\psi_{ij}(t) = \begin{cases} -\rho^t s_{i1}^{\mathrm{OLN}}(t, c_0) + \delta \psi_{ii}(t+1), & j = i,\, t \neq T, \\ -\rho^T \eta, & j = i,\, t = T, \\ \delta \psi_{ij}(t+1), & j \neq i,\, t \neq T, \\ 0, & j \neq i,\, t = T, \end{cases}$$

$$c_i(t+1) = \delta c_i(t) - \sum_{j \in \mathcal{N}} \mu_{ij}(t) s_{j2}^{\mathrm{OLN}}(t, c_0), \quad t \neq T, \quad c_i(0) = c_{i0}.$$

From these relations we can immediately conclude that $\psi_{ij}(t) = 0$ for all $i \neq j$ and $t \in \mathcal{T}$. Considering this and some transformations, we conclude:

$$s_{i1}^{\text{OLN}}(t, c_0) = \frac{p - (n+1)c_i(t) + \sum_{j \in \mathcal{N}} c_j(t)}{n+1}, \tag{1.10}$$

$$s_{i2}^{\text{OLN}}(t, c_0) = -\frac{\alpha_i(t)\psi_{ii}(t+1)}{\rho^t \varepsilon_i(t)}, \tag{1.11}$$

where the first expression matches (1.7). Due to the strict concavity of the Hamiltonian $\mathcal{H}_i(t,\cdot)$ over a set of variables $(u_i(t), y_i(t))$, which follows from the fact that the minor determinants of the Hesse matrix are $-2\rho^t < 0$ and $2\varepsilon_i(t)\rho^{2t} > 0$, we conclude that the maximum value of the function is given by is the only maximum that satisfies the conditions (1.10) – (1.11).

We will search for costate variables in linear form $\psi_{ii}(t) = \ell_{i1}'(t)c(t) + \ell_{i2}(t)$ and show that $\ell_{i1}(t)$ and $\ell_{i2}(t)$ satisfy (1.5) – (1.6). Taking into account (1.8) and the linear representation of the costate variables, the state equation can be written as $c(t+1) = M^{-1}(t+1)(\delta c(t) + m(t+1))$, where $t \in \mathcal{T} \setminus \{T\}$, which is the same as (1.9). Then, taking into account (1.10) and (1.11), we get this:

$$\psi_{ii}(t) = -\rho^t \cdot \frac{p + (\mathbf{e} - (n+1)\mathbf{e}_i)'c(t)}{n+1} + \delta[\ell_{i1}'(t+1)c(t+1) + \ell_{i2}(t+1)] =$$

$$= -\rho^t \cdot \frac{p + (\mathbf{e} - (n+1)\mathbf{e}_i)'c(t)}{n+1} + \delta\big[\ell_{i1}'(t+1)M^{-1}(t+1)(\delta c(t) + m(t+1)) + \ell_{i2}(t+1)\big].$$

Following the method of undetermined coefficients, we conclude (1.5) – (1.6). □

1.2.2. Feedback Nash equilibrium

If the information structure of the Γ^{ex} model assumes that firms choose actions based not only on the current period, but also on information about the state of costs at that moment $c(t)$, then firms can focus on the feedback Nash equilibrium.

First, let us formally define a feedback strategy of the firm $i \in \mathcal{N}$, denoted by s_i^{FB}, usually $s_i^{\text{FB}}(t,c) : \mathcal{T} \setminus \{T\} \times \mathcal{C}^n \mapsto \mathbb{U}_i \times \mathbb{Y}_i$, which corresponds to every intermediate period and every set of costs of firms $c(t) = (c_1(t), \ldots, c_n(t))'$, which unambiguously corresponds to the feasible behavior of firm i, that is $s_i^{\text{FB}}(t, c(t)) = \big(s_{i1}^{\text{FB}}(t, c(t)), s_{i2}^{\text{FB}}(t, c(t))\big) = (u_i(t), y_i(t))$.

To determine the feedback Nash equilibrium, we use the Bellman recurrence relations [3, 31, 74]. The equilibrium can be found using the following theorem.

Theorem 1.2. *A set of strategies* $s^{\mathrm{FBN}} = \left(s_1^{\mathrm{FBN}}, \ldots, s_n^{\mathrm{FBN}}\right)$ *with components*

$$s_i^{\mathrm{FBN}}(t,c) = \left(s_{i1}^{\mathrm{FBN}}(t,c), s_{i2}^{\mathrm{FBN}}(t,c)\right) = (a_i'(t)c + v_i(t), b_i'(t)c + w_i(t)),$$

where $i \in \mathcal{N}$, $t \in \mathcal{T} \setminus \{T\}$ *and* $c = (c_1, \ldots, c_n)'$ *is a given profile of costs, is the unique feedback Nash equilibrium in* Γ^{ex} *if there is a unique solution to the following system of recurrent relations:*

$$a_i(t) = \frac{\mathbf{e} - (n+1)\mathbf{e}_i}{n+1}, \ \ b_i(t) = \frac{-1}{\rho^t \varepsilon_i(t)} \Big[\delta I - \sum_{j \in \mathcal{N}} \mu_j(t) b_j'(t)\Big]' K_i(t+1)\mu_i(t), \quad (1.12)$$

$$v_i(t) = \frac{p}{n+1}, \ \ w_i(t) = \frac{-1}{\rho^t \varepsilon_i(t)} \Big[k_i(t+1) - K_i(t+1) \sum_{j \in \mathcal{N}} \mu_j(t) w_j(t)\Big]' \mu_i(t), \quad (1.13)$$

$$\begin{aligned} K_i(t) = {} & 2\rho^t a_i(t) a_i'(t) - \rho^t \varepsilon_i(t) b_i(t) b_i(t) + \\ & + \left(\delta I - \sum_{j=1}^n \mu_j(t) b_j'(t)\right)' K_i(t+1) \left(\delta I - \sum_{j=1}^n \mu_j(t) b_j'(t)\right), \end{aligned} \quad (1.14)$$

$$\begin{aligned} k_i(t) = {} & 2\rho^t a_i(t) v_i(t) - \rho^t \varepsilon_i(t) b_i(t) w_i(t) + \\ & + \left(\delta I - \sum_{j=1}^n \mu_j(t) b_j'(t)\right)' \left(k_i(t+1) - K_i(t+1) \sum_{j=1}^n \mu_j(t) w_j(t)\right), \end{aligned} \quad (1.15)$$

$$\begin{aligned} \kappa_i(t) = {} & \rho^t \left(v_i^2(t) - \frac{\varepsilon_i(t)}{2} w_i^2(t)\right) + \kappa_i(t+1) - \\ & - \left(k_i(t+1) - \frac{1}{2} K_i(t+1) \sum_{j=1}^n \mu_j(t) w_j(t)\right)' \sum_{j=1}^n \mu_j(t) w_j(t), \end{aligned} \quad (1.16)$$

with the boundary conditions of $i \in \mathcal{N}$*:* $K_i(T) = 0$, $k_i(T) = -\rho^T \eta \mathbf{e}_i$, $\kappa_i(T) = \rho^T \eta_i$*; and with this difference* $\rho^t \varepsilon_i(t) - \mu_i(t)' K_i(t+1) \mu_i(t)$ *positive for* $i \in \mathcal{N}$, $T \in \mathcal{T} \setminus \{T\}$*. Besides,*

$$\mathcal{J}_i\left(s^{\mathrm{FBN}}\right) = \frac{1}{2} c_0' K_i(0) c_0 + k_i'(0) c_0 + \kappa_i(0), \quad i \in \mathcal{N}.$$

Proof. From the theory of dynamic games [31, 41, 47] it is known that s^{FBN} is a Nash equilibrium if and only if there exist functions $V_i(t,\cdot) : \mathcal{C}^n \mapsto \mathbb{R}$, $t \in \mathcal{T}$, $i \in \mathcal{N}$ which satisfy the Bellman recurrence relations. Then for the model Γ^{ex} we have:

$$\begin{aligned} V_i(t,c) = \max_{(u_i(t),y_i(t))\in\, \mathbb{U}_i\times\mathbb{Y}_i} \Big[\rho^t\Big(p - c_i - u_i(t) - \sum_{j\neq i} s_{j1}^{\text{FBN}}(t,c)\Big)u_i(t) - \\ -\rho^t\frac{\varepsilon_i(t)}{2}y_i^2(t) + V_i\Big(t+1, \delta c - \mu_i(t)y_i(t) - \sum_{j\neq i}\mu_j(t)s_{j2}^{\text{FBN}}(t,c)\Big)\Big]. \end{aligned} \tag{1.17}$$

For the class of linear-quadratic games, the Bellman function can be found in a special form $V_i(t,c) = \frac{1}{2}c'K_i(t)c + k_i(t)'c + \kappa_i(t)$ with the boundary condition $V_i(T,c) = \rho^T(\eta_i - \eta c_i)$. Assuming a linear structure of the equilibrium, i.e. $s_{i1}^{\text{FBN}}(t,c) = a_i'(t)c + v_i(t)$ and $s_{i2}^{\text{FBN}}(t,c)) = b_i'(t)c + w_i(t)$, and performining maximization in (1.17), we get

$$\begin{aligned} s_{i1}^{\text{FBN}}(t,c) &= \frac{p - c_i - \sum_{j\neq i} s_{j1}^{\text{FBN}}(t,c)}{2}, \\ s_{i2}^{\text{FBN}}(t,c) &= \frac{-1}{\rho^t\varepsilon_i(t)}\Big[\Big(\delta c - \sum_{j\in\mathcal{N}}\mu_j(t)s_{j2}^{\text{FBN}}(t,c)\Big)'K_i(t+1) + k_i'(t+1)\Big]\mu_i(t), \end{aligned}$$

or

$$\begin{aligned} a_i'(t)c + v_i(t) &= \frac{p - c_i - \sum_{j\neq i}(a_j'(t)c + v_j(t))}{2}, \\ b_i'(t)c + w_i(t) &= \frac{-1}{\rho^t\varepsilon_i(t)}\Big[\Big(\delta c - \sum_{j\in\mathcal{N}}\mu_j(t)(b_j'(t)c + w_j(t))\Big)'K_i(t+1) + k_i'(t+1)\Big]\mu_i(t). \end{aligned}$$

For each firm and each non-terminal period, the equation (1.17) allows the following representation:

$$V_i(t,c) = \max_{(u_i(t),y_i(t))\in\mathbb{U}_i\times\mathbb{Y}_i}\left[\begin{pmatrix}\rho^t p \\ 0\end{pmatrix}'\begin{pmatrix}u_i(t) \\ y_i(t)\end{pmatrix} - c'\begin{pmatrix}\rho^t\mathbf{e}_i & 0\end{pmatrix}\begin{pmatrix}u_i(t) \\ y_i(t)\end{pmatrix} - \right.$$

$$-\frac{1}{2}\begin{pmatrix}u_i(t)\\ y_i(t)\end{pmatrix}'\begin{pmatrix}2\rho^t & 0\\ 0 & \rho^t\varepsilon_i(t)\end{pmatrix}\begin{pmatrix}u_i(t)\\ y_i(t)\end{pmatrix}-\sum_{j\neq i}s_j^{\text{FBN}}(t,c)'\begin{pmatrix}\rho^t & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_i(t)\\ y_i(t)\end{pmatrix}+$$

$$\left.+V_i\left(t+1,\,\delta c-\begin{pmatrix}0 & \mu_i(t)\end{pmatrix}\begin{pmatrix}u_i(t)\\ y_i(t)\end{pmatrix}-\sum_{j\neq i}\begin{pmatrix}0 & \mu_j(t)\end{pmatrix}s_j^{\text{FBN}}(t,c)\right)\right].$$

Using the linear representation of firms' strategies and the quadratic form of the function V_i, we write out a matrix of quadratic form for an expression enclosed in square brackets:

$$\begin{pmatrix}-\rho^t & 0\\ 0 & \frac{1}{2}\big(\mu_i'(t)K_i(t+1)\mu_i(t)-\rho^t\varepsilon_i(t)\big)\end{pmatrix}.$$

Due to the conditions of the theorem, this matrix is negatively defined, which ensures the uniqueness of the solution of the corresponding maximization problem. Using the method of undetermined coefficients for all $i\in\mathcal{N}$ and $t\in\mathcal{T}\setminus\{T\}$, we obtain the system (1.12) – (1.13), the unique solution of which is relative to $a_i(t)$, $v_i(t)$, $b_i(t)$ and $w_i(t)$ ensures the uniqueness of the Nash equilibrium [47].

Since $F_i(t,\cdot)=\big(s_{i1}^{\text{FBN}}(t,c)\big)^2-\frac{\varepsilon_i(t)}{2}\big(s_{i2}^{\text{FBN}}(t,c)\big)^2$, the equation (1.17), taking into account the form of the function $V_i(t,c)$ and the equilibrium behavior, we rewrite as

$$V_i(t,c)=\rho^t\left[\big(s_{i1}^{\text{FBN}}(t,c)\big)^2-\frac{\varepsilon_i(t)}{2}\big(s_{i2}^{\text{FBN}}(t,c)\big)^2\right]+V_i\Big(t+1,\delta c-\sum_{j\in\mathcal{N}}\mu_j(t)s_{j2}^{\text{FBN}}(t,c)\Big)$$

or

$$\frac{1}{2}c'K_i(t)c+k_i'(t)c+\kappa_i(t)=\rho^t\left[\big(a_i'(t)c+v_i(t)\big)^2-\frac{\varepsilon_i(t)}{2}\big(b_i'(t)c+w_i(t)\big)^2\right]+$$

$$+\left[\frac{1}{2}\Big(\delta c-\sum_{j\in\mathcal{N}}\mu_j(t)\big(b_j'(t)c+w_j(t)\big)\Big)'K_i(t+1)+k_i'(t+1)\right]\times$$

$$\times\left(\delta c-\sum_{j\in\mathcal{N}}\mu_j(t)\big(b_j'(t)c+w_j(t)\big)\right)+\kappa_i(t+1).$$

Determining the unknown coefficients in the quadratic and linear parts, as well as in the summand independent of c, we obtain the relations (1.14), (1.15), and (1.16).

Since $V_i(t, c)$ represents the profit of the firm $i \in \mathcal{N}$ in the Nash equilibrium in Γ^{ex}, starting at time t with a set of unit costs c, then $\mathcal{J}_i\left(s^{\text{FBN}}\right) = V_i(0, c_0)$ for $i \in \mathcal{N}$. $\square$

Remark 1.1 (to Theorem 1.2)**.** *In the Nash equilibrium, the production volumes* $s_{i1}^{\text{FBN}}(t, c)$ *depend only on the set of firms' costs and not on the game period, while the investment* $s_{i2}^{\text{FBN}}(t, c)$ *depends both on the set of firms' costs and on the game period.*

1.3. Numerical simulations and comparative analysis

Let us move on to the results of the thesis that can be obtained using Theorems 1.1 – 1.2. To make a comparative analysis of the results obtained, we will keep the common input parameters, namely $T = 3$, $p = 500$, $\varepsilon_i(t) = 1\,000$ and $\eta_i = 100\,000$ are assumed to be the same for all firms, $\eta = 1\,000$; technological obsolescence parameter $\delta = 1.07$; the initial costs of the firms are also assumed to be the same and equal to $c_{i0} = 100$, the discount factor $\rho = 0.95$; the network parameters $\alpha_i(t) = 1.8$, $\beta_{ij}(t) = 1$, $\gamma_{ij}(t) = 0.5$ are also the same for all firms and are constant over time. Consider the case of the interaction of three and four firms ($n = 3$ or $n = 4$) and find their equilibrium behavior for four different networks $\mathbf{g}_j$ which have a constant structure in time, i.e. for each $j \in \{1, 2, 3, 4\}$ we have that $\mathbf{g}_j(0) = \mathbf{g}_j(1) = \mathbf{g}_j(2)$. We will call such networks permanent and identify them with their network structure.

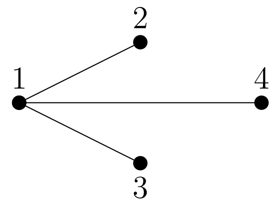
Tables 1.1 – 1.2 show the values of current equilibrium production $u_i^{\text{OLN}}(t) = s_{i1}^{\text{OLN}}(t, c_0)$, investment $y_i^{\text{OLN}}(t) = s_{i2}^{\text{OLN}}(t, c_0)$ and costs $c_i^{\text{OLN}}(t)$ for each firm $i \in \mathcal{N}$ at open-loop Nash equilibrium $s^{\text{OLN}}(t, c_0)$. The Table also show additional results: equilibrium profits $J_i^{\text{OLN}} := J_i(c_0, u^{\text{OLN}}, y^{\text{OLN}}) = \mathcal{J}_i(s^{\text{OLN}})$ and the current market price, which is formed according to the linear inverse demand function: In the Nash equilibrium, the market price is determined by the value $P^{\text{OLN}}(t) := p - \sum_{j \in \mathcal{N}} u_j^{\text{OLN}}(t)$, $t \in \mathcal{T} \setminus \{T\}$.

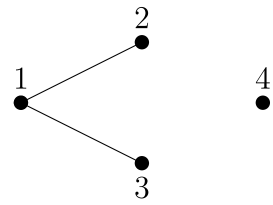
Tables 1.3 – 1.4 show the same values as in Tables 1.1 – 1.2, but for the feedback Nash equilibrium $s^{\text{FBN}}(t, c)$.

All values presented in this chapter are rounded to third decimal place.

Analyzing the data presented in Tables 1.1 – 1.4, we can conclude that for the considered network structures ($\mathbf{g}_1$ and $\mathbf{g}_4$ are star graphs, $\mathbf{g}_3$ and $\mathbf{g}_4$ are linear

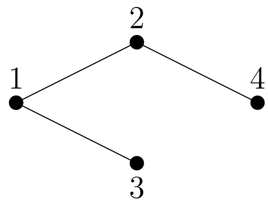
Table 1.1. **Nash equilibrium s^{OLN} in model Γ^{ex} for network structures $\mathbf{g}_1$ and $\mathbf{g}_2$, as well as the corresponding firm profits and prices.**

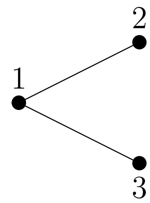




	Network $\mathbf{g}_1$				Network $\mathbf{g}_2$			
	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$u_1^{\mathrm{OLN}}(t)$	80.000	81.793	83.448	–	80.000	81.181	82.230	–
$u_2^{\mathrm{OLN}}(t)$	80.000	79.744	79.375	–	80.000	80.155	80.192	–
$u_3^{\mathrm{OLN}}(t)$	80.000	79.744	79.375	–	80.000	80.155	80.192	–
$u_4^{\mathrm{OLN}}(t)$	80.000	79.744	79.375	–	80.000	79.123	78.144	–
$y_1^{\mathrm{OLN}}(t)$	2.052	1.881	1.710	–	2.049	1.879	1.710	–
$y_2^{\mathrm{OLN}}(t)$	2.041	1.874	1.710	–	2.043	1.875	1.710	–
$y_3^{\mathrm{OLN}}(t)$	2.041	1.874	1.710	–	2.043	1.875	1.710	–
$y_4^{\mathrm{OLN}}(t)$	2.041	1.874	1.710	–	2.038	1.872	1.710	–
$c_1^{\mathrm{OLN}}(t)$	100.000	97.183	94.978	93.419	100.000	98.207	97.013	96.451
$c_2^{\mathrm{OLN}}(t)$	100.000	99.233	99.051	99.487	100.000	99.233	99.051	99.486
$c_3^{\mathrm{OLN}}(t)$	100.000	99.233	99.051	99.487	100.000	99.233	99.051	99.486
$c_4^{\mathrm{OLN}}(t)$	100.000	99.233	99.051	99.487	100.000	100.264	101.098	102.532
$P^{\mathrm{OLN}}(t)$	180.000	178.976	178.426	–	180.000	179.387	179.243	–
J_1^{OLN}	19 577.792				16 711.687			
J_2^{OLN}	13 496.491				13 669.861			
J_3^{OLN}	13 496.491				13 669.861			
J_4^{OLN}	13 496.491				10 627.026			

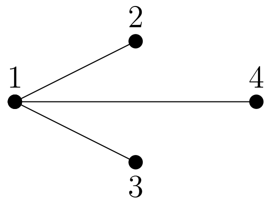
Table 1.2. **Nash equilibrium s^{OLN} in model Γ^{ex} for network structures $\mathbf{g}_3$ and $\mathbf{g}_4$, as well as the corresponding firm profits and prices.**

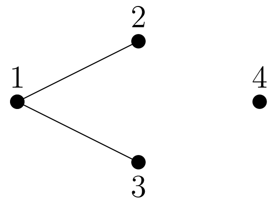




	Network $\mathbf{g}_3$				Network $\mathbf{g}_4$			
	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$u_1^{\mathrm{OLN}}(t)$	80.000	80.769	81.413	–	100.000	100.787	101.384	–
$u_2^{\mathrm{OLN}}(t)$	80.000	80.769	81.413	–	100.000	99.727	99.294	–
$u_3^{\mathrm{OLN}}(t)$	80.000	79.742	79.373	–	100.000	99.727	99.294	–
$u_4^{\mathrm{OLN}}(t)$	80.000	79.742	79.373	–	–	–	–	–
$y_1^{\mathrm{OLN}}(t)$	2.047	1.877	1.710	–	2.115	1.912	1.710	–
$y_2^{\mathrm{OLN}}(t)$	2.047	1.877	1.710	–	2.110	1.908	1.710	–
$y_3^{\mathrm{OLN}}(t)$	2.041	1.874	1.710	–	2.110	1.908	1.710	–
$y_4^{\mathrm{OLN}}(t)$	2.041	1.874	1.710	–	–	–	–	–
$c_1^{\mathrm{OLN}}(t)$	100.000	98.208	97.015	96.459	100.000	98.972	98.643	99.050
$c_2^{\mathrm{OLN}}(t)$	100.000	98.208	97.015	96.459	100.000	100.031	100.734	102.142
$c_3^{\mathrm{OLN}}(t)$	100.000	99.235	99.056	99.492	100.000	100.031	100.734	102.142
$c_4^{\mathrm{OLN}}(t)$	100.000	99.235	99.056	99.492	–	–	–	–
$P^{\mathrm{OLN}}(t)$	180.000	178.977	178.428	–	200.000	199.759	200.028	–
J_1^{OLN}	16 532.843				24 448.066			
J_2^{OLN}	16 532.843				21 234.905			
J_3^{OLN}	13 491.668				21 234.905			
J_4^{OLN}	13 491.668				–			

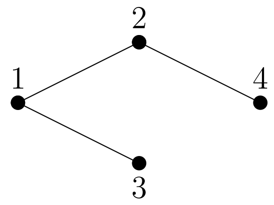
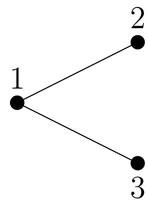
Table 1.3. **Nash equilibrium s^{FBN} in model Γ^{ex} for network structures $\mathbf{g}_1$ and $\mathbf{g}_2$, as well as the corresponding firm profits and prices.**





	Network $\mathbf{g}_1$				Network $\mathbf{g}_2$			
	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$u_1^{\mathrm{FBN}}(t)$	80.000	81.808	83.471	–	80.000	81.202	82.263	–
$u_2^{\mathrm{FBN}}(t)$	80.000	79.765	79.409	–	80.000	80.178	80.229	–
$u_3^{\mathrm{FBN}}(t)$	80.000	79.765	79.409	–	80.000	80.178	80.229	–
$u_4^{\mathrm{FBN}}(t)$	80.000	79.765	79.409	–	80.000	79.179	78.231	–
$y_1^{\mathrm{FBN}}(t)$	2.033	1.871	1.710	–	2.061	1.885	1.710	–
$y_2^{\mathrm{FBN}}(t)$	2.084	1.895	1.710	–	2.086	1.897	1.710	–
$y_3^{\mathrm{FBN}}(t)$	2.084	1.895	1.710	–	2.086	1.897	1.710	–
$y_4^{\mathrm{FBN}}(t)$	2.084	1.895	1.710	–	2.110	1.908	1.710	–
$c_1^{\mathrm{FBN}}(t)$	100.000	97.089	94.831	93.261	100.000	98.061	96.785	96.207
$c_2^{\mathrm{FBN}}(t)$	100.000	99.132	98.893	99.380	100.000	99.085	98.819	99.238
$c_3^{\mathrm{FBN}}(t)$	100.000	99.132	98.893	99.380	100.000	99.085	98.819	99.238
$c_4^{\mathrm{FBN}}(t)$	100.000	99.132	98.893	99.380	100.000	100.084	100.817	102.231
$P^{\mathrm{FBN}}(t)$	180.000	178.897	178.302	–	180.000	179.263	179.048	–
J_1^{FBN}	19 774.075				16 890.994			
J_2^{FBN}	13 523.290				13 764.255			
J_3^{FBN}	13 523.290				13 764.255			
J_4^{FBN}	13 523.290				10 691.319			

Table 1.4. **Nash equilibrium s^{FBN} in model Γ^{ex} for network structures $\mathbf{g}_3$ and $\mathbf{g}_4$, as well as the corresponding firm profits and prices.**

	Network $\mathbf{g}_3$ 				Network $\mathbf{g}_4$ 			
	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$u_1^{\mathrm{FBN}}(t)$	80.000	80.782	81.434	–	100.000	100.780	101.374	–
$u_2^{\mathrm{FBN}}(t)$	80.000	80.782	81.434	–	100.000	99.745	99.322	–
$u_3^{\mathrm{FBN}}(t)$	80.000	79.773	79.420	–	100.000	99.745	99.322	–
$u_4^{\mathrm{FBN}}(t)$	80.000	79.773	79.420	–	–	–	–	–
$y_1^{\mathrm{FBN}}(t)$	2.059	1.884	1.710	–	2.096	1.902	1.710	–
$y_2^{\mathrm{FBN}}(t)$	2.059	1.884	1.710	–	2.139	1.922	1.710	–
$y_3^{\mathrm{FBN}}(t)$	2.084	1.895	1.710	–	2.139	1.922	1.710	–
$y_4^{\mathrm{FBN}}(t)$	2.084	1.895	1.710	–	–	–	–	–
$c_1^{\mathrm{FBN}}(t)$	100.000	98.108	96.859	96.286	100.000	98.949	98.608	99.012
$c_2^{\mathrm{FBN}}(t)$	100.000	98.108	96.859	96.286	100.000	99.985	100.661	102.064
$c_3^{\mathrm{FBN}}(t)$	100.000	99.118	98.872	99.295	100.000	99.985	100.661	102.064
$c_4^{\mathrm{FBN}}(t)$	100.000	99.118	98.872	99.295	–	–	–	–
$P^{\mathrm{FBN}}(t)$	180.000	178.890	178.292	–	200.000	199.730	199.982	–
J_2^{FBN}	16 643.707				24 535.499			
J_2^{FBN}	16 643.707				21 223.474			
J_3^{FBN}	13 545.573				21 223.474			
J_4^{FBN}	13 545.573				–			

graphs, $\mathbf{g}_4$ there is a disconnected graph with an isolated vertex), the use of feedback strategies, that is, when firms adapt to the current costs of competitors and not only to the initial costs, allows all firms to reduce their costs in equilibrium. For the production behavior of firms in equilibrium, the following observation seems obvious.

Remark 1.2. *According to Theorems 1.1 – 1.2, the firm's production plans are $i \in \mathcal{N}$, i.e., $s_{i1}^{\mathrm{FBN}}(t,c)$ and $s_{i1}^{\mathrm{OLN}}(t,c_0)$, have the same functional form.*

At the same time, the number of competing firms in the market can play a key role. When there are four firms in the market, the feedback Nash equilibrium gives firms better profits compared to their profits in open-loop Nash equilibrium. In addition, when using feedback strategies, firms in equilibrium produce more at any given time (this is no longer the case when there are three firms in the market).

Note that the initial conditions for all firms were the same, and the only difference that broke the symmetry based on Tables 1.1 – 1.4 was their position in the network, i.e., it was determined by the corresponding network structure. This leads to the necessity of analyzing the results of numerical simulations in accordance with the peculiarities of $\mathbf{g}_j$.

1.4. The impact of network parameters and structures on equilibrium, equilibrium profits and externalities

Let us make a comparative analysis of the results presented in Tables 1.1 – 1.4. Note that all observations given in this section were made under the condition of asymmetry of firms in the network structure — to assess the influence of the network structure on the equilibrium behavior of firms and other indicators obtained under equilibrium conditions.

The only aspect in which symmetry between firms break for model Γ^{ex} were networks $\mathbf{g}_j$, with $j \in \{1, 2, 3, 4\}$. As a result, the data in Tables 1.1 – 1.4 allow us to assess the impact of the exogenous network structure on a number of important indicators obtained during the implementation of Nash equilibrium. At the same time, standard graph theory structures have been used for equilibrium modeling: linear (networks $\mathbf{g}_3$ and $\mathbf{g}_4$) and star (networks $\mathbf{g}_1$ and $\mathbf{g}_4$), and the case of a structure with one «isolated» firm (network $\mathbf{g}_2$) is also shown separately.

1.4.1. Equilibrium behavior

Based on Tables 1.1 – 1.4, a central difference in the investment behavior of firms can be observed when implementing Nash equilibrium.

Observation 1.1. *Let* $|\mathcal{N}_i(\mathbf{g})| > |\mathcal{N}_j(\mathbf{g})|$ *for* $c_{i0} = c_{j0}$ *for a pair of firms* $i, j \in \mathcal{N} : i \neq j$ *in a constant network* $\mathbf{g}$*, then*

- *in open-loop Nash equilibrium:* $u_i^{\mathrm{OLN}}(t) \geqslant u_j^{\mathrm{OLN}}(t)$, $y_i^{\mathrm{OLN}}(t) \geqslant y_j^{\mathrm{OLN}}(t)$*, then firm* i *produces and invests no less than firm* j*;*
- *in feedback Nash equilibrium:* $u_i^{\mathrm{FBN}}(t) \geqslant u_j^{\mathrm{FBN}}(t)$*, но* $y_i^{\mathrm{FBN}}(t) \leqslant y_j^{\mathrm{FBN}}(t)$*, then firm* i *produces and invests no more than firm* j*.*

This observation can be explained by the specifics of the strategies implemented, open-loop strategies as functions of time and feedback strategies as functions of time and the current state of unit costs.

Preposition 1.1. *In a market with a constant number of firms, regardless of their network and class of strategies, the firm with lower costs will produce a higher volume in Nash equilibrium.*

Proof. For the sake of generality, superscript «N» indicates the production behavior and costs of firms in Nash equilibrium (both open-loop and feedback). According to Remark 1.2 and Theorems 1.1 – 1.2 we have that

$$u_i^{\mathrm{N}}(t) = \frac{1}{n+1}\left(p - (n+1)c_i^{\mathrm{N}}(t) + \sum_{j=1}^{n} c_j^{\mathrm{N}}(t)\right)$$

for any firm $i \in \mathcal{N}$ and $t \in \mathcal{T} \setminus \{T\}$. Then from the inequality $c_i^{\mathrm{N}}(t) < c_j^{\mathrm{N}}(t)$, where $j \in \mathcal{N} \setminus \{i\}$, it follows that

$$u_i^{\mathrm{N}}(t) - u_j^{\mathrm{N}}(t) = c_j^{\mathrm{N}}(t) - c_i^{\mathrm{N}}(t) > 0, \text{ hence } u_i^{\mathrm{N}}(t) > u_j^{\mathrm{N}}(t).$$

For any pair of firms, the firm with lower unit costs in equilibrium produces a larger quantity of goods. This is because the cost advantage allows the firm to produce more efficiently. □

Preposition 1.2. *The Nash equilibrium behavior of each firm* $i \in \mathcal{N}$ *(both open-loop and feedback) has a nonlinear dependence on the network parameters* $\alpha_i(t)$*,* $\beta_{ij}(t)$*, and* $\gamma_{ij}(t)$*, where* $t \in \mathcal{T} \setminus \{T\}$*,* $j \in \mathcal{N} \setminus \{i\}$*.*

The validity of this statement follows from the functional type of equilibrium behavior of firms, presented in Theorems 1.1 – 1.2. As will be shown in the next subsection, this nonlinear dependence allows a good linear approximation for small values of the network parameters.

1.4.2. The price of goods like externality and equilibrium profits

Competition is usually non-trivial and often has effects that affect subjects not directly involved in the process. Let us agree to call such effects external to the competing parties. For the competition models studied in this thesis, the external effect can be the price of the product, since in this case the price is determined according to the current production behavior of the firms, and not by the consumers themselves. Thus, the price of a product to consumers depends on the competitive position of firms in the market.

Next, three options for the network are considered separately. By analyzing the transition from $\mathbf{g}_1$ to $\mathbf{g}_2$, one can understand the importance of a connection in a star, from $\mathbf{g}_3$ to $\mathbf{g}_2$ — the importance of a connection in a linear network when excluding interactions leads to an isolated firm, and finally, an analysis of the transition from $\mathbf{g}_2$ to $\mathbf{g}_4$ will show the effect of the number of firms in the market with a linear network. Note the effect of adding new connections to the network or removing existing ones on the firm's profit in Nash equilibrium. Understanding such effects will allow the firm, focusing on its profits, to review the current structure of relationships with its competitors — in the case when the network is not exogenous. At the same time, with an exogenous network, such effects can serve as a guideline for stabilizing competition and the market value of goods for consumers.

Observation 1.2. *For any pair of competing firms $i, j \in \mathcal{N} : i \neq j$, it follows from the condition that $|\mathcal{N}_i^{\mathrm{N}}(\mathbf{g})| > |\mathcal{N}_j^{\mathrm{N}}(\mathbf{g})|$ that $J_i^{\mathrm{N}} > J_j^{\mathrm{N}}$.*

In other words, of the two firms, the one with the most direct neighbors has the highest profit (all else being equal). This conclusion is based on a direct comparison of the profits, whose values are shown in the following Tables 1.1 – 1.4.

Observation 1.3. *The profitability of a firm can be influenced by all connections within the network, even those in which the firm is not directly involved. Firms that* *are directly involved in creating or removing a connection experience a significant change in profits.*

This can be seen from the data presented in Table 1.5, where $\Delta J_i^{\mathrm{OLN}}$ and $\Delta J_i^{\mathrm{FBN}}$ show the relative change in the profit of firm $i \in \mathcal{N}$, expressed as a percentage, when the network changes, when firms adhere to open-loop and feedback equilibria, respectively. For example, when switching from the $\mathbf{g}_2$ network to the $\mathbf{g}_3$ network, firms 2 and 4 (which establish a connection between themselves) receive a noticeable increase in profits, while firms 1 and 3 have slightly reduced profits.

Table 1.5. **Relative sensitivity of profits to network changes (%)**

Change	$\Delta J_1^{\mathrm{OLN}}$	$\Delta J_2^{\mathrm{OLN}}$	$\Delta J_3^{\mathrm{OLN}}$	$\Delta J_4^{\mathrm{OLN}}$	$\Delta J_1^{\mathrm{FBN}}$	$\Delta J_2^{\mathrm{FBN}}$	$\Delta J_3^{\mathrm{FBN}}$	$\Delta J_4^{\mathrm{FBN}}$
$\mathbf{g}_1 \to \mathbf{g}_2$	–14.640	1.285	1.285	–21.261	–14.580	1.782	1.782	–20.941
$\mathbf{g}_2 \to \mathbf{g}_1$	17.150	–1.268	–1.268	27.002	17.069	–1.751	–1.751	26.489
$\mathbf{g}_2 \to \mathbf{g}_3$	–1.070	20.944	–1.304	26.956	–1.464	20.920	–1.589	26.697
$\mathbf{g}_3 \to \mathbf{g}_2$	1.082	–17.317	1.321	–21.233	1.486	–17.301	1.614	–21.072
$\mathbf{g}_2 \to \mathbf{g}_4$	46.293	55.341	55.341	–	45.258	54.193	54.193	–
$\mathbf{g}_4 \to \mathbf{g}_2$	–31.644	–35.626	–35.626	–	–31.157	–35.146	–35.146	–

Observation 1.4. *The removal of the connection resulted in a greater increase in profits for firms whose neighbors did not lose the connection.*

For example, when moving $\mathbf{g}_3 \to \mathbf{g}_2$ from the network $\mathbf{g}_3$ to $\mathbf{g}_2$, we have $\Delta J_3^{\mathrm{FBN}} > \Delta J_1^{\mathrm{FBN}}$.

Observation 1.5. *Firms may benefit from reducing the number of competitors in the market and striving to capture a larger share of sales, potentially leading to a monopoly.*

The largest effect on the profit growth of firms 1, 2, and 3 is observed during the transition from $\mathbf{g}_2 \to \mathbf{g}_4$, which can also be interpreted as the exit of firm 4 from the market. And in the opposite direction: when a new firm enters the market, the profits of existing firms will decrease significantly, and their entry may be blocked.

Let us move on to the current value of the external effect (the price of the product) that results from adding new links to the network or removing existing ones when firms adhere to the Nash equilibrium. As noted earlier, understanding such effects allows us to infer the impact of the structure of firms' relationships on the cost of products to consumers. Using data from Tables 1.1 – 1.4, we move on to the next observation.

Observation 1.6. *The current market price in the case of firms implementing feedback Nash equilibrium turns out to be slightly lower than the corresponding price in the case of firms implementing open-loop Nash equilibrium.*

Next, based on the results of the numerical simulations, we calculate the relative changes in the current price, expressed as a percentage, when the network changes, if the firms remain in equilibrium. The corresponding values are denoted by $\Delta P^{\mathrm{OLN}}(t)$ and $\Delta P^{\mathrm{FBN}}(t)$ and are given in Table 1.6 below.

Table 1.6. **Sensitivity of current equilibrium prices to network changes (relative change, %)**

Change	$\Delta P^{\mathrm{OLN}}(0)$	$\Delta P^{\mathrm{OLN}}(1)$	$\Delta P^{\mathrm{OLN}}(2)$	$\Delta P^{\mathrm{FBN}}(0)$	$\Delta P^{\mathrm{FBN}}(1)$	$\Delta P^{\mathrm{FBN}}(2)$
$\mathbf{g}_1 \to \mathbf{g}_2$	0.000	0.230	0.457	0.000	0.205	0.418
$\mathbf{g}_2 \to \mathbf{g}_1$	0.000	–0.229	–0.455	0.000	–0.204	–0.417
$\mathbf{g}_2 \to \mathbf{g}_3$	0.000	–0.228	–0.454	0.000	–0.208	–0.422
$\mathbf{g}_3 \to \mathbf{g}_2$	0.000	0.229	0.456	0.000	0.208	0.424
$\mathbf{g}_2 \to \mathbf{g}_4$	11.111	11.356	11.596	11.111	11.417	11.692
$\mathbf{g}_4 \to \mathbf{g}_2$	–10.000	–10.198	–10.391	–10.000	–10.247	–10.468

Observation 1.7. *If the number of participants in the market remains the same, the network structure has no significant effect on the current price. If the number of participants in the market decreases (increases), it leads to a significant increase (decrease) in the current price of the product.*

The observation is based on the data in Table 1.6: in the case of $n = 4$, the change in current prices in equilibrium does not exceed 0.5 % when the network structure changes; the transition from network $\mathbf{g}_2$ to $\mathbf{g}_4$, i.e. the exit of firm 4 from the market, allows the remaining firms to increase current product prices by at least 11 %, regardless of the class of strategies. Note also that as the number of firms decreases (increases), the total quantity of goods produced in the market in Nash equilibrium decreases (increases), which follows from the nature of the inverse demand function.

Now let us evaluate the role of network parameters in the Nash equilibrium. Let us identify the «type» of a firm with the number of its direct neighbours in the network. Let us present the types of firms for the networks in question in the Table 1.7. Obviously, changing the value of a network parameter can have different effects on the behavior, costs, and profits of different types of firms. In particular, the effect of changing the parameter $\alpha_i(t)$ on the costs of firm $i \in \mathcal{N}$ can be estimated for any network structure, since it is the same for each firm and does not depend on the network structure. However, changing the parameter $\beta_{ij}(t)$ or $\gamma_{ij}(t)$ with $j \in \mathcal{N} \setminus \{i\}$ may affect the costs of different types of firms in different ways. Therefore, it seems reasonable to consider the impact of changes in network parameters on the profits of firms for networks $\mathbf{g}_1$ and $\mathbf{g}_2$. Based on the data from Table 1.1, we will present the changes in the form of graphs[3]. The dependence of the firms' profits in the open-loop equilibrium on changes in the network coefficients is shown in Figure. 1.1.

Table 1.7. **Types of firms in model Γ^{ex} for the considered exogenous networks**

$i \setminus \lvert\mathcal{N}_i(\mathbf{g}_j)\rvert$	$\lvert\mathcal{N}_i(\mathbf{g}_1)\rvert$	$\lvert\mathcal{N}_i(\mathbf{g}_2)\rvert$	$\lvert\mathcal{N}_i(\mathbf{g}_3)\rvert$	$\lvert\mathcal{N}_i(\mathbf{g}_4)\rvert$
1	3	2	2	2
2	1	1	2	1
3	1	1	1	1
4	1	0	1	-

Remark 1.3. *The profit function is nonlinear from the network parameters in equilibrium, and according to the Figure 1.1 the profits of firms in equilibrium allow a linear approximation.*

[3] Constructing similar graphs for feedback Nash equilibrium according to Table 1.3 leads to similar conclusions (Remark 1.4), in structure of which we will limit ourselves only for open-loop Nash equilibrium.

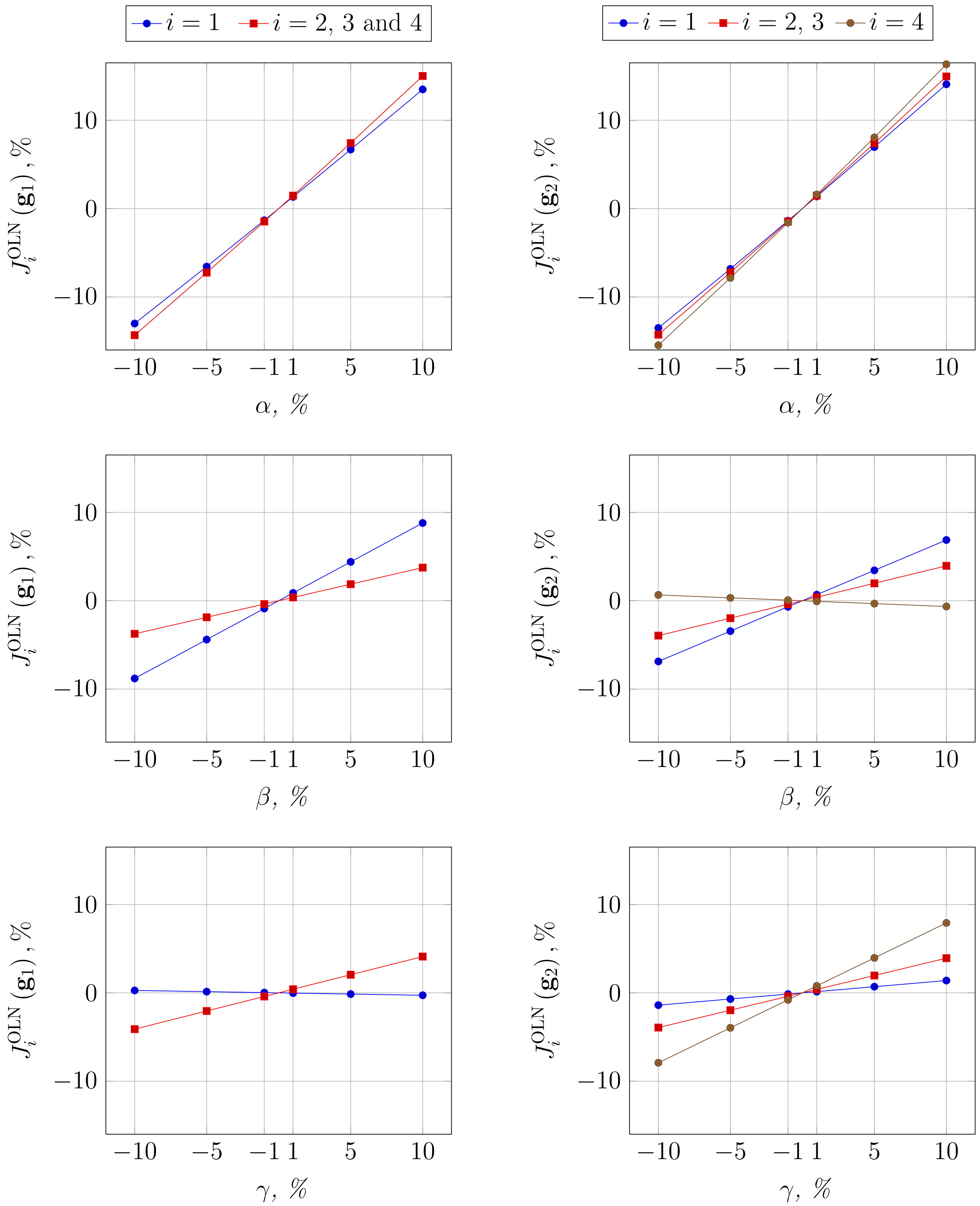


Figure 1.1. **The effect of changing the value of the network parameter on the profits of firms, in percentage terms**

Remark 1.4. *The sensitivity of the firm to a change in the value of the parameter $\alpha_i(t)$ or $\beta_{ij}(t)$ turns out to be directly proportional to its type.*

Given the equation (1.2), this remark seems natural. A similar situation is observed when, in the above remark, an isolated firm is replaced by a firm of maximum type and the parameter $\beta_{ij}(t)$ is replaced by $\gamma_{ij}(t)$.

1.4.3. Dynamics of competitiveness

Considering the competitiveness of each firm $i \in \mathcal{N}$ as the ability to outperform its competitors in profit — is typical for game-theoretic problems. Note that the competitiveness of each firm in model Γ^{ex} can be reduced to an assessment of the state of its unit costs relative to the unit costs of its competitors. Since it is the state of a firm's unit costs that determines its cost of producing a unit of goods, given equal unit costs, a firm with lower unit costs produces more. At the same time, regardless of the class of strategies — according to Remark 1.2 and the formula (1.10), the production behavior of each firm in equilibrium functionally depends on the state of unit costs of firms, further determining the current market price. The validity of the above consideration is confirmed by the results of numerical simulations (see Tables 1.1–1.4). It should also be noted that assessing the competitiveness of firms through the state of their unit costs, rather than through profit, will further generalize the results to the case when the network is endogenous, and the expression of the firm's profit will include the cost of its existing network connections.

Let us evaluate in Table 1.8 the sensitivity of the firm's unit cost in Nash equilibrium to changes in the network structure based on the data from Tables 1.1 – 1.4, from which it is easy to see that the type of firm in equilibrium plays a leading role in the dynamics of firms' competitiveness.

Table 1.8. **Sensitivity of current equilibrium costs to network changes (relative change, %)**

Change	Firm i	$\Delta c_i^{\mathrm{OLN}}(1)$	$\Delta c_i^{\mathrm{OLN}}(2)$	$c_i^{\mathrm{OLN}}(3)$	$\Delta c_i^{\mathrm{FBN}}(1)$	$\Delta c_i^{\mathrm{FBN}}(2)$	$\Delta c_i^{\mathrm{FBN}}(3)$
$\mathbf{g}_1 \to \mathbf{g}_2$	1	1.053	2.142	3.245	1.002	2.061	3.159
	2	–0.000	–0.000	–0.000	–0.048	–0.075	–0.080
	3	–0.000	–0.000	–0.000	–0.048	–0.075	–0.080
	4	1.039	2.067	3.061	0.961	1.945	2.934
$\mathbf{g}_2 \to \mathbf{g}_1$	1	–1.042	–2.097	–3.143	–0.992	–2.019	–3.062
	2	0.000	0.000	0.000	0.048	0.075	0.080
	3	0.000	0.000	0.000	0.048	0.075	0.080
	4	–1.028	–2.025	–2.970	–0.952	–1.908	–2.850
$\mathbf{g}_2 \to \mathbf{g}_3$	1	0.001	0.002	0.002	0.048	0.076	0.082
	2	–1.033	–2.056	–3.049	–0.986	–1.984	–2.975
	3	0.003	0.005	0.005	0.033	0.053	0.057
	4	–1.026	–2.020	–2.966	–0.966	–1.930	–2.873
$\mathbf{g}_3 \to \mathbf{g}_2$	1	–0.001	–0.002	–0.002	–0.048	–0.076	–0.082
	2	1.043	2.099	3.145	0.995	2.024	3.067
	3	–0.003	–0.005	–0.005	–0.033	–0.053	–0.057
	4	1.036	2.062	3.056	0.975	1.968	2.958
$\mathbf{g}_2 \to \mathbf{g}_4$	1	0.779	1.681	2.695	0.905	1.883	2.916
	2	0.805	1.699	2.669	0.908	1.863	2.847
	3	0.805	1.699	2.669	0.908	1.863	2.847
	4	–	–	–	–	–	–
$\mathbf{g}_4 \to \mathbf{g}_2$	1	–0.773	–1.653	–2.624	–0.897	–1.848	–2.833
	2	–0.799	–1.670	–2.600	–0.900	–1.829	–2.768
	3	–0.799	–1.670	–2.600	–0.900	–1.829	–2.768
	4	–	–	–	–	–	–

Observation 1.8. *In feedback Nash equilibrium, a firm's costs are less sensitive to a break of connection in which it participates than in open-loop Nash equilibrium. In open-loop NE (Nash equilibrium), a firm's costs are more sensitive to a break of connection in which it does not participate than in feedback NE.*

In fact, it is sufficient to observe the transitions from one network to another in which a link has been removed: from $\mathbf{g}_1$ to $\mathbf{g}_2$ and from $\mathbf{g}_3$ to $\mathbf{g}_2$. Similar to Observation 1.8, it is possible to formulate a similar result in the opposite direction, replacing the removal of a connection with its addition.

Observation 1.9. *Each firm should strive to have the highest possible type in the network, while also ensuring that other firms have the lowest possible type. This benefits the entire network.*

The network $\mathbf{g}_2$ is more profitable for firm 3 than $\mathbf{g}_1$, in which it is asymmetric to firm 2 — due to the fact that they have different types in these networks. It may seem that for firm 3 there can be no preference between networks $\mathbf{g}_1$ and $\mathbf{g}_2$, since in both its type is preserved and there is exactly one connection in which firm 3 does not participate — (2; 4) and (1; 4), respectively. However, in the network $\mathbf{g}_1$ the connection (2; 4) leads to two firms of the second type, and in the network $\mathbf{g}_2$ to one firm of the second type. This means that it is more profitable for each firm to have the largest number of firms of a lower type than its own. For the same reasons, firm 4 benefits from the network $\mathbf{g}_2$ in which its connection does not increase the number of firms with a higher type than it has.

1.5. Conclusions to Chapter 1

In the dynamic model of competition with exogenous network formation Γ^{ex}, a Nash equilibrium is obtained for two variants of the information structure — open-loop and feedback. The uniqueness of the Nash equilibrium is also proved for each variant of the information structure. Numerical simulations for several network structures (networks) using the computer program are given [26]. A comparative analysis of the results is performed, which allows us to assess the role and influence of the network structure, as well as network parameters on the behavior of firms, their profits and the external effect in equilibrium.

The main results described in Chapter 1 are presented in the publication [27] together with assumptions such as the restriction of the network parameters — for each firm $i \in \mathcal{N}$: $0 \leqslant \gamma_{ij}(t) < \beta_{ij}(t) < \alpha_i(t)$ with $j \in \mathcal{N} \setminus \{i\}$ and $t \in \mathcal{T} \setminus \{T\}$, the condition on a terminal period: firms exit the market at that moment, earning a profit equal to the market value of their production. However, it is worth noting that for the results presented (uniqueness and functional type of behavior of firms in equilibrium) for model Γ^{ex}, the assumptions on network parameters presented in the publication do not seem to be fundamental. First, the absence of interaction between firms may be more profitable than its presence (the variation in the interpretation of the exogenous network is diverse and depends on the problem under consideration). Second, the effect of a direct neighbor's investment in the network may be greater than the effect of its own investment. Thus, in the case of the effect of substitutability (submodularity), if firms from the set $\mathcal{N}_i\left(\mathbf{g}(t)\right)$ increase the volume of their investments, then firm i may decrease its — reliance on its neighbors in the network structure [66, 69]. Thus, the presented solutions have the property of adaptability to problems with non-positive network parameters (influence coefficients). Moreover, if at the final moment of model Γ^{ex} the condition for firms to exit the market is abandoned and the functional of (1.4) is rewritten in Lagrange form, it is enough to set $\eta_i = \eta = 0$ for each firm $i \in \mathcal{N}$, then the functional type of behavior of firms in the Nash equilibrium remains unchanged. It is known from the theory of optimal control that a functional written in the form of Lagrange can be equivalently represented in the form of Mayer and vice versa, which indicates that the presented methods of solving the model are applicable to various formulations and formal expression of firms' profits with the only condition — preservation of the quadratic form of the functional.

Despite the fact that the model Γ^{ex} is described as economic and mathematical and the results of the analysis are interpreted accordingly, it seems obvious that the results obtained during the study are not limited to application in economics and can be adapted as theoretical (identification or justification of patterns and phenomena), and to practical tasks in other scientific fields, examples of such areas are well described in [33].

Note that the relation (1.2) is adapted to two variants of network interaction of firms — if there is a connection between them in the network and in its absence, at the appropriate time. However, the conditions presented in Theorems 1.1–1.2 remain valid even when the model takes into account a greater variety of network interaction options. For example, it can be assumed that firms $i, j \in \mathcal{N}$ for which $(i, j) \notin \mathbf{g}(t)$ benefit from each other's investments with a coefficient of $\omega^{d_{ij}(\mathbf{g}(t))}(t)$, where $0 < \omega < 1$, and $d_{ij}(\mathbf{g}(t)) > 0$ — the length of the shortest path from firm i to j in network $\mathbf{g}(t)$. In this setting, we get $\mu_{ij}(t, \mathbf{g}(t)) = \omega^{d_{ij}(\mathbf{g}(t))}(t)$ at $i \neq j$, and it will be beneficial for each firm to reduce its distance to each competitor in the network — as a way to avoid decay in positive effects of competitors' investments. At the same time, the choice of strategies by firms in following Theorem 1.1 or Theorem 1.2 — depending on the information structure, allows them to adhere to the Nash equilibrium.

Chapter 2.
Dynamic models with endogenous network formation

Notwithstanding that [14] defines network games to be a rather young line in game theory, a certain portion of the generally accepted classification has already been well established. According to [15, 32, 36] network games can be divided into two classes: games on network and network formation games. According to the presented classification, the model discussed in Chapter 1 is a game on network. This chapter will discuss models in which firms' networking is formed only as part of their strategic behavior. This allows this chapter to be considered as a logical continuation of Chapter 1, where the model Γ^{ex} is to extended by introducing the firms' capability of participating in network formation. The previous analysis is supplemented by the matters of firms' equilibrium multicomponent behavior under various types of their network interaction (Sections 2.1 and 2.4). It is worth noting that games with multicomponent behavior have already been discussed by game theorists, where the work [42] can be exemplified, however numerous issues concerning the principles and conditions for the formation of network interactions still remain unresolved.

This chapter will focus on the extended version of the model Γ^{ex} (Section 2.2) and its variations (Sections 2.3 and 2.4) under the assumption that firms independently form their network structure each decision period, i.e. the network formation procedure becomes endogenous. For the firm's network behavior, conditions will be defined consistently with the interaction formation rule under consideration that ensure the «stability» of network interactions being formed. The chapter will also present the results of numerical simulations and benchmarking (Section 2.5), which will allow firms to assess the prospects of long-term and short-term networking with competitors. For the case of long-term networking between firms, the types of one-time and recurrent network communications costs will also be considered.

2.1. The strategic nature of network behavior and the formalization of network formation rules

Let us consider the possibility of endogenous formation of firms' network interactions, i.e., when each firm's choice of direct neighbors in the network is the result of its strategic behavior. Let us assume that by choosing its network behavior at a non-terminal period, each firm has knowledge of the costs associated with its potential connections. To do this, we define for each period $t \in \mathcal{T} \setminus \{T\}$ the matrix of network costs (or interaction costs of firms) as $\Pi(t) = \{\pi_{ij}(t)\}$, where for each pair of firms $i, j \in \mathcal{N}$ we have that $\pi_{ij}(t) \geqslant 0$ — the communication cost $(i{,}j)$ that firm i bears for communicating with firm j if $(i{,}j)$ belongs to the network structure $\mathbf{g}(t)$ formed at time t, while $\pi_{ii}(t) = 0$. We assume that the sequence of communication cost matrices $\{\Pi(t)\}_{t=0}^{T-1}$ is always given and is common by known knowledge.

Let us consider for each time $t \in \mathcal{T} \setminus \{T\}$ and the firm $i \in \mathcal{N}$ the vector $g_i(t) = (g_{i1}(t), \ldots, g_{in}(t)) \in \mathbb{G}_i$, which we will call the network behavior of the firm i in the current period, and whose elements are

$$g_{ij}(t) = \begin{cases} 1, & \text{if } i \text{ offers a connection to the firm } j \in \mathcal{N} \setminus \{i\} \text{ at the time of } t \neq T, \\ 0, & \text{otherwise;} \end{cases}$$

and be interpreted as an offer or consent of the firm i to establish a network connection with the firm j in the current period, while we assume that $g_{ii}(t) \equiv 0$. Thus, we can assume that the network behavior of firm i at each nonterminal period of the model is determined by a binary vector, and define the set of feasible types of its network behavior as $\mathbb{G}_i = \{0, 1\}^n$ — space $n-$dimensional binary vectors. Set $g_i = (g_i(0), g_i(1), \ldots, g_i(T-1))$ to define the network behavior of firm i in each of the models discussed below.

We will say that a bilateral network connection is formed between a pair of firms i and j at time t if and only if $g_{ij}(t) = g_{ji}(t) = 1$, i.e. both firms at the current time agree to establish a connection (i, j) with each other, such that $(i, j) = (j, i)$ and $(i{,}j)$ belong to the network structure $\mathbf{g}(t)$ formed in the current period. The described rule of forming bilateral links in network structures is of interest to many researchers and is often found in the analysis of network games, for example in [15, 16, 22, 39].

Based on the rule of bilateral network interaction, we determine the dynamics of the unit cost of the firm $i \in \mathcal{N}$ by changing the equation (1.2) as follows:

$$c_i(t+1)=\delta c_i(t) - \alpha_i(t) y_i(t) - \\ -\sum_{j \neq i} \Big(\beta_{ij}(t)\, g_{ij}(t) g_{ji}(t) + \gamma_{ij}(t)\, (1 - g_{ij}(t) g_{ji}(t)) \Big) y_j(t), \tag{2.1}$$

where $t \in \mathcal{T} \setminus \{T\}$ и $c_i(0) = c_{i0}$, $c_i(t) \in \mathcal{C}$, $j \in \mathcal{N} \setminus \{i\}$. Describing the current network behavior of firms with the vector $g(t) = (g_1(t), \ldots, g_n(t))$, we will clarify the current profits of firm i with the following function:

$$F_i\,(t, g(t), c_i(t), u(t), y_i(t)) = \\ = \left(p - c_i(t) - \sum_{j=1}^{n} u_j(t) \right) u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t) - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(t) g_{ji}(t). \tag{2.2}$$

Thus, the gameplay described in Section 1.1 is complemented by the fact that now the behavior of a firm $i \in \mathcal{N}$ in period $t \in \mathcal{T} \setminus \{T\}$ is described by a feasible set $(g_i(t), u_i(t), y_i(t)) \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$; the network structure $\mathbf{g}(t)$ is built according to the rule of bilateral interaction (unless another rule is explicitly specified for the formation of its links) unit costs are calculated according to (2.1), and current profits are determined according to (2.2).

2.2. Open-loop Nash equilibrium

To formally represent a model with endogenous network formation as a dynamic game, and also Nash equilibrium as its solution, following [47] let us start by defining the strategies of competing firms. An open-loop strategy of firm $i \in \mathcal{N}$ is called the mapping $s_i(t, c_0) : \mathcal{T} \setminus \{T\} \mapsto \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$, which corresponds to each intermediate period and the initial values of the unit costs of all firms the feasible behavior of firm i of the following type $s_i(t, c_0) = (g_i(t), u_i(t), y_i(t))$. Since the vector c_0 is fixed, we will omit the dependence of the strategy on it and consider it only as a function of time. For the same reason, we will consider the firms' open-loop strategies as a function of time until the end of the thesis.

Following [20, 47], we define a dynamic model of competition with endogenous formation of bilateral network interaction of firms as a game in normal form:

$$\Gamma^{\mathrm{en}} = \Big\langle \mathcal{N}, \{\mathcal{S}_i\}_{i \in \mathcal{N}}, \{\mathcal{J}_i\}_{i \in \mathcal{N}} \Big\rangle,$$

where $\mathcal{S}_i = \{s_i \mid s_i(t, c_0) = (g_i(t), u_i(t), y_i(t)), t \in \mathcal{T} \setminus \{T\}\}$ — the set of strategies of the firm $i \in \mathcal{N}$, $s = (s_1, \ldots, s_n)$ — a strategy profile and the payoff function $\mathcal{J}_i(s) = J_i(c_0, \mathbf{g}, u, y)$ — the discounted profit of the firm i, determined by the following expression:

$$J_i(c_0, \mathbf{g}, u, y) = \sum_{t=0}^{T-1} \rho^t F_i(t, \mathbf{g}(t), c_i(t), u(t), y_i(t)) + \rho^T (\eta_i - \eta c_i(T)), \tag{2.3}$$

where current profits are determined according to (2.2), with a set of network interaction structures $\mathbf{g} = \{\mathbf{g}(t)\}_{t=0}^{T-1}$ that firms managed to generate, while the dynamics of unit costs are described by (2.1).

The following theorem characterizes the Nash equilibrium in the model Γ^{en}.

Theorem 2.1. *A set of strategies, $s^{\mathrm{N}} = (s_1^{\mathrm{N}}, \ldots, s_n^{\mathrm{N}})$, whose components $s_i^{\mathrm{N}}(t) = (g_i^{\mathrm{N}}(t), u_i^{\mathrm{N}}(t), y_i^{\mathrm{N}}(t))$, $i \in \mathcal{N}$, $t \in \mathcal{T} \setminus \{T\}$ have the form*

$$g_{ij}^{\mathrm{N}}(t) = \begin{cases} 1, & \pi_{ij}(t) < \pi_{ij}^{\mathrm{N}}(t),\ \pi_{ji}(t) < \pi_{ji}^{\mathrm{N}}(t), \qquad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \text{other}, \end{cases} \tag{2.4}$$

$$u_i^{\mathrm{N}}(t) = \frac{p - (n+1)c_i^{\mathrm{N}}(t) + \sum_{j \in \mathcal{N}} c_j^{\mathrm{N}}(t)}{n+1}, \tag{2.5}$$

$$y_i^{\mathrm{N}}(t) = -\frac{\alpha_i(t)\phi_i(t+1)}{\rho^t \varepsilon_i(t)}, \tag{2.6}$$

where

$$\pi_{ij}^{\mathrm{N}}(t) = \frac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^{\mathrm{N}}(t)\, y_j^{\mathrm{N}}(t),$$

is a Nash equilibrium in model Γ^{en}. Here $c_i^{\mathrm{N}}(t)$ satisfies (2.1) with the initial condition $c_i^{\mathrm{N}}(0) = c_{i0}$, and $\phi_i(t)$ satisfies the relation $\phi_i(t) = -\rho^t u_i^{\mathrm{N}}(t) + \delta\phi_i(t+1)$ with the boundary condition $\phi_i(T) = -\rho^T \eta$ for $i \in \mathcal{N}$.

Proof. First of all, following [51, 53, 72], let us suppose that instead of the n-dimensional binary vector $g_i(t)$ the firm $i \in \mathcal{N}$ currently chooses $t \neq T$ the n-dimensional vector $z_i(t)$, whose components are $z_{ij}(t) \in [0, 1]$. We will characterize such a network behavior of the firm i as its tendency to form a connection with the firm j in this period. In extreme cases, i.e. when $z_{ij}(t) = 0$ or $z_{ij}(t) = 1$, the firm i does not offer and offers a connection to the firm j, respectively. Let $z(t) = (z_1(t), \ldots, z_n(t))$, σ_i — a strategy of the firm i, i.e, $\sigma_i(t) = (z_i(t), u_i(t), y_i(t))$, and $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a set of strategies. Using standard dynamic game theory techniques, taking into account (2.1) and (2.3) for the firm i, the Hamiltonian takes the form:

$$\begin{aligned}
&\mathcal{H}_i\left(t, c(t), z(t), u(t), y(t), \psi_i(t+1)\right) = \\
&= \rho^t\Big[\Big(p - c_i(t) - \sum_{j\in\mathcal{N}} u_j(t)\Big)u_i(t) - \frac{\varepsilon_i(t)}{2}y_i^2(t) - \sum_{j\neq i}\pi_{ij}(t)z_{ij}(t)z_{ji}(t)\Big] + \sum_{j\in\mathcal{N}}\psi_{ij}(t+1)\times \\
&\times\Big[\delta c_j(t) - \alpha_j(t)y_j(t) - \sum_{r\neq j}\Big(\beta_{jr}(t)z_{jr}(t)z_{rj}(t) + \gamma_{jr}(t)(1 - z_{jr}(t)z_{rj}(t))\Big)y_r(t)\Big],
\end{aligned}$$

where $\psi_i(t) = (\pi_{i1}(t), \ldots, \psi_{in}(t))$, $t \in \mathcal{T}\backslash\{0\}$, — a set of costate variables. According to the Pontryagin maximum principle [23, 47], if the set of strategies σ^{N} is a Nash equilibrium, then there exist non-zero vectors $\psi_i(t)$ at $t \in \mathcal{T}\setminus\{0\}$, $i \in \mathcal{N}$, which satisfy the system of recurrence relations:

$$\begin{aligned}
\sigma_i^{\mathrm{N}}(t) = \arg &\max_{z_i(t)\in[0,1]^n,\; u_i(t)\in\mathbb{U}_i,\; y_i(t)\in\mathbb{Y}_i} \\
&\mathcal{H}_i\left(t, c^{\mathrm{N}}(t), \left(z_{-i}^{\mathrm{N}}(t)\,|\,z_i(t)\right), \left(u_{-i}^{\mathrm{N}}(t)\,|\,u_i(t)\right), \left(y_{-i}^{\mathrm{N}}(t)\,|\,y_i(t)\right), \psi_i(t+1)\right),
\end{aligned}$$

$$\psi_{ij}(t) = \begin{cases} -\rho^t u_i^{\mathrm{N}}(t) + \delta\psi_{ii}(t+1), & j = i,\, t \neq T, \\ -\rho^T\eta, & j = i,\, t = T, \\ 0, & j \neq i, \end{cases}$$

$$\begin{aligned}
c_i^{\mathrm{N}}(t+1) = \delta c_i^{\mathrm{N}}(t) - \alpha_i(t)y_i^{\mathrm{N}}(t) - &\\
- \sum_{j\neq i}\Big(\beta_{ij}(t)z_{ij}^{\mathrm{N}}(t)z_{ji}^{\mathrm{N}}(t) &+ \gamma_{ij}(t)(1 - z_{ij}^{\mathrm{N}}(t)z_{ji}^{\mathrm{N}}(t))\Big)y_j^{\mathrm{N}}(t), \quad t \neq T, \\
c_i^{\mathrm{N}}(0) = c_{i0}.&
\end{aligned}$$

Considering the equality $\pi_{ij}(t) = 0$ for $i \neq j$ and the linearity of the Hamiltonian with respect to the variables $z_{ij}(t)$ for the Nash equilibrium σ^{N}, it is necessary that:

$$z_{ij}^{\mathrm{N}}(t) = \begin{cases} 1, & \Big(\rho^t\pi_{ij}(t) + \psi_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t))y_j^{\mathrm{N}}(t)\Big)z_{ji}^{\mathrm{N}}(t) < 0 \text{ and } j \neq i, \\ 0, & \text{other,} \end{cases}$$

$$u_i^{\mathrm{N}}(t) = \frac{p - (n+1)c_i^{\mathrm{N}}(t) + \sum_{j\in\mathcal{N}} c_j^{\mathrm{N}}(t)}{n+1},$$

$$y_i^{\mathrm{N}}(t) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)}\psi_{ii}(t+1). \tag{2.7}$$

Thus, if σ^{N} is a Nash equilibrium, then $s^{\mathrm{N}} = \sigma^{\mathrm{N}}$, where $g_i^{\mathrm{N}} = z_i^{\mathrm{N}}$. Therefore, the Nash equilibrium dictates that two firms i and j establish a connection at time $t \neq T$, i.e., choose in their network behavior $g_{ij}^{\mathrm{N}}(t) = g_{ji}^{\mathrm{N}}(t) = 1$, if the following inequalities are satisfied:

$$\rho^t \pi_{ij}(t) + \psi_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t))y_j^{\mathrm{N}}(t) < 0,$$
$$\rho^t \pi_{ji}(t) + \psi_{jj}(t+1)(\beta_{ji}(t) - \gamma_{ji}(t))y_i^{\mathrm{N}}(t) < 0.$$

Considering (2.7), in Nash equilibrium two firms i and j establish a connection in period t if $\pi_{ij}(t) < \pi_{ij}^{\mathrm{N}}(t)$ and $\pi_{ji}(t) < \pi_{ji}^{\mathrm{N}}(t)$. Setting $\phi_i(t) = \psi_{ii}(t)$ we get the expressions (2.4) – (2.6).

The existence of nonzero costate variables guarantees for each firm a nonzero investment behavior determined by (2.7). The Hessian of the Hamiltonian $\mathcal{H}_i$ is negative definite: $-2\rho^t \left(u_i^{\mathrm{N}}(t)\right)^2 - \rho^t \varepsilon_i(t) \left(y_i^{\mathrm{N}}(t)\right)^2 < 0$. Therefore, it remains to conclude that s^{N} will be an open-loop Nash equilibrium. □

Remark 2.1. *Following [27], the conditions* (2.4) – (2.6) *can also be represented in an alternative recurrent form. To avoid repeating similar steps, we leave this out. In addition, we note that when moving from model* Γ^{ex} *to* Γ^{en}*, the functional type of production and investment behavior of firms* (2.5) – (2.6) *in Nash equilibrium is preserved.*

2.3. Open-loop Nash equilibrium for models with constant network interaction

The considered model Γ^{en} allows firms to rebuild their network interaction with competitors at each decision period, which can be interpreted as a variant of strategic behavior in which firms are guided by short-term relationships, an example of which may be short-term agreements or contracts. It also seems natural to consider options for network interaction when connections in the network are established for a long time, in other words, when the network has a permanent structure in the model. At the same time, the interaction costs that firms take into account when choosing direct neighbors in the network can be regular — firms bear the costs of their existing connections in the network at every period $t \in \mathcal{T} \setminus \{T\}$ (Subsection 2.3.1) as well as one-time — firms bear the costs of their existing connections only at the time of network formation (Subsection 2.3.2).

A comparative analysis of the Nash equilibrium conditions obtained for each of the types of network interaction of competing firms in a dynamic process described in this Section (Section 2.5) will rise a number of important questions, among which the central one is — which option of interaction with common input parameters (short-term or long-term) is more profitable for competing parties?

2.3.1. A model with the cost of establishing and maintaining network connections

Consider a model in which firms choose their network behavior once in the initial period, but bear the cost of establishing network connections with their direct neighbors at each time $t \in \mathcal{T} \setminus \{T\}$ — as they receive the effect of investments from all firms, according to the constructed structure of network interaction. Thus, the network in the model has a constant structure, so $\mathbf{g} = \mathbf{g}(0) := \mathbf{g}_0$. At the same time, the cost of the firm's network interaction is borne by $t \in \mathcal{T} \setminus \{T\}$ at each period. This assumption can be interpreted as the cost of forming and maintaining network interaction over a long period of time.

Call the feasible behavior of the firm $i \in \mathcal{N}$ in the model: three actions $(g_i(0), u_i(0), y_i(0)) \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$ for $t = 0$ and a pair of actions $(u_i(t), y_i(t)) \in$

$\mathbb{U}_i \times \mathbb{Y}_i$ for $t \in \mathcal{T} \setminus \{0, T\}$. The total profit of the firm i is set to

$$J_i\left(c_0, \mathbf{g}_0, u, y\right) = \\ = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i(t) - \sum_{j=1}^{n} u_j(t) \right) u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t) - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(0) g_{ji}(0) \right] + \\ + \rho^T (\eta_i - \eta c_i(T)), \quad (2.8)$$

where the current unit cost satisfies the recurrence relation

$$c_i(t+1) = \delta c_i(t) - \alpha_i(t) y_i(t) \\ - \sum_{j \neq i} \Big(\beta_{ij}(t) g_{ij}(0) g_{ji}(0) + \gamma_{ij}(t)(1 - g_{ij}(0) g_{ji}(0)) \Big) y_j(t) \quad (2.9)$$

for $t \in \mathcal{T} \setminus T$, with initial unit cost $c_i(0) = c_{i0}$.

Then, the dynamic model of competition with endogenous formation of long-term network interaction and regular interaction costs can be represented as a dynamic game in normal form:

$$\Gamma_{01}^{\text{en}} = \left\langle \mathcal{N}, \{\mathcal{S}_i\}_{i \in \mathcal{N}}, \{\mathcal{J}_i\}_{i \in \mathcal{N}} \right\rangle,$$

where $\mathcal{S}_i$ is the set of strategies of the firm i such that

$$s_i(t) = \begin{cases} (g_i(0), u_i(0), y_i(0)), & t = 0, \\ (u_i(t), y_i(t)), & t \in \mathcal{T} \setminus \{0, T\}; \end{cases} \quad (2.10)$$

the payoff function $\mathcal{J}_i(s) = J_i\left(c_0, \mathbf{g}_0, u, y\right)$ — discounted profit determined according to the functional (2.8), where $s = (s_1, \ldots, s_n)$ — a strategy profile in the game in which the network structure of interaction between firms is $\mathbf{g}_0$, and the dynamics of unit costs is determined by (2.9) with the initial condition $c(0) = c_0$.

The following theorem characterizes the open-loop Nash equilibrium for model Γ_{01}^{en}.

Theorem 2.2. *In model Γ_{01}^{en} a set of strategies $s^* = (s_1^*, \ldots, s_n^*)$, whose components*

$$s_i^*(t) = \begin{cases} (g_i^*(0), u_i^*(0), y_i^*(0)), & t = 0, \\ (u_i^*(t), y_i^*(t)), & t \in \mathcal{T} \setminus \{0, T\}, \end{cases}$$

for $i \in \mathcal{N}$, have the form

$$g^*_{ij}(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \pi^*_{ij},\ \sum\limits_{t=0}^{T-1} \rho^t \pi_{ji}(t) < \pi^*_{ji}, \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.11}$$

$$u^*_i(t) = \frac{p - (n+1)c^*_i(t) + \sum\limits_{j \in \mathcal{N}} c^*_j(t)}{n+1},$$

$$y^*_i(t) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)} \phi_i(t+1), \tag{2.12}$$

is a Nash equilibrium, where

$$\pi^*_{ij} = \sum_{t=0}^{T-1} \rho^t \frac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y^*_i(t)\, y^*_j(t).$$

Here $c^*_i(t)$ satisfies the relation (2.9) *with the initial condition $c^*_i(0) = c_{i0}$, and $\phi_i(t)$ satisfies the relation $\phi_i(t) = -\rho^t u^*_i(t) + \delta\phi_i(t+1)$ with the boundary condition $\phi_i(T) = -\rho^T \eta$ for $i \in \mathcal{N}$.*

Proof. Since the current profit for each firm $i \in \mathcal{N}$ depends, among other things, on the network behavior chosen at the beginning, standard methods based on Pontryagin's maximum principle are not applicable. Therefore, a different approach is required to prove the theorem.

First, as in the proof of Theorem 2.1, we assume that instead of the n-dimensional binary vector $g_i(0)$, firm i chooses the n-dimensional vector $z_i(0)$ in the initial period, whose components $z_{ij}(0) \in [0; 1]$ characterize the tendency of i to form a connection with $j \in \mathcal{N} \setminus \{i\}$ at $t = 0$. Let $z(0) = (z_1(0), \dots, z_n(0))$, σ_i — a strategy of firm i, a $\sigma = (\sigma_1, \dots, \sigma_n)$ — a set of strategies.

Assuming that the strategies of all firms except i are fixed, to find the best response to these strategies, the firm i must maximize (2.8) taking into account the equation of the dynamics of unit costs (2.9). To find the best response for firm i, we write out the Lagrange function:

$$\mathcal{L}_i(c, z(0), u, y, \lambda_i) = \sum_{t=0}^{T-1} \rho^t \Bigg[\Bigg(p - \sum_{j=1}^{n} u_j(t) \Bigg) u_i(t) - c_i(t) u_i(t) -$$

$$- \frac{\varepsilon_i(t)}{2} y_i^2(t) - \sum_{j \neq i} \pi_{ij}(t) z_{ij}(0) z_{ji}(0) \Bigg] + \rho^T \left(\eta_i - \eta c_i(T) \right) -$$

$$- \sum_{t=0}^{T-1} \sum_{j=1}^{n} \lambda_{ij}(t+1) \Bigg[c_j(t+1) - \delta c_j(t) + \alpha_j(t) y_j(t) +$$

$$+ \sum_{r \neq j} \Big(\beta_{jr}(t) z_{jr}(0) z_{rj}(0) + \gamma_{jr}(t)(1 - z_{jr}(0) z_{rj}(0)) \Big) y_r(t) \Bigg],$$

where $\lambda_i = (\lambda_i(1), \ldots, \lambda_i(T))$ when $\lambda_i(t) = (\lambda_{i1}(t), \ldots, \lambda_{in}(t))$, $t \in \mathcal{T} \setminus \{0\}$, — a set of Lagrange multipliers. If the set of strategies σ^* is a Nash equilibrium, then there exist non-zero sets λ_i, $i \in \mathcal{N}$ satisfying the system of recurrence relations:

$$z_{ij}^*(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \Big(\rho^t \pi_{ij}(t) + \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t)) y_j^*(t) \Big) z_{ji}^*(0) < 0 \text{ and } j \neq i, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_i^*(t) = \frac{p - (n+1) c_i^*(t) + \sum\limits_{j=1}^{n} c_j^*(t)}{n+1}, \qquad t \neq T,$$

$$y_i^*(t) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)} \lambda_{ii}(t+1), \qquad t \neq T, \tag{2.13}$$

$$\lambda_{ij}(t) = \begin{cases} -\rho^t u_i^*(t) + \delta \lambda_{ii}(t+1), & j = i, \quad t \neq T, \\ -\rho^T \eta, & j = i, \quad t = T, \\ 0, & j \neq i, \end{cases}$$

$$c_i^*(t+1) = \delta c_i^*(t) - \alpha_i(t) y_i^*(t) -$$

$$- \sum_{j \neq i} \Big(\beta_{ij}(t) z_{ij}^*(0) z_{ji}^*(0) + \gamma_{ij}(t)(1 - z_{ij}^*(0) z_{ji}^*(0)) \Big) y_j^*(t), \quad t \neq T,$$

$$c_i^*(0) = c_{i0}.$$

If σ^* is a Nash equilibrium, then $s^* = \sigma^*$, where $g_i^*(0) = z_i^*(0)$. We conclude that the Nash equilibrium requires two different firms i and j to establish a connection in the initial period, i.e. to choose in their network behavior $g_{ij}^*(0) = g_{ji}^*(0) = 1$, if the inequalities are satisfied:

$$\sum_{t=0}^{T-1} \Big(\rho^t \pi_{ij}(t) + \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t))y_j^*(t)\Big) < 0,$$
$$\sum_{t=0}^{T-1} \Big(\rho^t \pi_{ji}(t) + \lambda_{jj}(t+1)(\beta_{ji}(t) - \gamma_{ji}(t))y_i^*(t)\Big) < 0.$$

Considering the above expression for $y_i^*(t)$, in Nash equilibrium, two firms i and j establish a connection at the initial time $\sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \pi_{ij}^*$ and $\sum\limits_{t=0}^{T-1} \rho^t \pi_{ji}(t) < \pi_{ji}^*$. Setting $\phi_i(t) = \lambda_{ii}(t)$ we get the expressions (2.11)–(2.12).

The existence of non-zero Lagrange multipliers guarantees for each firm non-zero investment behavior, determined according to (2.13), in the context of which the Hessian of the Lagrange function $\mathcal{L}_i$ is negative definite:

$$-2\sum_{t=1}^{T-1} \rho^t u_i^*(t) c_i^*(t) - \sum_{t=0}^{T-1} \rho^t \left[2(u_i^*(t))^2 + \varepsilon_i(t)(y_i^*(t))^2\right] < 0.$$

Therefore, we can conclude that s^* will be an open-loop Nash equilibrium for model Γ_{01}^{en}. □

2.3.2. A model with one-time network cost

Now consider a model in which firms also choose and implement their network behavior once at an initial period, but only incur the costs of direct network connections at that period.

The network behavior of firms in such a model can be characterized by the costs of network interaction, which are necessary only for the provision and/or formation of links in the conditions of long-term network interaction of firms. As an example, we can cite a strategy profile in which a couple of firms decide to jointly invest in a project with a duration of $\mathcal{T}$, while the costs of network interaction itself are reduced here to the costs of negotiating, drafting, and signing a general agreement between the firms.

A dynamic competition model with endogenous formation of long-term network interaction and one-time network interaction costs is referred to as a dynamic game in normal form:

$$\Gamma_{02}^{\text{en}} = \left\langle \mathcal{N}, \{\mathcal{S}_i\}_{i\in\mathcal{N}}, \{\mathcal{J}_i\}_{i\in\mathcal{N}} \right\rangle,$$

where $\mathcal{S}_i$ is a set of strategies of the firm i, prescribing to it feasible behavior of the form (2.10), and the payoff function $\mathcal{J}_i(s) = J_i\left(c_0, \mathbf{g}_0, u, y\right)$ — discounted profit of the firm i, determined according to the following expression

$$J_i\left(c_0, \mathbf{g}_0, u, y\right) = \\ = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i(t) - \sum_{j=1}^{n} u_j(t)\right) u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t)\right] - \sum_{j\neq i} \pi_{ij}(0) g_{ij}(0) g_{ji}(0) + \\ + \rho^T \left(\eta_i - \eta c_i(T)\right), \quad (2.14)$$

where a network of $\mathbf{g}_0$ is formed from the network behavior of firms at $t = 0$, and the dynamics of unit costs is given by the (2.9) with $c(0) = c_0$.

The Nash equilibrium in Γ_{02}^{en} is given by the following theorem.

Theorem 2.3. *In the model Γ_{02}^{en} a set of strategies $s^{**} = (s_1^{**}, \ldots, s_n^{**})$, whose components*

$$s_i^{**}(t) = \begin{cases} (g_i^{**}(0), u_i^{**}(0), y_i^{**}(0)), & t = 0, \\ (u_i^{**}(t), y_i^{**}(t)), & t \in \mathcal{T} \setminus \{0, T\}, \end{cases}$$

for $i \in \mathcal{N}$, have the form

$$g_{ij}^{**}(0) = \begin{cases} 1, & \pi_{ij}(0) < \pi_{ij}^{**},\ \pi_{ji}(0) < \pi_{ji}^{**}, \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.15}$$

$$u_i^{**}(t) = \frac{p - (n+1)c_i^{**}(t) + \sum\limits_{j=1}^{n} c_j^{**}(t)}{n+1},$$

$$y_i^{**}(t) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)} \phi_i(t+1), \tag{2.16}$$

is a Nash equilibrium, where

$$\pi_{ij}^{**} = \sum_{t=0}^{T-1} \rho^t \frac{\varepsilon_i(t)}{\alpha_i(t)} \left(\beta_{ij}(t) - \gamma_{ij}(t)\right) y_i^{**}(t)\, y_j^{**}(t).$$

Here $c_i^{**}(t)$ satisfies the relation (2.9) *with the initial condition $c_i^{**}(0) = c_{i0}$, and $\phi_i(t)$ satisfies the relation $\phi_i(t) = -\rho^t u_i^{**}(t) + \delta\phi_i(t+1)$ with the boundary condition $\phi_i(T) = -\rho^T \eta$ for the firm i.*

Proof. Since the current profit of the firm also depends on the network behavior chosen in the initial period, we use a similar proof of Theorem 2.2. First, let us assume that instead of the n-dimensional binary vector $g_i(0)$, the firm $i \in \mathcal{N}$ chooses the n-dimensional vector $z_i(0)$ with components $z_{ij}(0) \in [0,1]$ at the initial time. Let $z(0) = (z_1(0), \dots, z_n(0))$, σ_i firm strategy i, a $\sigma = (\sigma_1, \dots, \sigma_n)$ — a set of strategies.

Assuming that the strategies of all firms except i are fixed, to find the best response to those strategies, the firm i must maximize (2.14) taking into account (2.9). Let us write the Lagrange function:

$$\mathcal{L}_i(c, z(0), u, y, \lambda_i) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - \sum_{j=1}^{n} u_j(t) \right) u_i(t) - c_i(t) u_i(t) - \frac{\varepsilon_i(t)}{2} y_i^2(t) \right] -$$
$$- \sum_{j \neq i} \pi_{ij}(0) z_{ij}(0) z_{ji}(0) + \rho^T \left(\eta_i - \eta c_i(T) \right) -$$
$$- \sum_{t=0}^{T-1} \sum_{j=1}^{n} \lambda_{ij}(t+1) \Bigg[c_j(t+1) - \delta c_j(t) + \alpha_j(t) y_j(t) +$$
$$+ \sum_{r \neq j} \Big(\beta_{jr}(t) z_{jr}(0) z_{rj}(0) + \gamma_{jr}(t)(1 - z_{jr}(0) z_{rj}(0)) \Big) y_r(t) \Bigg],$$

where $\lambda_i = (\lambda_i(1), \dots, \lambda_i(T))$ when $\lambda_i(t) = (\lambda_{i1}(t), \dots, \lambda_{in}(t))$, $t \in \mathcal{T} \setminus \{0\}$, — a set of Lagrange multipliers. If the set of strategies σ^{**} is a Nash equilibrium, then there exist non-zero sets λ_i, $i \in \mathcal{N}$ satisfying the system of recurrence relations:

$$z_{ij}^{**}(0) = \begin{cases} 1, & \left(\pi_{ij}(0) + \sum\limits_{t=0}^{T-1} \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t)) y_j^{**}(t) \right) z_{ji}^{**}(0) < 0, \quad j \neq i, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_i^{**}(t) = \frac{p - (n+1) c_i^{**}(t) + \sum\limits_{j=1}^{n} c_j^{**}(t)}{n+1}, \qquad t \neq T,$$
$$y_i^{**}(t) = -\frac{\alpha_i(t)}{\rho^t \varepsilon_i(t)} \lambda_{ii}(t+1), \qquad t \neq T,$$

$$\lambda_{ij}(t) = \begin{cases} -\rho^t u_i^{**}(t) + \delta\lambda_{ii}(t+1), & j = i,\, t \neq T, \\ -\rho^T \eta, & j = i,\, t = T, \\ 0, & j \neq i, \end{cases}$$

$$c_i^{**}(t+1) = \delta c_i^{**}(t) - \alpha_i(t) y_i^{**}(t) - \sum_{j \neq i} \Big(\beta_{ij}(t) z_{ij}^{**}(0) z_{ji}^{**}(0) + \\ + \gamma_{ij}(t)(1 - z_{ij}^{**}(0) z_{ji}^{**}(0)) \Big) y_j^{**}(t), \quad t \neq T,$$

$$c_i^{**}(0) = c_{i0}.$$

If σ^{**} is a Nash equilibrium, then $s^{**} = \sigma^{**}$, where $g_i^{**}(0) = z_i^{**}(0)$. Thus, the Nash equilibrium dictates that any pair of different firms i and j should establish a connection at the initial period, i.e., choose as their network behavior $g_{ij}^{**}(0) = g_{ji}^{**}(0) = 1$, in the case that the inequalities are true

$$\pi_{ij}(0) + \sum_{t=0}^{T-1} \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t)) y_j^{**}(t) < 0, \\ \pi_{ji}(0) + \sum_{t=0}^{T-1} \lambda_{jj}(t+1)(\beta_{ji}(t) - \gamma_{ji}(t)) y_i^{**}(t) < 0.$$

Considering the above expression for $y_i^{**}(t)$, in Nash equilibrium two firms i and j establish a connection at the initial time $\pi_{ij}(0) < \pi_{ij}^{**}$ and $\pi_{ji}(0) < \pi_{ji}^{**}$. Putting $\phi_i(t) = \lambda_{ii}(t)$ we get (2.15) – (2.16).

Since the existence of non-zero Lagrange multipliers guarantees non-zero investment behavior for each firm at every period $t \in \mathcal{T} \setminus \{T\}$, the Hessian of the Hessian Lagrange function $\mathcal{L}_i$ turns out to be negative definite:

$$-2 \sum_{t=1}^{T-1} \rho^t u_i^{**}(t) c_i^{**}(t) - \sum_{t=0}^{T-1} \rho^t \Big(2\big(u_i^{**}(t)\big)^2 + \varepsilon_i(t) \big(y_i^{**}(t)\big)^2 \Big) < 0.$$

Therefore, we can conclude that s^{**} will be an open-loop Nash equilibrium for model $\Gamma_{02}^{\mathrm{en}}$. □

2.4. Nash equilibrium with unilateral network interaction

Let us move on to the rule of forming a unilateral network interaction of competing firms, when the network connection between a pair of firms $i \in \mathcal{N}$ and $j \in \mathcal{N} \setminus \{i\}$ is directed $((i,j) \neq (j,i))$ in $\mathbf{g}(t)$, at $t \in \mathcal{T} \setminus \{T\}$. In this case, let us agree to understand by notation (i,j) an arc in $\mathbf{g}(t)$, indicating j as a firm from whose investments firm i receives an effect with a coefficient $\beta_{ij}(t)$. Note that for the unilateral formation of a network interaction, the desire of only one firm is sufficient, i.e. if firm i chooses $g_{ij}(t) = 1$, then $(i,j) \in \mathbf{g}(t)$.

Note that when switching in the previously considered models Γ^{en}, Γ^{en}_{01} and Γ^{en}_{02} from the bilateral network rule to unilateral, the methodology for finding a open-loop Nash equilibrium does not change. However, the condition for network interaction in equilibrium takes a different form. Since unilateral network interaction does not require firms to focus on the network behavior of their competitors, they can choose it based only on their own interest. In this context, we will only discuss those components of the models in Nash equilibrium that take on a different appearance than they do in equilibrium with bilateral interaction.

■ *At an open-loop Nash equilibrium for model* Γ^{en} *with unilateral network interaction, we have*:

$$J_i\left(c_0, \mathbf{g}^{\text{N}}, u^{\text{N}}, y^{\text{N}}\right) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i^{\text{N}}(t) - \sum_{j=1}^{n} u_j^{\text{N}}\right) u_i^{\text{N}} - \frac{\varepsilon_i(t)}{2}\left(y_i^{\text{N}}(t)\right)^2 - \right.$$
$$\left. - \sum_{j\neq i} \pi_{ij}(t) g_{ij}^{\text{N}}(t)\right] + \rho^T\left(\eta_i - \eta c_i^{\text{N}}(T)\right),$$

$$c_i^{\text{N}}(t+1) = \delta c_i^{\text{N}}(t) - \alpha_i(t) y_i^{\text{N}}(t) - \sum_{j\neq i}\Big(\beta_{ij}(t)\, g_{ij}^{\text{N}}(t) + \gamma_{ij}(t)\left(1 - g_{ij}^{\text{N}}(t)\right)\Big)\, y_j^{\text{N}}(t),$$

$$g_{ij}^{\text{N}}(t) = \begin{cases} 1, & \pi_{ij}(t) < \dfrac{\varepsilon_i(t)(\beta_{ij}(t) - \gamma_{ij}(t))}{\alpha_i(t)}\, y_i^{\text{N}}(t)\, y_j^{\text{N}}(t), \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \text{otherwise.} \end{cases}$$

■ *At an open-loop Nash equilibrium for model* Γ^{en}_{01} *with unilateral network interac-*

tion, we have:

$$J_i\left(c_0, \mathbf{g}^*, u^*, y^*\right) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i^*(t) - \sum_{j=1}^{n} u_j^* \right) u_i^* - \frac{\varepsilon_i(t)}{2} \left(y_i^*(t)\right)^2 - \right.$$
$$\left. - \sum_{j \neq i} \pi_{ij}(t) g_{ij}^*(0) \right] + \rho^T \left(\eta_i - \eta c_i^*(T)\right),$$

$$c_i^*(t+1) = \delta c_i^*(t) - \alpha_i(t)\, y_i^*(t) - \sum_{j \neq i} \Big(\beta_{ij}(t) g_{ij}^*(0) + \gamma_{ij}(t)(1 - g_{ij}^*(0)) \Big)\, y_j^*(t),$$

$$g_{ij}^*(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \sum\limits_{t=0}^{T-1} \rho^t \dfrac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^*(t)\, y_j^*(t), \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \text{otherwise.} \end{cases}$$

■ *At an open-loop Nash equilibrium for model* Γ_{02}^{en} *with unilateral network interaction, we have:*

$$J_i\left(c_0, \mathbf{g}^{**}, u^{**}, y^{**}\right) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i^{**}(t) - \sum_{j=1}^{n} u_j^{**} \right) u_i^{**} - \frac{\varepsilon_i(t)}{2} \left(y_i^{**}(t)\right)^2 \right] -$$
$$- \sum_{j \neq i} \pi_{ij}(0) g_{ij}^{**}(0) + \rho^T \left(\eta_i - \eta c_i^{**}(T)\right),$$

$$c_i^{**}(t+1) = \delta c_i^{**}(t) - \alpha_i(t)\, y_i^{**}(t) - \sum_{j \neq i} \Big(\beta_{ij}(t) g_{ij}^{**}(0) + \gamma_{ij}(t)(1 - g_{ij}^{**}(0)) \Big)\, y_j^{**}(t),$$

$$g_{ij}^{**}(0) = \begin{cases} 1, & \pi_{ij}(0) < \sum\limits_{t=0}^{T-1} \rho^t \dfrac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^{**}(t)\, y_j^{**}(t), \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we will move on to numerical simulations. This will enable us to compare the results obtained at open-loop Nash equilibrium with common parameters for all the models considered.

2.5. Numerical simulations and comparative analysis of results

To illustrate the theoretical results in Sections 2.2 – 2.4, we will use the same data that were employed for numerical simulations in Chapter 1 (Section 1.3). Recall them:

$n=4$	The parameters are universal across all firms					
$T=3$	$\eta_i = 100\,000$	$\delta = 1.07$	constants over time			
$p=500$	$\eta = 1000$	$\rho = 0.95$	$\alpha = 1.8$	$\beta = 1$	$\gamma = 0.5$	$\varepsilon = 1000$
	$c_{i0} = 100,\ i = \overline{1,n}$					

Cost matrices for potential connections between firms in network structures will be consistent across all models and remain constant over time.

$$\Pi(t) = \begin{pmatrix} 0 & 800 & 800 & 800 \\ 800 & 0 & 800 & 800 \\ 900 & 900 & 0 & 900 \\ 1100 & 1100 & 1100 & 0 \end{pmatrix}.$$

In Table 2.1, for each firm $i \in \mathcal{N}$, its current feasible behavior is given by $\big(g_i^{\mathrm{N}}(t), u_i^{\mathrm{N}}(t), y_i^{\mathrm{N}}(t)\big)$ in open-loop Nash equilibrium, $s^{\mathrm{N}} = \big(g^{\mathrm{N}}, u^{\mathrm{N}}, y^{\mathrm{N}}\big)$, and the corresponding unit cost $c_i^{\mathrm{N}}(t)$. At the same time, the table contains the results of numerical simulations obtained for Γ^{en} with two different types of network structure formation — bilateral and unilateral network interaction. The table also shows the network structures $\mathbf{g}^{\mathrm{N}}(t)$ prescribed by the Nash equilibrium, current unit prices in the market $P^{\mathrm{N}}(t) := p - \sum\limits_{j \in \mathcal{N}} u_j^{\mathrm{N}}(t)$ and firm profits $J_i^{\mathrm{N}} := J_i(c_0, g^{\mathrm{N}}, u^{\mathrm{N}}, y^{\mathrm{N}})$.

Similar in structure to Table 2.1 are the Tables 2.2 – 2.3, which present the results of numerical simulations corresponding to models of long-term network interaction, $\Gamma_{01}^{\mathrm{en}}$ and $\Gamma_{02}^{\mathrm{en}}$. The results are given in Tables 2.1 – 2.3, obtained by implementing the conditions of Theorems 2.1 – 2.3 using the program [26]. All values in the tables in this Section are rounded to the third decimal place.

First, we note that the results presented in Section 1.4 remain valid in the case of endogenous formation of network interaction between firms, which can be directly verified according to Tables 2.1 – 2.3.

Table 2.1. **Open-loop Nash equilibrium s^{N} and corresponding profits and unit costs of firms, as well as current prices in Γ^{en}.**

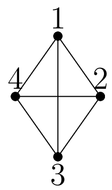
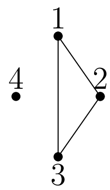
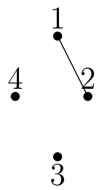
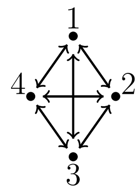
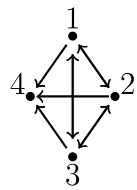
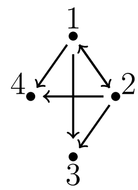
Γ^{en}:	bilateral interaction				unilateral interaction			
t :	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}^{\mathrm{N}}(t)$				–				–
$g_1^{\mathrm{N}}(t)$	(0,1,1,1)	(0,1,1,0)	(0,1,0,0)	–	(0,1,1,1)	(0,1,1,1)	(0,1,1,1)	–
$g_2^{\mathrm{N}}(t)$	(1,0,1,1)	(1,0,1,0)	(1,0,0,0)	–	(1,0,1,1)	(1,0,1,1)	(1,0,1,1)	–
$g_3^{\mathrm{N}}(t)$	(1,1,0,1)	(1,1,0,0)	(0,0,0,0)	–	(1,1,0,1)	(1,1,0,1)	(0,0,0,0)	–
$g_4^{\mathrm{N}}(t)$	(1,1,1,0)	(0,0,0,0)	(0,0,0,0)	–	(1,1,1,0)	(0,0,0,0)	(0,0,0,0)	–
$u_1^{\mathrm{N}}(t)$	80.000	80.564	81.193	–	80.000	80.564	81.570	–
$u_2^{\mathrm{N}}(t)$	80.000	80.564	81.193	–	80.000	80.564	81.570	–
$u_3^{\mathrm{N}}(t)$	80.000	80.564	81.193	–	80.000	80.564	81.570	–
$u_4^{\mathrm{N}}(t)$	80.000	80.561	79.309	–	80.000	80.561	78.745	–
$y_1^{\mathrm{N}}(t)$	2.046	1.877	1.710	–	2.046	1.878	1.710	–
$y_2^{\mathrm{N}}(t)$	2.046	1.877	1.710	–	2.046	1.878	1.710	–
$y_3^{\mathrm{N}}(t)$	2.046	1.877	1.710	–	2.046	1.878	1.710	–
$y_4^{\mathrm{N}}(t)$	2.043	1.874	1.710	–	2.042	1.873	1.710	–
$c_1^{\mathrm{N}}(t)$	100.000	97.183	95.917	96.133	100.000	97.182	94.977	93.417
$c_2^{\mathrm{N}}(t)$	100.000	97.183	95.917	96.133	100.000	97.182	94.977	93.417
$c_3^{\mathrm{N}}(t)$	100.000	97.183	95.917	96.988	100.000	97.182	94.977	95.982
$c_4^{\mathrm{N}}(t)$	100.000	97.186	97.801	99.004	100.000	97.186	97.801	99.004
$P^{\mathrm{N}}(t)$	180.000	177.747	177.110	–	180.000	177.747	176.545	–
J_1^{N}	12 103.590				12 280.904			
J_2^{N}	12 103.590				12 280.904			
J_3^{N}	11 602.535				11 662.737			
J_4^{N}	10 723.252				10 645.829			

Table 2.2. **Open-loop Nash equilibrium s^* and corresponding profits and unit costs of firms, as well as current prices in $\Gamma_{01}^{\mathrm{en}}$**

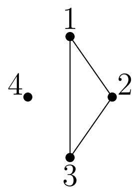
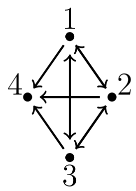
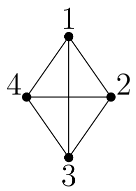
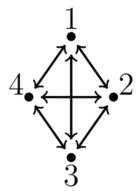
$\Gamma_{01}^{\mathrm{en}}$:	bilateral interaction				unilateral interaction			
t :	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}_0^*$				–				–
$g_1^*(t)$	$(0,1,1,0)$	–	–	–	$(0,1,1,1)$	–	–	–
$g_2^*(t)$	$(1,0,1,0)$	–	–	–	$(1,0,1,1)$	–	–	–
$g_3^*(t)$	$(1,1,0,0)$	–	–	–	$(1,1,0,1)$	–	–	–
$g_4^*(t)$	$(0,0,0,0)$	–	–	–	$(0,0,0,0)$	–	–	–
$u_1^*(t)$	80.000	80.771	81.416	–	80.000	81.181	82.230	–
$u_2^*(t)$	80.000	80.771	81.416	–	80.000	81.181	82.230	–
$u_3^*(t)$	80.000	80.771	81.416	–	80.000	81.181	82.230	–
$u_4^*(t)$	80.000	78.711	77.325	–	80.000	78.095	76.102	–
$y_1^*(t)$	2.047	1.877	1.710	–	2.049	1.879	1.710	–
$y_2^*(t)$	2.047	1.877	1.710	–	2.049	1.879	1.710	–
$y_3^*(t)$	2.047	1.877	1.710	–	2.049	1.879	1.710	–
$y_4^*(t)$	2.036	1.870	1.710	–	2.033	1.868	1.710	–
$c_1^*(t)$	100.000	98.205	97.010	96.448	100.000	97.182	94.977	93.418
$c_2^*(t)$	100.000	98.205	97.010	96.448	100.000	97.182	94.977	93.418
$c_3^*(t)$	100.000	98.205	97.010	96.448	100.000	97.182	94.977	93.418
$c_4^*(t)$	100.000	100.266	101.101	102.535	100.000	100.268	101.106	102.540
$P^*(t)$	180.000	178.976	178.426	–	180.000	178.362	177.208	–
J_1^*	11 973.707				12 466.088			
J_2^*	11 973.707				12 466.088			
J_3^*	11 403.207				11 610.338			
J_4^*	10 454.336				10 199.379			

Table 2.3. **Open-loop Nash equilibrium s^{**} and corresponding profits and unit costs of firms, as well as current prices in Γ_{02}^{en}**

Γ_{02}^{en}:	bilateral interaction				unilateral interaction			
t :	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}_0^{**}$				–				–
$g_1^{**}(t)$	$(0,1,1,0)$	–	–	–	$(0,1,1,1)$	–	–	–
$g_2^{**}(t)$	$(1,0,1,0)$	–	–	–	$(1,0,1,1)$	–	–	–
$g_3^{**}(t)$	$(1,1,0,0)$	–	–	–	$(1,1,0,1)$	–	–	–
$g_4^{**}(t)$	$(1,1,1,0)$	–	–	–	$(1,1,1,0)$	–	–	–
$u_1^{**}(t)$	80.000	80.564	81.005	–	80.000	80.564	81.005	–
$u_2^{**}(t)$	80.000	80.564	81.005	–	80.000	80.564	81.005	–
$u_3^{**}(t)$	80.000	80.564	81.005	–	80.000	80.564	81.005	–
$u_4^{**}(t)$	80.000	80.564	81.005	–	80.000	80.564	81.005	–
$y_1^{**}(t)$	2.045	1.877	1.710	–	2.045	1.877	1.710	–
$y_2^{**}(t)$	2.045	1.877	1.710	–	2.045	1.877	1.710	–
$y_3^{**}(t)$	2.045	1.877	1.710	–	2.045	1.877	1.710	–
$y_4^{**}(t)$	2.045	1.877	1.710	–	2.045	1.877	1.710	–
$c_1^{**}(t)$	100.000	97.182	94.976	93.417	100.000	97.182	94.976	93.417
$c_2^{**}(t)$	100.000	97.182	94.976	93.417	100.000	97.182	94.976	93.417
$c_3^{**}(t)$	100.000	97.182	94.976	93.417	100.000	97.182	94.976	93.417
$c_4^{**}(t)$	100.000	97.182	94.976	93.417	100.000	97.182	94.976	93.417
$P^{**}(t)$	180.000	177.745	175.981	–	180.000	177.745	175.981	–
J_1^{**}	16 648.020				16 648.020			
J_2^{**}	16 648.020				16 648.020			
J_3^{**}	16 348.020				16 348.020			
J_4^{**}	15 748.020				15 748.020			

Next, we introduce the concept of an upper bound on the interaction cost that the firm $i \in \mathcal{N}$ is willing to pay for to interaction with the firm $j \in \mathcal{N} \setminus \{i\}$ at period $t \in \mathcal{T} \setminus \{T\}$. Let us call the upper limit of the allowable communication costs of a firm i with a firm j in the network structure $\mathbf{g}(t)$ the minimum value of the communication costs (i,j), starting from which the condition of network interaction in Nash equilibrium for the firm i in relation to the firm j imposes the choice of $g^{\mathrm{N}}_{ij}(t) = 0$. For the models considered, the upper bounds of feasible communication costs (network interaction or just interaction) are set by the values $\pi^{\mathrm{N}}_{ij}(t)$, $\pi^{*}_{ij}(t)$ and $\pi^{**}_{ij}(t)$. Let us turn to Table 2.4 with the values $\pi^{\mathrm{N}}_{ij}(t)$ for model Γ^{en}.

Table 2.4. **The upper limits of feasible costs of interaction in model Γ^{en}**

t	Γ^{en}	bilateral interaction				unilateral interaction			
	$i \backslash j$	1	2	3	4	1	2	3	4
0	1	0	1162.573	1162.573	1160.709	0	1163.320	1163.320	1160.520
	2	1162.573	0	1162.573	1160.709	1163.320	0	1163.320	1160.520
	3	1162.573	1162.573	0	1160.709	1163.320	1163.320	0	1160.520
	4	1160.709	1160.709	1160.709	0	1160.520	1160.520	1160.520	0
1	1	0	978.705	978.705	977.025	0	979.376	979.376	976.857
	2	978.705	0	978.705	977.025	979.376	0	979.376	976.857
	3	978.705	978.705	0	977.025	979.376	979.376	0	976.857
	4	977.025	977.025	977.025	0	976.857	976.857	976.857	0
2	1	0	812.250	812.250	812.250	0	812.250	812.250	812.250
	2	812.250	0	812.250	812.250	812.250	0	812.250	812.250
	3	812.250	812.250	0	812.250	812.250	812.250	0	812.250
	4	812.250	812.250	812.250	0	812.250	812.250	812.250	0

Analyze the data of Tables 2.1 and 2.4 for model Γ^{en} with bilateral network interaction. Since the chosen parameters $(\varepsilon_i(t), \alpha_i(t), \beta_{ij}(t), \gamma_{ij}(t))$ are common to firms, then $\pi^{\mathrm{N}}_{ij}(t) = \pi^{\mathrm{N}}_{ji}(t)$ for all $t \in \mathcal{T} \setminus \{T\}$. The decrease in investment volume over time according to Table 2.1 implies a decrease in the upper bounds of feasible network interaction costs at which firms will be interested in interacting. Then, in the Nash equilibrium, firms i and j establish a relationship in the current period if $\pi^{\mathrm{N}}_{ij}(t) > \max\{\pi_{ij}(t), \pi_{ji}(t)\}$. According to the found equilibrium

s^{N} at time $t = 0$, links are established between arbitrary pairs of firms. In fact, even the communication cost of firm 4, which is the highest compared to the other firms ($\pi_{4j}(t) = 1100$, $j = 1, 2, 3$, $t = 0, 1, 2$), does not exceed the minimum communication cost $\pi^{\mathrm{N}}_{4j}(0) = 1160.709$. At $t = 1$, firms 1, 2, and 3 maintain network connections with each other, but connections with firm 4 are excluded. This is due to a change in the upper bound of feasible communication costs for firm 4, since $\pi^{\mathrm{N}}_{4j}(0)$ =977.025, which means $\pi_{4j}(1) < \pi^{\mathrm{N}}_{4j}(0)$, making firm 4 connections with other firms unprofitable. Finally, at $t = 2$ in network $\mathbf{g}(2)$, only link $(1, 2)$ is observed, since it becomes unprofitable for firms 3 and 4 to interact: their communication costs (900 and 1100, respectively) turn out to be above the upper bound of feasible communication costs, equal to 812,250.

For model Γ^{en} with unilateral network interaction of firms, describing the network behavior of firms based on data from Tables 2.1 and 2.4 can be done in the same way as it was done with bilateral network interaction. However, in this case, the network behavior of firms is more personalized in the sense that, according to the rule of forming connections in the network, with unilateral interaction, firms may not focus on the upper limits of feasible interaction costs of competitors when choosing their direct environment in each network structure. We explain this by going back to Tables 2.1 and 2.4. let us look at the network structures of Table 2.1 for unilateral network interaction of firms, namely for the following links: for $t = 1$ links (1, 4), (2, 4), and (3, 4), and for $t = 2$ links (1, 3), (1, 4), (2, 3), (2, 4). For example, at $t = 1$, firm 1 can afford the connection $(1, 4)$, since $\pi_{14}(1) = 800 < 976.857 = \pi^{\mathrm{N}}_{14}(1)$, therefore chooses $g^{\mathrm{N}}_{14}(1) = 1$, although firm 4 cannot afford the $(4, 1)$ connection in the $\mathbf{g}^{\mathrm{N}}(1)$ network structure, so $\pi_{41}(1) = 1100 > 976.857 = \pi^{\mathrm{N}}_{41}(1)$. At the same time, with unilateral network interaction, firm 1 receives the effect of $\beta_{14}(1)\, y^{\mathrm{N}}_4(1) = 1.873$ from the investments of firm 4 the current period, and with bilateral network interaction, the effect of firm 1 from the investments of firm 4 will be $\gamma_{14}(1)\, y^{\mathrm{N}}_4(1) = 0.937$. This ensures an improvement in the competitive position of firm 1 in the market compared to its competitive position with bilateral interaction of firms.

To evaluate the change in the competitiveness of firms, let us turn to Table 2.5. Here we look at the change in the competitiveness of firms in Nash equilibrium at $t = 1, 2, 3$. To do this, we introduce the value $\Delta c^{\mathrm{N}}_{ij}(t) := c^{\mathrm{N}}_j(t) - c^{\mathrm{N}}_i(t)$, which characterizes the competitive advantage of firm i over firm j in the Nash equilibrium: if $\Delta c_{ij}^{\mathrm{N}}(t) > 0$, then i has a competitive advantage over j estimated by $\Delta c_{ij}^{\mathrm{N}}(t)$, and if $\Delta c_{ij}^{\mathrm{N}}(t) < 0$, then j has a competitive advantage over i estimated by $|\Delta c_{ij}^{\mathrm{N}}(t)| = \Delta c_{ji}^{\mathrm{N}}(t)$. Also note for each firm i the value $\mathcal{D}_i^{\mathrm{N}}(t) := \left(\max\limits_{j \in \mathcal{N}} c_j^{\mathrm{N}}(t) - c_i^{\mathrm{N}}(t)\right) / \sum\limits_{j=1}^{n} c_j^{\mathrm{N}}(t) \times 100$, which characterizes in percentage terms the current competitiveness of firm i relative to all firms in the market.

Table 2.5. **The relationship between competitiveness and the competitive position of firms in the market in equilibrium s^{N} in Γ^{en}**

t	Γ^{en}	bilateral interaction					unilateral interaction				
	i	$\Delta c_{i1}^{\mathrm{N}}(t)$	$\Delta c_{i2}^{\mathrm{N}}(t)$	$\Delta c_{i3}^{\mathrm{N}}(t)$	$\Delta c_{i4}^{\mathrm{N}}(t)$	$\mathcal{D}_i^{\mathrm{N}}(t)$	$\Delta c_{i1}^{\mathrm{N}}(t)$	$\Delta c_{i2}^{\mathrm{N}}(t)$	$\Delta c_{i3}^{\mathrm{N}}(t)$	$\Delta c_{i4}^{\mathrm{N}}(t)$	$\mathcal{D}_i^{\mathrm{N}}(t)$
1	1	-	0	0	0.003	0.001	-	0	0	0.004	0.001
	2	0	-	0	0.003	0.001	0	-	0	0.004	0.001
	3	0	0	-	0.003	0.001	0	0	-	0.004	0.001
	4	-0.003	-0.003	-0.003	-	0.000	-0.004	-0.004	-0.004	-	0.000
2	1	-	0	0	1.884	0.489	-	0	0	2.824	0.738
	2	0	-	0	1.884	0.489	0	-	0	2.824	0.738
	3	0	0	-	1.884	0.489	0	0	-	2.824	0.738
	4	-1.884	-1.884	-1.884	-	0.000	-2.824	-2.824	-2.824	0	0.000
3	1	-	0	0.855	2.871	0.739	-	0	2.565	5.587	1.463
	2	0	-	0.855	2.871	0.739	0	-	2.565	5.587	1.463
	3	-0.855	-0.855	-	2.016	0.519	-2.565	-2.565	-	3.022	0.792
	4	-2.871	-2.871	-2.016	-	0.000	-5.587	-5.587	-3.022	-	0.000

From Table 2.5 it can be concluded that the network connections, which exist for Γ^{en} with unilateral interaction of firms in equilibrium and are not present in a similar position with bilateral interaction, have a significant impact on the change in the competitive position of firms in the market. This makes such connections very valuable, since in the case under consideration there is also a higher competitive advantage for all firms over firm 4, while firms 1, 2, and 3 can afford high interaction costs and firm 4 can only afford lower ones.

For the models $\Gamma_{01}^{\mathrm{en}}$ and $\Gamma_{02}^{\mathrm{en}}$ for the bilateral network interaction of firms, referring to the Tables 2.2 – 2.3, we conclude that the Nash equilibrium requires firms i and j to establish a link at the initial time if $\pi_{ij}^* > \max\left\{\sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t),\ \sum\limits_{t=0}^{T-1} \rho^t \pi_{ji}(t)\right\}$

or $\pi_{ij}^{**} > \max\{\pi_{ij}(0),\, \pi_{ji}(0)\}$ respectively. At the same time, the discounted sums of interaction costs for each firm are 2282,000, 2282,000, 2567,250, and 3137,750, respectively. Comparing these values with the data in Table 2.6, we see that it is impractical for firm 4 to establish links with other firms, while the rest of the firms establish all links among themselves. And in model $\Gamma_{02}^{\mathrm{en}}$, since the communication cost of any firm does not exceed the interaction cost limit equal to 2824.706, at the initial time, links are established between all pairs of firms.

Table 2.6. **The upper limits of feasible costs of interaction in models $\Gamma_{01}^{\mathrm{en}}$ and $\Gamma_{02}^{\mathrm{en}}$**

	bilateral interaction				unilateral interaction			
$i\backslash j$	1	2	3	4	1	2	3	4
				model $\Gamma_{01}^{\mathrm{en}}$				
1	0	2826.619	2826.619	2817.107	0	2830.405	2830.405	2816.144
2	2826.619	0	2826.619	2817.107	2830.405	0	2830.405	2816.144
3	2826.619	2826.619	0	2817.107	2830.405	2830.405	0	2816.144
4	2817.107	2817.107	2817.107	0	2816.144	2816.144	2816.144	0
				model $\Gamma_{02}^{\mathrm{en}}$				
1	0	2824.706	2824.706	2824.706	0	2824.706	2824.706	2824.706
2	2824.706	0	2824.706	2824.706	2824.706	0	2824.706	2824.706
3	2824.706	2824.706	0	2824.706	2824.706	2824.706	0	2824.706
4	2824.706	2824.706	2824.706	0	2824.706	2824.706	2824.706	0

The results of the observations made for model Γ^{en} with bilateral and unilateral network interaction can be transferred to the case of $\Gamma_{01}^{\mathrm{en}}$ and $\Gamma_{02}^{\mathrm{en}}$, so they are not given.

Comparing the results obtained with the bilateral network interaction of firms for models Γ^{en} (Table 2.1) and $\Gamma_{01}^{\mathrm{en}}$ (Table 2.2), we evaluate the advantages and disadvantages of short-term and long-term network interactions, respectively, under the same conditions represented by the input parameters of the models. The following indicators will be compared:

■ **The profits of the firms.** Long-term interactions reduces the profit of each firm, while for firms that have connections in long-term interaction (firms 1, 2, 3) the decrease in profit is about 1%, but for firm 4 a similar decrease in profit is about 2.5%. It can be concluded that the profits of firms whose number of network connections does not depend on the duration of interaction turn out to be less sensitive to the duration of network interaction.

■ **The production volume and the price of the product.** Since in the considered models the volume of goods supplied to the market is linearly related to the current price for units of goods, we evaluate these indicators together. Thus, in the case of short-term interaction, the firms produce a larger total volume of goods, which causes a decrease in the price of goods. As a result, each firm produces a larger volume of goods in short-term interaction with its competitors than in the long-term case. Note, however, that the situations compared differ by less than 1%.

■ **The volume of investments.** The presence and the number of network connections have a significant impact on the investment behavior of firms. Thus, the investments of firms 1, 2, and 3 are almost the same for both types of the duration of network interaction of firms. And for firm 4, which is deprived of network connections in long-term interaction, it turns out to be more profitable to reduce the volume of its investments, at $t = 0$ the decrease is $y_4^{\mathrm{N}}(0) - y_4^*(0) = 0.007$, a at $t = 1$ we have $y_4^{\mathrm{N}}(1) - y_4^*(1) = 0.004$. At the same time, a slightly larger volume of investment is implemented by firms that have connections with long-term interaction, while it is more profitable for firm 4 to reduce the total volume of its investment in equilibrium. Thus, it can be assumed that the total investment volume of each firm is influenced more by its network environment than by the duration of the interaction.

■ **The relationship between competitiveness and the competitive position of firms in the market.** According to Table 2.5 it can be noted that in the Nash equilibrium with unilateral network interaction, firms 1, 2, and 3 have a sharper increase in their competitive position in the market than in the case of bilateral interaction. Under these conditions, it is more profitable for these firms if the network interaction is unilateral. In fact, with bilateral network interaction we have $\mathcal{D}_i(2) = 0.489$, where $i = 1, 2, 3$, and with unilateral $\mathcal{D}_i(2) = 0.738$, for the next period, $t = 3$, we have $\mathcal{D}_1(3) = \mathcal{D}_2(3) = 0.739, \mathcal{D}_3(3) = 0.519$, and $\mathcal{D}_1(3) = \mathcal{D}_2(3) = 1.463, \mathcal{D}_3(3) = 0.792$. The same cannot be said for firm 4, since its competitiveness relative to other firms becomes lower in the unilateral network interaction than in the bilateral network interaction. Thus, for firm 4, the case of bilateral networking of firms turns out to be preferable, since in this case the change in the growth of competitive advantage among other firms is slower.

As a result of the analysis of competition models with endogenous network formation, the equilibrium network behavior of firms turned out to be similar in models Γ^{en}, Γ^{en}_{01} and Γ^{en}_{02}: a firm will offer a connection to its competitor if the cost of establishing and maintaining that connection (or the discounted sum of such costs) does not exceed a certain threshold. Note that in the absence of connection establishment and maintenance costs $\pi_{ij}(t) = 0$ for all $i, j \in \mathcal{N}$ and $t \in \mathcal{T} \setminus \{T\}$ firms establish all possible connections in Nash equilibrium, which is true for any model.

In conclusion, we would like to note that for the considered models with endogenous formation of a permanent network, the obtained Nash equilibria ensure «stability» of networks over time — no firm over time will be ready to abandon its existing network connection, nor will it strive to form a connection that is not prescribed to it by the Nash equilibrium.

2.6. Conclusions to Chapter 2

The analysis of competition models with endogenous network formation structure resulted in obtaining open-loop Nash equilibria. The equilibrium production and investment behavior of each firm is established, and the conditions under which the firm is interested in forming a network interaction with certain competitors are found. Although this chapter analyses three dynamic models with endogenous network formation, where connections can be both bilateral and unilateral, the Nash equilibria found in them share several similarities.

1. For all models, the structure of the firm's production and investment behavior in equilibrium is the same as the corresponding structure in exogenous network interaction.

2. The conditions for equilibrium network behavior are similar. A firm engages in network interaction if the costs of establishing and maintaining it, or the

discounted amount of such costs, do not exceed a certain threshold, which is the upper limit of feasible network interaction costs.

3. In a Nash equilibrium where there are no costs for forming and maintaining connections in the network, firms establish all types of connections. This applies to each model.

In conclusion, it is worth noting that in models with endogenous formation of a permanent network structure, the Nash equilibria obtained ensure the stability of the network interaction structures being formed. Over time, no firm will be willing to give up any of its existing network connections, and at the same time, it will not strive to form a connection that was not prescribed by the Nash equilibrium.

The main results of the chapter are published in [28].

Chapter 3.

Adaptation and application of game theoretic models to analyze the equilibrium behavior of competing firms

This Chapter analyzes the equilibrium behavior of competing firms in dynamics, taking into account common practical conditions. Models with endogenous formation of network interaction are considered, based on the following assumptions:

1. For each firm $i \in \mathcal{N}$ we have $u_i(t) = u_i \in \mathbb{U}_i$ at $t \in \mathcal{T} \setminus \{T\}$, i.e. the production volume is constant over time;

2. For each pair of firms i and $j \in \mathcal{N} \setminus \{i\}$, and for each time $t \in \mathcal{T} \setminus \{T\}$, we assume that $\beta_{ij}(t) > \gamma_{ij}(t) \geqslant 0$. That is, the network interaction between firms ensures an increase in their positive impact on each other's unit costs. This means that the relationship between the firms can be considered as a interaction, and the firms themselves i and j are called partners;

3. Firms can implement their investment behavior in two ways, which we will call variable (risky) — $y_i(t) \in \mathbb{Y}_i$, and constant (cautious) — $y_i(t) = y_i \in \mathbb{Y}_i$, at $t \in \mathcal{T} \setminus \{T\}$.

The assumptions presented allow us to move from the game-theoretic models of the Chapter 2 to practice-oriented models. At the same time, we maintain the general concept of the models studied and adapt the theoretical results obtained to the analysis of market competition in conditions close to practice. Indeed, in practice, the constant volume of production of the firm on a small planning horizon is natural. In fact, in the economic planning of production, a production plan is often drawn up and approved, which determines the volume of production of the firm for a certain period of time. The production plan simplifies the economic analysis of both the firm's activities and its sales market. It also allows for effective management. Often, the production plan of the firm provides for the production of a constant volume of goods, which is explained both by stabilization and optimization of attracted and spent resources, for example, the number of working personnel (human capital). It is also worth noting that the constant volume of production on the considered planning horizon ensures uniform saturation of the market with goods at a constant cost per unit of goods for the consumer. According to the considered model concept, such a volume of goods is determined by the value of $\sum_{j=1}^{n} u_j$, and the price per unit of goods is — the value of $P(t) = p - \sum_{j=1}^{n} u_j > 0$, if $p > \sum_{j=1}^{n} u_j$. The second assumption also seems quite natural. Any interaction always implies interaction under conditions that are formally regulated and legally formalized. This is the basis for the security and benefits of interaction. The third assumption allows us to consider two variants of common investment behavior in practice. We will also be interested in comparing these variants of equilibrium behavior. A more detailed description and analysis of these types of investment behavior will be given in Sections 3.1 and 3.2, respectively.

Let us note another feature of this Chapter. The characterization for the equilibrium behavior of firms will be based on a open-loop information structure. For each model, the corresponding conditions for the behavior of firms in open-loop Nash equilibrium are presented. This allows firms to choose their behavior with a minimum requirement for a set of information needed to make a decision. This restriction is due to two aspects. First, as shown in the first Chapter 1, the profits of firms in open-loop and feedback Nash equilibria can be quite close. Second, in practice it is quite difficult to know the unit costs of competitors at all times. In most cases, such information is private to each firm and not subject to disclosure. At the same time, the open-loop information structure allows each firm participating in the game to build a calendar financial and economic plan that stabilizes its economic activity and contributes to the organization of effective management with information that is always available.

Thus, game-theoretic models of dynamic competition with network interactions between firms, which are the subject of this Chapter, turn out to be quite popular in practice and, from an economic point of structure, more feasible for the analysis of real economic processes.

3.1. Analysis of short-term network interactions of competing firms

In the context of a dynamic process, the main interest of endogenous network interaction is the ability of firms to rewire the network structure at each decision period. In order to do this adequately, we will stick to scenarios in which firms implement their investments at each non-terminal period. At the same time, we consider two different types of investment behavior — when firms' investments are implemented in variable and constant volumes.

3.1.1. Variable investment

According to the assumptions presented at the beginning of this Chapter, each firm $i \in \mathcal{N}$ plans for a set of time periods $\mathcal{T}$ a constant volume of production $u_i \in \mathbb{U}_i$, while $u = (u_1, \ldots, u_n) \in \mathbb{U}_1 \times \ldots \times \mathbb{U}_n$. We will say that we are considering a model with constant production and variable (risky) investment, in which the strategy of the firm i prescribes to it, at each decision period an feasible behavior $(g_i(t), u_i, y_i(t)) \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$. Considering here and further the strategy of each firm as a function of time — as done in Chapter 2, we get that $\mathcal{S}_i = \{s_i(t) \mid s_i(t) = (g_i(t), u_i, y_i(t)), t \in \mathcal{T} \setminus \{T\}\}$ — the set of feasible strategies of firm i, $s = (s_1, \ldots, s_n) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ — strategy profile, $y = (y(0), \ldots, y(T-1))$, $y(t) = (y_1(t), \ldots, y_n(t)) \in \mathbb{Y}_1 \times \ldots \times \mathbb{Y}_n$ at $t \in \mathcal{T} \setminus \{T\}$. The evolution of the unit costs of the firm i with bilateral interaction is described by the equation (2.1), and its discounted profit given by the following function

$$J_i(c_0, \mathbf{g}, u, y) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - \sum_{j=1}^{n} u_j \right) u_i - c_i(t) u_i - \frac{\varepsilon_i(t)}{2} y_i^2(t) - \right. \\ \left. - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(t) g_{ji}(t) \right] + \rho^T (\eta_i - \eta c_i(T)), \tag{3.1}$$

where $\mathbf{g} = \{\mathbf{g}(t)\}_{t=0}^{T-1}$ — a set of networks of bilateral interaction formed as a result of the implementation by all firms of their network behavior at appropriate game periods.

Let us denote the presented dynamic model of competition with endogenous formation of network interaction of firms with constant production volume and risky investment behavior by $\bar{\Gamma}^{\text{en}}$. The open-loop Nash equilibrium for bilateral network interaction is characterized by the following theorem.

Theorem 3.1. *In model $\bar{\Gamma}^{\text{en}}$, the open-loop Nash equilibrium is a set of strategies $s^{\text{N}} = (s_1^{\text{N}}, \ldots, s_n^{\text{N}})$, whose components $s_i^{\text{N}}(t) = (g_i^{\text{N}}(t), u_i^{\text{N}}, y_i^{\text{N}}(t))$ for $i \in \mathcal{N}$, $t \neq T$*

have the form:

$$g_{ij}^{\mathrm{N}}(t) = \begin{cases} 1, & \pi_{ij}(t) < \pi_{ij}^{\mathrm{N}}(t),\, \pi_{ji}(t) < \pi_{ji}^{\mathrm{N}}(t), \quad j \neq i, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.2}$$

$$u_i^{\mathrm{N}} = \frac{p - \sum\limits_{\tau=0}^{T-1} \rho^\tau \left((n+1) c_i^{\mathrm{N}}(\tau) - \sum\limits_{j \in \mathcal{N}} c_j^{\mathrm{N}}(\tau) \right) \left(\sum\limits_{\tau=0}^{T-1} \rho^\tau \right)^{-1}}{n+1}, \tag{3.3}$$

$$y_i^{\mathrm{N}}(t) = -\frac{\alpha_i(t)\phi_i(t+1)}{\rho^t \varepsilon_i(t)}, \tag{3.4}$$

where

$$\pi_{ij}^{\mathrm{N}}(t) = \frac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^{\mathrm{N}}(t)\, y_j^{\mathrm{N}}(t), \tag{3.5}$$

$$\phi_i(t) = \begin{cases} -\rho^t \left(u_i^{\mathrm{N}} \cdot \sum\limits_{\tau=0}^{T-t-1} (\rho\delta)^\tau + \eta(\rho\delta)^{T-t} \right), & t \neq T, \\ -\rho^T \eta, & t = T. \end{cases} \tag{3.6}$$

The unit cost of $c_i^{\mathrm{N}}(t)$ *satisfies* (2.1) *given* $c_i^{\mathrm{N}}(0) = c_{i0}$.

Proof. We start by assuming that each firm i chooses the n-dimensional vector $z_i(t)$ as its network behavior with components $z_{ij}(t) \in [0,1]$. Let $z(t) = (z_1(t), \ldots, z_n(t))$, $z = (z(0), \ldots, z(T-1))$; σ_i — the strategy of the firm is i, and $\sigma = (\sigma_1, \ldots, \sigma_n)$ — a set of strategies. Then $\sigma_i(t) = (z_i(t), u_i, y_i(t))$. Let us fix the strategies of all firms except i. To find the best response to these strategies, the firm i must maximize (3.1) taking into account (2.1).

Introduce the Lagrange function to take this into account

$$\begin{aligned} &\mathcal{L}_i(c, z, u, y, \lambda_i) = \\ &\quad = \sum_{t=0}^{T-1} \rho^t \left[\left(p - c_i(t) - \sum_{j \in \mathcal{N}} u_j \right) u_i - \frac{\varepsilon_i(t)}{2} y_i^2(t) - \sum_{j \neq i} \pi_{ij}(t) z_{ij}(t) z_{ji}(t) \right] + \\ &\qquad + \rho^T (\eta_i - \eta c_i(T)) - \sum_{t=0}^{T-1} \sum_{j \in \mathcal{N}} \lambda_{ij}(t+1) \Bigg[c_j(t+1) - \delta c_j(t) + \alpha_j(t) y_j(t) + \\ &\qquad\qquad + \sum_{r \neq j} \Big(\beta_{jr}(t) z_{jr}(t) z_{rj}(t) + \gamma_{jr}(t)(1 - z_{jr}(t) z_{rj}(t)) \Big) y_r(t) \Bigg], \end{aligned}$$

where $\lambda_i = (\lambda_i(1), \ldots, \lambda_i(T))$ when $\lambda_i(t) = (\lambda_{i1}(t), \ldots, \lambda_{in}(t))$, $t \in \mathcal{T} \setminus \{0\}$, — a set of Lagrange multipliers. The best response of firm i to the fixed strategies of its competitors is a strategy whose components satisfy the following system (taking into account the linearity of $\mathcal{L}_i$ with variables $z_{ij}(t)$):

$$z_{ij}(t) = \begin{cases} 1, & \big(\rho^t \pi_{ij}(t) + \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t)) y_j(t)\big) z_{ji}(t) < 0, \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \text{otherwise}, \end{cases},$$

$$u_i = \frac{1}{2}\left(p - \sum_{j \neq i} u_j - \sum_{\tau=0}^{T-1} \rho^\tau c_i(\tau) \cdot \left(\sum_{\tau=0}^{T-1} \rho^\tau \right)^{-1} \right),$$

$$y_i(t) = -\frac{\alpha_i(t) \lambda_{ii}(t+1)}{\rho^t \varepsilon_i(t)},$$

where

$$\lambda_{ij}(t) = \begin{cases} -\rho^t u_i + \delta \lambda_{ii}(t+1), & j = i,\, t \neq T, \\ -\rho^T \eta, & j = i,\, t = T, \\ 0, & j \neq i. \end{cases} \tag{3.7}$$

Therefore, if σ^{N} is a Nash equilibrium, then $s^{\mathrm{N}} = \sigma^{\mathrm{N}}$ and $g_i^{\mathrm{N}} = z_i^{\mathrm{N}}$. Thus, the Nash equilibrium dictates that firms i and j establish a partnership in the current period $t \in \mathcal{T} \setminus \{T\}$ while satisfying the inequalities

$$\rho^t \pi_{ij}(t) + \lambda_{ii}(t+1)(\beta_{ij}(t) - \gamma_{ij}(t)) y_j^{\mathrm{N}}(t) < 0 \text{ and}$$
$$\rho^t \pi_{ji}(t) + \lambda_{jj}(t+1)(\beta_{ji}(t) - \gamma_{ji}(t)) y_i^{\mathrm{N}}(t) < 0.$$

From the recurrence relation (3.7), in which $\phi_i(t) = \lambda_{ii}(t)$ is set in equilibrium, we get (3.5) and (3.6), which leads to the expressions (3.2), (3.3) and (3.4).

Since the Hessian of the Lagrange function $\mathcal{L}_i$ is definite negative:

$$-2u_i^{\mathrm{N}} \cdot \sum_{t=1}^{T-1} \rho^t c_i^{\mathrm{N}}(t) - \sum_{t=0}^{T-1} \rho^t \Big(2\left(u_i^{\mathrm{N}}\right)^2 + \varepsilon_i(t) \left(y_i^{\mathrm{N}}(t)\right)^2 \Big) < 0,$$

then it remains to conclude that s^{N} will be a Nash equilibrium in the model $\bar{\Gamma}^{\mathrm{en}}$. □

In analyzing the open-loop Nash equilibrium found, we will make two remarks. First, unlike the model Γ^{en}, in which the current production volume was determined by the firms' current unit costs, in the model $\bar{\Gamma}^{\text{en}}$ with constant production (but variable investment), the production volume of each firm (3.3) is expressed in terms of the weighted average cost of the firms on the horizon under consideration (the weight at the time t is assumed to be equal to ρ^t, i.e. current costs are discounted at the beginning of the game). And second, the ratio (3.6) in the Nash equilibrium sets a linear relationship between the current investment level of the firm and its production volume:

$$y_i^{\text{N}}(t) = \begin{cases} \rho \dfrac{\alpha_i(t)}{\varepsilon_i(t)} \left(u_i^{\text{N}} \cdot \sum\limits_{\tau=0}^{T-t-2} (\rho\delta)^{\tau} + \eta(\rho\delta)^{T-t-1} \right), & t \neq T-1, \\ \rho\eta \dfrac{\alpha_i(T-1)}{\varepsilon_i(T-1)}, & t = T-1. \end{cases} \tag{3.8}$$

Thus, with the values $\alpha_i(t)$ and $\varepsilon_i(t)$ constant over time, the firm's investment volume in the Nash equilibrium $y_i^{\text{N}}(t)$ will be a monotonically decreasing function of time at $u_i^{\text{N}} > \eta(1-\rho\delta)$. Moreover, the time-invariant parameters $\beta_{ij}(t)$ and $\gamma_{ij}(t)$ lead to a monotonic decrease of the upper bounds of the allowable costs of network interaction $\pi_{ij}^{\text{N}}(t)$ in (3.5), below which firms are willing to interactions. In other words, the latter means that the number of network partnerships does not increase over time.

Corollary 3.1 (from Theorem 3.1). *Let $\alpha_i(t) = \alpha(t)$ and $\varepsilon_i(t) = \varepsilon(t)$ for any firm $i \in \mathcal{N}$. Then, in the model $\bar{\Gamma}^{\text{en}}$, when implementing the Nash equilibrium, for arbitrary firms i and $j \in \mathcal{N} \setminus \{i\}$ the following three conditions are equivalent:*

$$\frac{\sum\limits_{\tau=0}^{T-1} \rho^{\tau} c_i^{\text{N}}(\tau)}{\sum\limits_{\tau=0}^{T-1} \rho^{\tau}} < \frac{\sum\limits_{\tau=0}^{T-1} \rho^{\tau} c_j^{\text{N}}(\tau)}{\sum\limits_{\tau=0}^{T-1} \rho^{\tau}} \Leftrightarrow u_i^{\text{N}} > u_j^{\text{N}} \Leftrightarrow y_i^{\text{N}}(t) > y_j^{\text{N}}(t), \ t \neq T-1.$$

Proof. According to (3.3), in the Nash equilibrium the difference in production volumes is equal to the weighted difference in current costs, i.e.

$$u_i^{\text{N}} - u_j^{\text{N}} = \sum_{\tau=0}^{T-1} \rho^{\tau} \left(c_j^{\text{N}}(\tau) - c_i^{\text{N}}(\tau) \right) / \sum_{\tau=0}^{T-1} \rho^{\tau},$$

and the linear ratio is (3.8) allows you to relate the difference in production volumes to current investment volumes at a time $t \in \mathcal{T} \setminus \{T-1, T\}$:

$$y_i^{\mathrm{N}}(t) - y_j^{\mathrm{N}}(t) = \rho\eta \frac{\alpha(t)}{\varepsilon(t)} \left(u_i^{\mathrm{N}} - u_j^{\mathrm{N}}\right) \sum_{\tau=0}^{T-t-2} (\rho\delta)^{\tau}.$$

In addition, we note that equality is valid

$$y_i^{\mathrm{N}}(T-1) = y_j^{\mathrm{N}}(T-1) = \rho\eta \frac{\alpha(T-1)}{\varepsilon(T-1)}.$$

□

Similar to what was shown in Section 2.4 for the model $\bar{\Gamma}^{\mathrm{en}}$ with unilateral network interaction of firms, it is sufficient to simply change the equation of the dynamics of unit costs, the functional type of profit of firms, and the conditions of their network interaction in Nash equilibrium accordingly. In this regard, in this Chapter we will focus only on the condition of network interaction in open-loop Nash equilibrium, which we will present as a corollary from the corresponding equilibrium theorem for bilateral interaction of firms and give without proof.

Corollary 3.2 (from Theorem 3.1)**.** *If we assume that the interaction of firms is one-sided, then the equilibrium behavior of firms in terms of production and investment behavior is preserved and determined according to* (3.3) – (3.4)*, and the network behavior of firm $i \in \mathcal{N}$ in Nash equilibrium takes the following form:*

$$g_{ij}^{\mathrm{N}}(t) = \begin{cases} 1, & \pi_{ij}(t) < \dfrac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^{\mathrm{N}}(t)\, y_j^{\mathrm{N}}(t), \\ 0, & \textit{otherwise}, \end{cases} \qquad j \in \mathcal{N} \setminus \{i\}.$$

3.1.2. Constant investment

Now consider a model in which firms implement their investments in equal amounts (contributions), i.e. the investment behavior of each firm is constant over time. The relevance of the model under consideration is due to the approach to investment activity, the main idea of which is to average the cost of investments during market fluctuations, which protects investors from significant losses and is therefore often called constant in practice. Another example of such investment behavior is when investments can be understood as sponsoring, for example, a research laboratory. In this case, the sponsorship is also often made in equal amounts over a period of time.

As before, the strategy of the firm $i \in \mathcal{N}$ is called the function $s_i(t)$, which prescribes to it a feasible behavior $(g_i(t), u_i, y_i) \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$ at each decision period $t \in \mathcal{T} \setminus \{T\}$. Then we redefine the set of strategies of firm i as $\mathcal{S}_i = \{s_i(t) \,|\, s_i(t) = (g_i(t), u_i, y_i), t \in \mathcal{T} \setminus \{T\}\}$, $s = (s_1, \ldots, s_n)$ — strategy profile, $y = (y_1, \ldots, y_n) \in \mathbb{Y}_1 \times \ldots \times \mathbb{Y}_n$ at $t \in \mathcal{T} \setminus \{T\}$. Note that the network behavior of firms remains tied to a particular period. The evolution of the unit costs of firm i with bilateral interaction is described by the equation

$$c_i(t+1) = \delta c_i(t) - \alpha_i(t) y_i - \sum_{j \neq i} \Big(\beta_{ij}(t) g_{ij}(t) g_{ji}(t) + \gamma_{ij}(t)(1 - g_{ij}(t) g_{ji}(t))\Big) y_j, \quad (3.9)$$

where $c_i(0) = c_{i0}$, and its profit is given by

$$J_i(c_0, \mathbf{g}, u, y) = \sum_{t=0}^{T-1} \rho^t \Bigg[\Bigg(p - \sum_{j=1}^{n} u_j\Bigg) u_i - c_i(t) u_i - \frac{\varepsilon_i(t)}{2} y_i^2 - \\ - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(t) g_{ji}(t)\Bigg] + \rho^T (\eta_i - \eta c_i(T)).$$

Denote the presented dynamic model of competition with endogenous network formation with constant production volume and constant investment behavior by $\bar{\bar{\bar{\Gamma}}}^{\mathrm{en}}$. The open-loop Nash equilibrium with bilateral interaction for this model is characterized by the following theorem.

Theorem 3.2. *The open-loop Nash equilibrium for model* $\bar{\bar{\bar{\Gamma}}}^{\mathrm{a}}$ *is a set of strategies* $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n)$*, whose components* $\hat{s}_i(t) = (\hat{g}_i(t), \hat{u}_i, \hat{y}_i)$ *for* $i \in \mathcal{N}$ *and* $t \neq T$ *have the form:*

$$\hat{g}_{ij}(t) = \begin{cases} 1, & \pi_{ij}(t) < \hat{\pi}_{ij}(t),\ \pi_{ji}(t) < \hat{\pi}_{ji}(t), \quad j \neq i, \\ 0, & \textit{otherwise}, \end{cases} \quad (3.10)$$

$$\hat{u}_i = \frac{p - \sum\limits_{\tau=0}^{T-1} \rho^\tau \Big((n+1)\hat{c}_i(\tau) - \sum\limits_{j \in \mathcal{N}} \hat{c}_j(\tau)\Big) \cdot \Big(\sum\limits_{\tau=0}^{T-1} \rho^\tau\Big)^{-1}}{n+1}, \quad (3.11)$$

$$\hat{y}_i = -\frac{\sum\limits_{t=0}^{T-1} \alpha_i(t)\phi_i(t+1)}{\sum\limits_{t=0}^{T-1} \rho^t \varepsilon_i(t)}, \tag{3.12}$$

where

$$\hat{\pi}_{ij}(t) = \frac{\varepsilon_i(t)}{\alpha_i(t)}(\beta_{ij}(t) - \gamma_{ij}(t))\,\hat{y}_i\,\hat{y}_j, \tag{3.13}$$

$$\phi_i(t) = \begin{cases} -\rho^t\left(\hat{u}_i \cdot \sum\limits_{\tau=0}^{T-t-1}(\rho\delta)^\tau + \eta(\rho\delta)^{T-t}\right), & t \neq T, \\ -\rho^T\eta, & t = T. \end{cases}$$

The unit cost of $\hat{c}_i(t)$ satisfies (3.9), *given* $\hat{c}_i(0) = c_{i0}$.

Proof. The proof of this theorem largely repeats the steps of the proof of the theorem 3.1, so it will be omitted. Note only that since the Hessian of the Lagrange function $\mathcal{L}_i$ is negative definite:

$$-2\hat{u}_i \cdot \sum_{t=1}^{T-1} \rho^t \hat{c}_i(t) - \sum_{t=0}^{T-1} \rho^t \left(2\,(\hat{u}_i)^2 + \varepsilon_i(t)\,(\hat{y}_i)^2\right) < 0,$$

then it remains to conclude that $\hat{s}$ will be a open-loop Nash equilibrium in $\bar{\bar{\Gamma}}^{\mathrm{en}}$. □

The above theorem allows us to make a number of observations. First, the existence of non-zero Lagrange multipliers guarantees the non-zero investment behavior of $\hat{y}_i$ of each firm. Second, in the Nash equilibrium, there is also a linear relationship between the firm's investment volume and its production volume, which is expressed in the following form

$$\hat{y}_i = \frac{\sum\limits_{t=0}^{T-2} \rho^{t+1}\left(\hat{u}_i \cdot \sum\limits_{\tau=0}^{T-t-2}(\rho\delta)^\tau + \eta(\rho\delta)^{T-t-1}\right)\alpha_i(t) + \rho^T\eta\alpha_i(T-1)}{\sum\limits_{\tau=0}^{T-1} \rho^t \varepsilon_i(t)}. \tag{3.14}$$

The time invariant parameters $\alpha_i(t)$, $\beta_{ij}(t)$, $\gamma_{ij}(t)$, and $\varepsilon_i(t)$ imply a constant upper bound on the allowable cost of network interaction $\hat{\pi}_{ij}(t)$ in (3.13), which, if exceeded, will cause firm i to refuse to interaction with firm j. If this is true for all pairs of firms, then in Nash equilibrium none of the established partnerships will be broken over time.

A corollary similar to Collary 3.1, which links the ratios between the weighted average costs, production volumes, and investments of two firms, is given without proof.

Corollary 3.3 (from Theorem 3.2). *Let $\alpha_i(t) = \alpha$ and $\varepsilon_i(t) = \varepsilon$ for arbitrary firms. Then, in Nash equilibrium, for any firms i and j, the following three conditions are equivalent:*

$$\frac{\sum\limits_{\tau=0}^{T-1} \rho^\tau \hat{c}_i(\tau)}{\sum\limits_{\tau=0}^{T-1} \rho^\tau} < \frac{\sum\limits_{\tau=0}^{T-1} \rho^\tau \hat{c}_j(\tau)}{\sum\limits_{\tau=0}^{T-1} \rho^\tau} \quad \Leftrightarrow \quad \hat{u}_i > \hat{u}_j \quad \Leftrightarrow \quad \hat{y}_i > \hat{y}_j.$$

Here is another corollary that reveals the aspect of unilateral network interaction of firms in Nash equilibrium for the model $\bar{\bar{\Gamma}}^{\text{en}}$.

Corollary 3.4 (from Theorem 3.2). *If we assume that the interaction of firms is one-sided, then the equilibrium behavior of firms in terms of production and investment behavior is preserved and determined according to* (3.11) – (3.12), *and the network behavior of firm $i \in \mathcal{N}$ in Nash equilibrium takes the following form:*

$$\hat{g}_{ij}(t) = \begin{cases} 1, & \pi_{ij}(t) < \dfrac{\varepsilon_i(t)}{\alpha_i(t)}(\beta_{ij}(t) - \gamma_{ij}(t))\, \hat{y}_i\, \hat{y}_j, \\ 0, & \textit{otherwise}, \end{cases} \qquad j \in \mathcal{N} \setminus \{i\}. \tag{3.15}$$

3.1.3. Numerical simulations of equilibrium behavior for short-term network interactions

Compare the equilibrium results of modeling in Γ^{en} and $\bar{\Gamma}^{\text{en}}$, while evaluating how much the adaptation of the theoretical model of Γ^{en} turns out to be feasible for practical applications in real conditions of market competition. Also compare and evaluate two variants of investment behavior of firms, which are provided in practical game models $\bar{\Gamma}^{\text{en}}$ and $\bar{\bar{\Gamma}}^{\text{en}}$.

As input parameters for presenting the results of numerical simulations of open-loop Nash equilibrium under the conditions of Theorems 3.1–3.2, we use the parameters from the Chapter 2 (p. 61). With these data we present the result of the modeling: $s^{\mathrm{N}} = \left(g^{\mathrm{N}}, u^{\mathrm{N}}, y^{\mathrm{N}}\right)$ or $\hat{s} = (\hat{g}, \hat{u}, \hat{y})$ for models $\bar{\Gamma}^{\mathrm{en}}$ and $\bar{\bar{\Gamma}}^{\mathrm{en}}$ in Table 3.1 – 3.2 respectively. These Tables have a common structure and contain for each firm $i \in \mathcal{N}$ its feasible behavior in equilibrium, $\left(g_i^{\mathrm{N}}(t), u_i^{\mathrm{N}}, y_i^{\mathrm{N}}(t)\right)$ and $(\hat{g}_i(t), \hat{u}_i, \hat{y}_i)$, with unilateral and bilateral interaction of firms. The Tables also contain information about the current price of the product, $P^{\mathrm{N}} := p - \sum_{j=1}^{n} u_j^{\mathrm{N}}$ or $\hat{P} := p - \sum_{j=1}^{n} \hat{u}_j$, the current unit costs of the firms, $c_i^{\mathrm{N}}(t)$ and $\hat{c}_i(t)$, and their equilibrium profits, $J_i^{\mathrm{N}} := J_i\left(c_0, \mathbf{g}^{\mathrm{N}}, u^{\mathrm{N}}, y^{\mathrm{N}}\right)$ and $\hat{J}_i := J_i(c_0, \hat{\mathbf{g}}, \hat{u}, \hat{y})$. For the sake of clarity, the following Tables illustrate the network structures of interaction between firms in an equilibrium strategy profile, $\mathbf{g}^{\mathrm{N}}(t)$ and $\hat{\mathbf{g}}(t)$. All values in Tables are rounded to three decimal places.

In addition, we present Tables 3.3 – 3.4, which indicate the upper bounds of feasible costs of network interaction of firms in models $\bar{\Gamma}^{\mathrm{en}}$ and $\bar{\bar{\Gamma}}^{\mathrm{en}}$.

According to the data from Tables 3.1 – 3.4, it is possible to explain the network behavior of firms in equilibrium for the models $\bar{\Gamma}^{\mathrm{en}}$ and $\bar{\bar{\Gamma}}^{\mathrm{en}}$ for both unilateral and bilateral network interaction of firms, similar to what was done in Section 2.5 for the game-theoretic model Γ^{en}, which is why we omit it here.

Evaluate the results of modeling the Nash equilibrium behavior in the models Γ^{en} and $\bar{\Gamma}^{\mathrm{en}}$. As one can see from Tables 2.1 and 3.1, the network structures are preserved, regardless of their type (bilateral or unilateral). A peculiarity of the cautious investment behavior of firms is that the upper bounds of the allowable costs of interaction turn out to be constant over time (see Table 3.4), which seems natural based on the conditions of network interaction (3.15) and constant investment volumes in equilibrium, as well as constant network parameters. At the same time, we note that the total investment volume[1] of firms changes slightly in the transition from a variable to a constant production volume (or from Γ^{en} to $\bar{\Gamma}^{\mathrm{en}}$):

- In Γ^{en} with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.633, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.627,$$

  and with unilateral network interaction

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.634, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.625;$$

[1] Talking about the total investment volume of firms, discounting is not taken into account here and further.

Table 3.1. **Open-loop Nash equilibrium s^{N} and corresponding profits and unit costs of firms, as well as current prices in the modal $\bar{\Gamma}^{\mathrm{en}}$.**

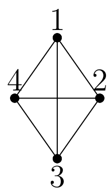
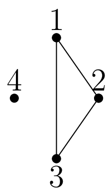
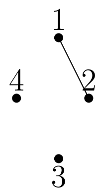
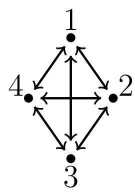
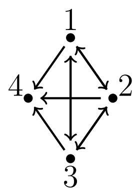
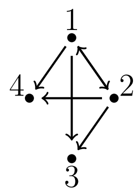
	bilateral interaction				unilateral interaction			
$t:$	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}^{\mathrm{N}}(t)$				–				–
$g_1^{\mathrm{N}}(t)$	(0,1,1,1)	(0,1,1,0)	(0,1,0,0)	–	(0,1,1,1)	(0,1,1,1)	(0,1,1,1)	–
$g_2^{\mathrm{N}}(t)$	(1,0,1,1)	(1,0,1,0)	(1,0,0,0)	–	(1,0,1,1)	(1,0,1,1)	(1,0,1,1)	–
$g_3^{\mathrm{N}}(t)$	(1,1,0,1)	(1,1,0,0)	(0,0,0,0)	–	(1,1,0,1)	(1,1,0,1)	(0,0,0,0)	–
$g_4^{\mathrm{N}}(t)$	(1,1,1,0)	(0,0,0,0)	(0,0,0,0)	–	(1,1,1,0)	(0,0,0,0)	(0,0,0,0)	–
u_1^{N}	80.564	80.564	80.564	–	80.683	80.683	80.683	–
u_2^{N}	80.564	80.564	80.564	–	80.683	80.683	80.683	–
u_3^{N}	80.564	80.564	80.564	–	80.683	80.683	80.683	–
u_4^{N}	79.969	79.969	79.969	–	79.791	79.791	79.791	–
$y_1^{\mathrm{N}}(t)$	2.045	1.876	1.710	–	2.045	1.876	1.710	–
$y_2^{\mathrm{N}}(t)$	2.045	1.876	1.710	–	2.045	1.876	1.710	–
$y_3^{\mathrm{N}}(t)$	2.045	1.876	1.710	–	2.045	1.876	1.710	–
$y_4^{\mathrm{N}}(t)$	2.043	1.875	1.710	–	2.042	1.875	1.710	–
$c_1^{\mathrm{N}}(t)$	100.000	97.188	95.924	96.141	100.000	97.187	94.986	93.426
$c_2^{\mathrm{N}}(t)$	100.000	97.188	95.924	96.141	100.000	97.187	94.986	93.426
$c_3^{\mathrm{N}}(t)$	100.000	97.188	95.924	96.996	100.000	97.187	94.986	95.991
$c_4^{\mathrm{N}}(t)$	100.000	97.189	97.803	99.007	100.000	97.189	97.804	99.007
P^{N}	178.338	178.338	178.338	–	177.980	177.980	177.980	–
J_1^{N}	12 099.364				12 276.380			
J_2^{N}	12 099.364				12 276.380			
J_3^{N}	11 598.308				11 658.213			
J_4^{N}	10 717.991				10 638.375			

Table 3.2. **Open-loop Nash equilibrium $\hat{s}$ and corresponding profits and unit costs of firms, as well as current prices in the model $\bar{\bar{\Gamma}}^{\text{en}}$**

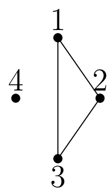
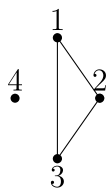
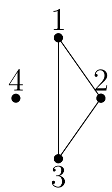
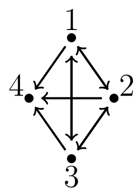
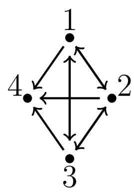
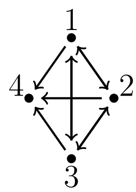
	bilateral interaction				unilateral interaction			
$t:$	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\hat{\mathbf{g}}(t)$				–				–
$\hat{g}_1(t)$	(0,1,1,0)	(0,1,1,0)	(0,1,1,0)	–	(0,1,1,1)	(0,1,1,1)	(0,1,1,1)	–
$\hat{g}_2(t)$	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)	–	(1,0,1,1)	(1,0,1,1)	(1,0,1,1)	–
$\hat{g}_3(t)$	(1,1,0,0)	(1,1,0,0)	(1,1,0,0)	–	(1,1,0,1)	(1,1,0,1)	(1,1,0,1)	–
$\hat{g}_4(t)$	(1,1,1,0)	(0,0,0,0)	(0,0,0,0)	–	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	–
$\hat{u}_1$	80.589	80.589	80.589	–	80.961	80.961	80.961	–
$\hat{u}_2$	80.589	80.589	80.589	–	80.961	80.961	80.961	–
$\hat{u}_3$	80.589	80.589	80.589	–	80.961	80.961	80.961	–
$\hat{u}_4$	78.725	78.725	78.725	–	78.166	78.166	78.166	–
$\hat{y}_1$	1.883	1.883	1.883	–	1.883	1.883	1.883	–
$\hat{y}_2$	1.883	1.883	1.883	–	1.883	1.883	1.883	–
$\hat{y}_3$	1.883	1.883	1.883	–	1.883	1.883	1.883	–
$\hat{y}_4$	1.879	1.879	1.879	–	1.878	1.878	1.878	–
$\hat{c}_1(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$\hat{c}_2(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$\hat{c}_3(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$\hat{c}_4(t)$	100.000	100.793	101.642	102.550	100.000	100.794	101.644	102.553
$\hat{P}$	179.508	179.508	179.508	–	178.951	178.951	178.951	–
$\hat{J}_1$	11 921.435				12 401.633			
$\hat{J}_2$	11 921.435				12 401.633			
$\hat{J}_3$	11 350.935				11 545.883			
$\hat{J}_4$	10 454.836				10 207.999			

Table 3.3. **The upper limits of feasible costs of interaction in the model $\bar{\Gamma}^{\text{en}}$**

t	$\bar{\Gamma}^{\text{en}}$ $i\backslash j$	bilateral interaction 1	2	3	4	unilateral interaction 1	2	3	4
0	1	0	1161.331	1161.331	1160.166	0	1161.797	1161.797	1160.049
	2	1161.331	0	1161.331	1160.166	1161.797	0	1161.797	1160.049
	3	1161.331	1161.331	0	1160.166	1161.797	1161.797	0	1160.049
	4	1160.166	1160.166	1160.166	0	1160.049	1160.049	1160.049	0
1	1	0	977.583	977.583	977.053	0	977.795	977.795	977.000
	2	977.583	0	977.583	977.053	977.795	0	977.795	977.000
	3	977.583	977.583	0	977.053	977.795	977.795	0	977.000
	4	977.053	977.053	977.053	0	977.000	977.000	977.000	0
2	1	0	812.250	812.250	812.250	0	812.250	812.250	812.250
	2	812.250	0	812.250	812.250	812.250	0	812.250	812.250
	3	812.250	812.250	0	812.250	812.250	812.250	0	812.250
	4	812.250	812.250	812.250	0	812.250	812.250	812.250	0

Table 3.4. **The upper limits of feasible costs of interaction in the model $\bar{\bar{\Gamma}}^{\text{en}}$**

t	$\bar{\bar{\Gamma}}^{\text{en}}$ $i\backslash j$	bilateral interaction 1	2	3	4	unilateral interaction 1	2	3	4
0, 1, 2	1	0	984.555	984.555	982.821	0	985.247	985.247	982.647
	2	984.555	0	984.555	982.821	985.247	0	985.247	982.647
	3	984.555	984.555	0	982.821	985.247	985.247	0	982.647
	4	982.821	982.821	982.821	0	982.647	982.647	982.647	0

- In $\bar{\Gamma}^{\text{en}}$ with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.631, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.628,$$

and with unilateral network interaction

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.631, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.627.$$

At the same time, the change in the competitive position of firms in the market over time ($\mathcal{D}_i(t), t = 1, 2, 3$) will be more pronounced in Γ^{en} (Table 2.5) than in $\bar{\Gamma}^{\text{en}}$ (presented on page 87).

It is also worth noting that the results of the analysis based on the numerical simulation of the equilibrium in Γ^{en} remain valid for the model $\bar{\Gamma}^{\text{en}}$ and are therefore omitted. However, it is noteworthy in this example that the equilibrium profits of firms (as well as the total volume of goods produced) obtained for the models Γ^{en} and $\bar{\Gamma}^{\text{en}}$ turn out to be close enough. In fact, we estimate the relative change in the equilibrium profit of firms as a percentage during their transition from a variable to a constant volume of production. Using data from Tables 2.1 and 3.2 for this purpose, we conclude: for firms 1 and 2, the decrease in profit will be 0.035 %, for firm 3, the decrease in profit will be 0.036 %, and for firm 4, the decrease in profit will be 0.049 % — with bilateral interaction. If the interaction of the firms is assumed to be unilateral, then the decrease in equilibrium profit for all firms is not much larger — for firms 1 and 2 by 0.037 %, for firm 3 by 0.039 %, and for firm 4 by 0.070 %. Such a small decrease in equilibrium profit (less than one percent) indicates, on the one hand, that the theoretical results obtained in Chapters 1-2 are quite feasible in practice and confirms the rationality of choosing a constant volume of production in the behavior of producers of goods.

To compare the possibilities of investment behavior of firms, we go to the results of equilibrium modeling for the models $\bar{\Gamma}^{\text{en}}$ and $\bar{\bar{\Gamma}}^{\text{en}}$. According to the fact that for constant investments the upper bounds of the allowable costs of interaction between firms are constant and at the beginning of the period lower than the corresponding bounds for risky investments — we conclude from Table 3.3 and 3.4, then for firm 4, in the condition of bilateral interaction, we have $\pi_{4j}(0) = 1100 > 982.821 = \hat{\pi}_{4j}(0)$, for $j = 1, 2$ and 3. In the condition of unilateral interaction, the strategy profile is similar, since $\pi_{4j}(0) = 1100 > 982.647 = \hat{\pi}_{4j}(0)$. Therefore, regardless of the type of interaction in the model $\bar{\bar{\Gamma}}^{\text{en}}$, firm 4 has no partners. And for $t = 2$ with unilateral interaction for the firm 3, we have $\pi_{3j}(2) < \min\{\hat{\pi}_{3j}(2), \hat{\pi}_{34}(2)\}$, $j = 1, 2$, which allows it to interact with all firms. Thus, the type of investment behavior can affect firms' partnerships in different ways, despite the fact that in $\bar{\Gamma}^{\text{en}}$ and $\bar{\bar{\Gamma}}^{\text{en}}$, firms' total investment volumes are quite close:

- In $\bar{\Gamma}^{\text{en}}$ with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.631, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.628,$$

and with unilateral network interaction

$$\sum_{t=0}^{2} y_1^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_2^{\mathrm{N}}(t) = \sum_{t=0}^{2} y_3^{\mathrm{N}}(t) = 5.631, \quad \sum_{t=0}^{2} y_4^{\mathrm{N}}(t) = 5.627;$$

- In $\bar{\bar{\Gamma}}^{\text{en}}$ with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} \hat{y}_1(t) = \sum_{t=0}^{2} \hat{y}_2(t) = \sum_{t=0}^{2} \hat{y}_3(t) = 5.649, \quad \sum_{t=0}^{2} \hat{y}_4(t) = 5.637,$$

and with unilateral network interaction

$$\sum_{t=0}^{2} \hat{y}_1(t) = \sum_{t=0}^{2} \hat{y}_2(t) = \sum_{t=0}^{2} \hat{y}_3(t) = 5.649, \quad \sum_{t=0}^{2} \hat{y}_4(t) = 5.634.$$

Let us move on to assessing changes in the competitiveness of firms, for which we will create a Table 3.5, where $\widehat{\mathcal{D}}_i(t) := \left(\max\limits_{j \in \mathcal{N}} \hat{c}_j(t) - \hat{c}_i(t) \right) / \sum\limits_{j=1}^{n} \hat{c}_j(t) \times 100$, $t = 1, 2, 3$.

It is interesting to note that as firms become more cautious in their investment behavior, the change in their competitive position on the market becomes more pronounced. One can check this by comparing the data in Table 3.5 and the following indicators in the equilibrium of the model $\bar{\Gamma}^{\text{en}}$:
with bilateral interaction $\mathcal{D}_1^{\mathrm{N}}(1) = \mathcal{D}_2^{\mathrm{N}}(1) = \mathcal{D}_3^{\mathrm{N}}(1) = 0.0003$, $\mathcal{D}_1^{\mathrm{N}}(2) = \mathcal{D}_2^{\mathrm{N}}(2) = \mathcal{D}_3^{\mathrm{N}}(2) = 0.487$, $\mathcal{D}_1^{\mathrm{N}}(3) = \mathcal{D}_2^{\mathrm{N}}(3) = 0.738, \mathcal{D}_3^{\mathrm{N}}(3) = 0.518$, $\mathcal{D}_4^{\mathrm{N}}(t) = 0$ — for each t, since firm 4 had the highest unit cost at each period; in the case of unilateral interaction of firms, we have $\mathcal{D}_1^{\mathrm{N}}(1) = \mathcal{D}_2^{\mathrm{N}}(1) = \mathcal{D}_3^{\mathrm{N}}(1) = 0.0005$, $\mathcal{D}_1^{\mathrm{N}}(2) = \mathcal{D}_2^{\mathrm{N}}(2) = \mathcal{D}_3^{\mathrm{N}}(2) = 0.736$, $\mathcal{D}_1^{\mathrm{N}}(3) = \mathcal{D}_2^{\mathrm{N}}(3) = 1.462, \mathcal{D}_3^{\mathrm{N}}(3) = 0.790$, $\mathcal{D}_4^{\mathrm{N}}(t) = 0$.

Thus, it can be concluded that the change in the competitive position of firms in the market is faster with cautious investment behavior. At the same time, bilateral network interaction with risky investment behavior of firms provides the lowest rate of change of competitive position of firms in equilibrium.

Table 3.5. **The relationship between competitiveness and the competitive position of firms in the market, with their equilibrium behavior in the model $\overline{\overline{\Gamma}}^{\mathrm{en}}$**

t	$\overline{\overline{\Gamma}}^{\mathrm{en}}$	bilateral interaction					unilateral interaction				
	i	$\Delta\hat{c}_{i1}(t)$	$\Delta\hat{c}_{i2}(t)$	$\Delta\hat{c}_{i3}(t)$	$\Delta\hat{c}_{i4}(t)$	$\widehat{\mathcal{D}}_i(t)$	$\Delta\hat{c}_{i1}(t)$	$\Delta\hat{c}_{i2}(t)$	$\Delta\hat{c}_{i3}(t)$	$\Delta\hat{c}_{i4}(t)$	$\widehat{\mathcal{D}}_i(t)$
1	1	-	0	0	1.887	0.475	-	0	0	2.829	0.717
	2	0	-	0	1.887	0.475	0	-	0	2.829	0.717
	3	0	0	-	1.887	0.475	0	0	-	2.829	0.717
	4	-1.887	-1.887	-1.887	-	0.000	-2.829	-2.829	-2.829	-	0.000
2	1	-	0	0	3.906	0.989	-	0	0	5.856	1.505
	2	0	-	0	3.906	0.989	0	-	0	5.856	1.505
	3	0	0	-	3.906	0.989	0	0	-	5.856	1.505
	4	-3.906	-3.906	-3.906	-	0.000	-5.856	-5.856	-5.856	0	0.000
3	1	-	0	0	6.066	1.547	-	0	0	9.095	2.375
	2	0	-	0	6.066	1.547	0	-	0	9.095	2.375
	3	0	0	-	6.066	1.547	0	0	-	9.095	2.375
	4	-6.066	-6.066	-6.066	-	0.000	-9.095	-9.095	-9.095	-	0.000

3.2. Analysis of long-term network interactions of competing firms

Let us move on to dynamic models in which firms enter into long-term network interaction, forming a network structure of interaction once and for all at $t = 0$. Following the notation used in Chapter 2, we denote such a structure as $\mathbf{g}(0) = \mathbf{g}_0$. In the thesis, the equilibrium behavior of firms is of interest, and its comparison with the equilibrium behavior in short-term interaction and with the results obtained for the models $\Gamma^{\mathrm{en}}_{01}$ (Subsection 2.3.1), $\Gamma^{\mathrm{en}}_{02}$ (Subsection 2.3.2).

3.2.1. Variable investment

We start with a model in which firms implement variable investment volumes or risky investment behavior. By the strategy of the firm $i \in \mathcal{N}$ we will understand a function that prescribes to it, am each decision period, $t \in \mathcal{T} \setminus \{T\}$ feasible behavior of the form

$$s_i(t) = \begin{cases} (g_i(0), u_i, y_i(0)) & \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i, \quad t = 0, \\ (u_i, y_i(t)) & \in \mathbb{U}_i \times \mathbb{Y}_i, \qquad\quad t \in \mathcal{T} \setminus \{0, T\}. \end{cases} \tag{3.16}$$

The change in the unit cost of firm i is described by the equation (2.9), at $c_i(0) = c_{i0}$, and its profit under the network structure of bilateral long-term interaction $\mathbf{g}_0$, which is formed at the initial time, is given by

$$J_i\left(c_0, \mathbf{g}_0, u, y\right) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - \sum_{j=1}^{n} u_j\right) u_i - c_i(t)u_i - \frac{\varepsilon_i(t)}{2} y_i^2(t) - \right.$$

$$\left. - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(0) g_{ji}(0)\right] + \rho^T \left(\eta_i - \eta c_i(T)\right),$$

where $y = (y(0), \ldots, y(T-1))$ and $y(t) = (y_1(t), \ldots, y_n(t))$, $t \in \mathcal{T} \setminus \{T\}$.

Let us denote the presented dynamic model of competition with endogenous formation of long-term network interaction with constant production volume and risky investment behavior by $\bar{\Gamma}_0^{\text{en}}$. The open-loop Nash equilibrium is characterized by the following theorem.

Theorem 3.3. *The open-loop Nash equilibrium for the model* $\bar{\Gamma}_0^{\text{en}}$ *is a set of strategies* $s^* = (s_1^*, \ldots, s_n^*)$, *whose components satisfy* (3.16) *for* $i \in \mathcal{N}$, $t \neq T$ *and have the form:*

$$g_{ij}^*(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \pi_{ij}^*, \ \sum\limits_{t=0}^{T-1} \rho^t \pi_{ji}(t) < \pi_{ji}^*, \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.17}$$

$$u_i^* = \frac{p - \sum\limits_{\tau=0}^{T-1} \rho^\tau \left((n+1)c_i^*(\tau) - \sum\limits_{j \in \mathcal{N}} c_j^*(\tau)\right) \cdot \left(\sum\limits_{\tau=0}^{T-1} \rho^\tau\right)^{-1}}{n+1}, \tag{3.18}$$

$$y_i^*(t) = -\frac{\alpha_i(t)\phi_i(t+1)}{\rho^t \varepsilon_i(t)}, \tag{3.19}$$

where

$$\pi_{ij}^*(t) = \sum_{t=0}^{T-1} \rho^t \frac{\varepsilon_i(t)}{\alpha_i(t)} \left(\beta_{ij}(t) - \gamma_{ij}(t)\right) y_i^*(t)\, y_j^*(t),$$

$$\phi_i(t) = \begin{cases} -\rho^t \left(u_i^* \cdot \sum\limits_{\tau=0}^{T-t-1} (\rho\delta)^\tau + \eta(\rho\delta)^{T-t}\right), & t \neq T, \\ -\rho^T \eta, & t = T. \end{cases}$$

The unit cost of $c_i^*(t)$ *satisfies* (2.9) *given* $c_i^*(0) = c_{i0}$.

Proof. The methodology of proving this theorem largely repeats the steps of proving Theorem 2.2, so it is omitted. □

Note the validity of the functional dependence (3.8) by replacing $y_i^{\mathrm{N}}(t)$ with $y_i^*(t)$ in it, and the corollary 3.1 is also in equilibrium for the model $\bar{\Gamma}_0^{\mathrm{en}}$. Next, we consider the condition of network interaction in equilibrium with unilateral interaction in the network $\mathbf{g}_0$.

Corollary 3.5 (from Theorem 3.3). *If we assume that the interaction of firms is one-sided, then the equilibrium behavior of firms in terms of production and investment behavior is preserved and determined according to* (3.18) – (3.19)*, and the network behavior of firm $i \in \mathcal{N}$ in Nash equilibrium takes the following form:*

$$g_{ij}^*(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \sum\limits_{t=0}^{T-1} \rho^t \dfrac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^*(t)\, y_j^*(t), \\ 0, & \textit{otherwise}, \end{cases} \qquad j \in \mathcal{N} \setminus \{i\}.$$

3.2.2. Constant investment

Let us move on to a model in which firms implement a constant volume of investments or a constant investment behavior. By the strategy of the firm $i \in \mathcal{N}$ we will understand a function that prescribes to it in an unambiguous way mt each decision period $t \in \mathcal{T} \setminus \{T\}$ feasible behavior of the form

$$s_i(t) = \begin{cases} (g_i(0), u_i, y_i) & \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i, \quad t = 0, \\ (u_i, y_i) & \in \mathbb{U}_i \times \mathbb{Y}_i, \quad t \in \mathcal{T} \setminus \{0, T\}. \end{cases} \tag{3.20}$$

The change in the unit cost of firm i is described by the equation

$$\begin{aligned} c_i(t+1) = \delta c_i(t) - \alpha_i(t) y_i - \\ - \sum_{j \neq i} \Big(\beta_{ij}(t) g_{ij}(0) g_{ji}(0) + \gamma_{ij}(t)(1 - g_{ij}(0) g_{ji}(0)) \Big) y_j, \end{aligned} \tag{3.21}$$

at $c_i(0) = c_{i0}$, and its profit under the network structure of bilateral long-term interaction $\mathbf{g}_0$, which is formed at the initial time, is given by

$$J_i\left(c_0, \mathbf{g}_0, u, y\right) = \sum_{t=0}^{T-1} \rho^t \left[\left(p - \sum_{j=1}^{n} u_j \right) u_i - c_i(t) u_i - \frac{\varepsilon_i(t)}{2} y_i^2 - \right.$$
$$\left. - \sum_{j \neq i} \pi_{ij}(t) g_{ij}(0) g_{ji}(0) \right] + \rho^T \left(\eta_i - \eta c_i(T) \right),$$

where $y = (y_1, \ldots, y_n) \in \mathbb{Y}_1 \times \ldots \times \mathbb{Y}_n$ for each $t \in \mathcal{T} \setminus \{T\}$. Note that the costs of network interaction of firms in $\mathbf{g}_0$ are time-bound.

Denote the presented dynamic model of competition with endogenous formation of long-term network interaction with constant production volume and cautious investment behavior by $\bar{\bar{\Gamma}}_0^{\text{en}}$. The open-loop Nash equilibrium for this model is characterized by the following theorem.

Theorem 3.4. *The open-loop Nash equilibrium for the model* $\bar{\bar{\Gamma}}_0^{\text{en}}$ *is a set of strategies* $s^{**} = (s_1^{**}, \ldots, s_n^{**})$ *whose components satisfy* (3.20) *for* $i \in \mathcal{N}$, $t \neq T$ *and have the form:*

$$g_{ij}^{**}(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \pi_{ij}^{**},\ \sum\limits_{t=0}^{T-1} \rho^t \pi_{ji}(t) < \pi_{ji}^{**}, \quad j \in \mathcal{N} \setminus \{i\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.22}$$

$$u_i^{**} = \frac{p - \sum\limits_{\tau=0}^{T-1} \rho^\tau \left((n+1) c_i^{**}(\tau) - \sum\limits_{j \in \mathcal{N}} c_j^{**}(\tau) \right) \cdot \left(\sum\limits_{\tau=0}^{T-1} \rho^\tau \right)^{-1}}{n+1}, \tag{3.23}$$

$$y_i^{**} = -\frac{\sum\limits_{t=0}^{T-1} \alpha_i(t) \phi_i(t+1)}{\sum\limits_{t=0}^{T-1} \rho^t \varepsilon_i(t)}, \tag{3.24}$$

where

$$\pi_{ij}^{**}(t) = \sum_{t=0}^{T-1} \rho^t \frac{\varepsilon_i(t)}{\alpha_i(t)} \left(\beta_{ij}(t) - \gamma_{ij}(t) \right) y_i^{**} y_j^{**},$$

$$\phi_i(t) = \begin{cases} -\rho^t \left(u_i^{**} \cdot \sum\limits_{\tau=0}^{T-t-1} (\rho\delta)^\tau + \eta (\rho\delta)^{T-t} \right), & t \neq T, \\ -\rho^T \eta, & t = T. \end{cases}$$

The unit cost of $c_i^{**}(t)$ *satisfies* (3.21) *given* $c_i^{**}(0) = c_{i0}$.

Proof. The methodology of proving this theorem largely repeats the steps of proving Theorem 2.2, so it is omitted. $\square$

Note the validity of the functional dependence (3.14) replacing $\hat{y}_i$ by y_i^{**} and the corollary 3.1. Let us move on to the condition of network interaction in equilibrium with one-way interaction in $\mathbf{g}_0$.

Corollary 3.6 (from Theorem 3.4)**.** *If we assume that the interaction of firms is one-sided, then the equilibrium behavior of firms in terms of production and investment behavior is preserved and determined according to* (3.23) – (3.24)*, and the network behavior of firm* $i \in \mathcal{N}$ *in Nash equilibrium takes the following form:*

$$g_{ij}^{**}(0) = \begin{cases} 1, & \sum\limits_{t=0}^{T-1} \rho^t \pi_{ij}(t) < \sum\limits_{t=0}^{T-1} \rho^t \dfrac{\varepsilon_i(t)}{\alpha_i(t)} (\beta_{ij}(t) - \gamma_{ij}(t))\, y_i^{**}\, y_j^{**}, \\ 0, & \textit{otherwise}, \end{cases} \quad j \in \mathcal{N} \setminus \{i\}.$$

3.2.3. Numerical simulations of equilibrium behavior for long-term network interactions

Move on to the numerical simulations of the Nash equilibrium defined by the conditions of Theorems 3.3–3.4 for the models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$, respectively. The input parameters of the simulation remain the same (page 61). In this Section, we are interested in comparing the results of modeling the practical model $\bar{\Gamma}_0^{\text{en}}$ with the theoretical model Γ_0^{en} in terms of costs aimed at forming and maintaining network interaction (Subsection 2.3.1 and Section 2.4), as well as in comparing the data in equilibria for two types of investment behavior of firms.

Let us present the simulation results in Tables 3.6 – 3.7, which have a structure already familiar to the reader and contain the main emphasis on the feasible behavior of firms in a Nash equilibrium strategy profile. In addition, we will provide Table 3.8 showing the upper bounds of the feasible costs of interaction of firms in equilibrium for the models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$, respectively. All simulation results are rounded to the third decimal place as before.

According to the data from Tables 3.6 – 3.8, it is possible to explain the network behavior of firms in equilibrium for the models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$ for both unilateral and

Table 3.6. **The Nash equilibrium s^* and the corresponding profits and unit costs of the firms, as well as the market price in the model $\bar{\Gamma}_0^{\text{en}}$**

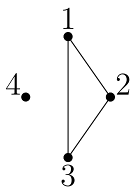
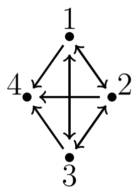
	bilateral interaction				unilateral interaction			
$t:$	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}_0^*$				–				
$g_1^*(0)$	(0,1,1,0)	–	–	–	(0,1,1,1)	–	–	–
$g_2^*(0)$	(1,0,1,0)	–	–	–	(1,0,1,1)	–	–	–
$g_3^*(0)$	(1,1,0,0)	–	–	–	(1,1,0,1)	–	–	–
$g_4^*(0)$	(0,0,0,0)	–	–	–	(0,0,0,0)	–	–	–
u_1^*	80.703	80.703	80.703	–	81.097	81.097	81.097	–
u_2^*	80.703	80.703	80.703	–	81.097	81.097	81.097	–
u_3^*	80.703	80.703	80.703	–	81.097	81.097	81.097	–
u_4^*	78.728	78.728	78.728	–	78.137	78.137	78.137	–
$y_1^*(t)$	2.045	1.876	1.710	–	2.047	1.877	1.710	–
$y_2^*(t)$	2.045	1.876	1.710	–	2.047	1.877	1.710	–
$y_3^*(t)$	2.045	1.876	1.710	–	2.047	1.877	1.710	–
$y_4^*(t)$	2.038	1.873	1.710	–	2.036	1.872	1.710	–
$c_1^*(t)$	100.000	98.209	97.018	96.456	100.000	97.187	94.986	93.427
$c_2^*(t)$	100.000	98.209	97.018	96.456	100.000	97.187	94.986	93.427
$c_3^*(t)$	100.000	98.209	97.018	96.456	100.000	97.187	94.986	93.427
$c_4^*(t)$	100.000	100.263	101.096	102.530	100.000	100.265	101.099	102.533
P^*	179.163	179.163	179.163	–	178.575	178.575	178.575	–
J_1^*	11 969.791				12 462.514			
J_2^*	11 969.791				12 462.514			
J_3^*	11 399.294				11 606.765			
J_4^*	10 448.161				10 187.284			

Table 3.7. **The Nash equilibrium s^{**} and the corresponding profits and unit costs of the firms, as well as the market price in the model $\bar{\bar{\Gamma}}_0^{\mathrm{en}}$**

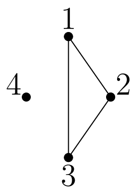
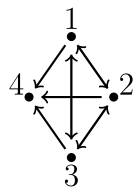
	bilateral interaction				unilateral interaction			
t :	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\mathbf{g}_0^{**}$				–				
$g_1^{**}(0)$	(0,1,1,0)	–	–	–	(0,1,1,1)	–	–	–
$g_2^{**}(0)$	(1,0,1,0)	–	–	–	(1,0,1,1)	–	–	–
$g_3^{**}(0)$	(1,1,0,0)	–	–	–	(1,1,0,1)	–	–	–
$g_4^{**}(0)$	(0,0,0,0)	–	–	–	(0,0,0,0)	–	–	–
u_1^{**}	80.589	80.589	80.589	–	80.961	80.961	80.961	–
u_2^{**}	80.589	80.589	80.589	–	80.961	80.961	80.961	–
u_3^{**}	80.589	80.589	80.589	–	80.961	80.961	80.961	–
u_4^{**}	78.725	78.725	78.725	–	78.166	78.166	78.166	–
y_1^{**}	1.883	1.883	1.883	–	1.883	1.883	1.883	–
y_2^{**}	1.883	1.883	1.883	–	1.883	1.883	1.883	–
y_3^{**}	1.883	1.883	1.883	–	1.883	1.883	1.883	–
y_4^{**}	1.879	1.879	1.879	–	1.878	1.878	1.878	–
$c_1^{**}(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$c_2^{**}(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$c_3^{**}(t)$	100.000	98.906	97.736	96.484	100.000	97.965	95.788	93.458
$c_4^{**}(t)$	100.000	100.793	101.642	102.550	100.000	100.794	101.644	102.553
P^{**}	179.508	179.508	179.508	–	178.951	178.951	178.951	–
J_1^{**}	11 921.435				12 401.633			
J_2^{**}	11 921.435				12 401.633			
J_3^{**}	11 350.935				11 545.883			
J_4^{**}	10 454.836				10 207.999			

Table 3.8. **The upper limits of feasible costs of interaction in $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$**

	bilateral interaction				unilateral interaction			
$i \backslash j$	1	2	3	4	1	2	3	4
				model $\bar{\Gamma}_0^{\text{en}}$				
1	0	2823.869	2823.869	2818.329	0	2826.078	2826.078	2817.771
2	2823.869	0	2823.869	2818.329	2826.078	0	2826.078	2817.771
3	2823.869	2823.869	0	2818.329	2826.078	2826.078	0	2817.771
4	2818.329	2818.329	2818.329	0	2817.771	2817.771	2817.771	0
				model $\bar{\bar{\Gamma}}_0^{\text{en}}$				
1	0	2808.443	2808.443	2803.498	0	2810.418	2810.418	2803.001
2	2808.443	0	2808.443	2803.498	2810.418	0	2810.418	2803.001
3	2808.443	2808.443	0	2803.498	2810.418	2810.418	0	2803.001
4	2803.498	2803.498	2803.498	0	2803.001	2803.001	2803.001	0

bilateral network interaction of firms, similar to what was done in Section 2.5 for the game-theoretic model Γ_{01}^{en}, which is why we omit it here.

Evaluate the results obtained by modeling equilibria in the models Γ_{01}^{en} and $\bar{\Gamma}_0^{\text{en}}$. As you can see from Tables 2.2 and 3.6, the network structures of interaction, regardless of its type (bilateral or unilateral), are preserved. Note that the total investment of the firms changes slightly during the transition from a variable production volume to a constant (in this case, from the model Γ_{01}^{en} to $\bar{\Gamma}_0^{\text{en}}$):

- In Γ_{01}^{en} with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} y_1^*(t) = \sum_{t=0}^{2} y_2^*(t) = \sum_{t=0}^{2} y_3^*(t) = 5.634, \quad \sum_{t=0}^{2} y_4(t) = 5.616,$$

  and with unilateral network interaction

$$\sum_{t=0}^{2} y_1^*(t) = \sum_{t=0}^{2} y_2^*(t) = \sum_{t=0}^{2} y_3^*(t) = 5.638, \quad \sum_{t=0}^{2} y_4^*(t) = 5.611;$$

- In $\bar{\Gamma}_0^{\text{en}}$ with bilateral network interaction of firms, we have

$$\sum_{t=0}^{2} y_1^*(t) = \sum_{t=0}^{2} y_2^*(t) = \sum_{t=0}^{2} y_3^*(t) = 5.631, \quad \sum_{t=0}^{2} y_4^*(t) = 5.621,$$

and with unilateral network interaction

$$\sum_{t=0}^{2} y_1^*(t) = \sum_{t=0}^{2} y_2^*(t) = \sum_{t=0}^{2} y_3^*(t) = 5.634, \quad \sum_{t=0}^{2} y_4^*(t) = 5.618.$$

At the same time, the change in the competitive position of firms in the market over time will be more noticeable in the model Γ_{01}^{en} than in $\bar{\Gamma}_0^{\text{en}}$, which is easy to verify. We also note that the results of the analysis based on the numerical simulations for Γ_{01}^{en} remain valid for the model $\bar{\Gamma}_0^{\text{en}}$, in which they are omitted.

It is noteworthy in this example that the profits of the firms (as well as the total volume of goods produced) in equilibrium for the models Γ_{01}^{en} and $\bar{\Gamma}_0^{\text{en}}$ with general simulations parameters, they turn out to be quite close. This can be easily verified by referring to Tables 2.2 and 3.6 and concluding that the change in profit is less than 1%. This allows us to conclude that the theoretical results obtained in Chapters 1-2, while maintaining the conceptual assumptions, turn out to be quite feasible for analyzing the behavior of competing firms in real conditions.

To compare the types of investment behavior, we look at the results of equilibrium simulations for the $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$. Despite the fact that the upper bounds of the allowable costs of interaction in Table 3.8 are different for the models, the network structure is preserved both with unilateral and bilateral network interaction. Thus, it can be concluded that the type of investment behavior in long-term interaction does not affect the partnerships of firms in the examples considered.

Referring to Tables 3.6 and 3.7, it is easy to make sure that the total investment of the firms in models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$ are also close, therefore we will immediately proceed to the assessment of the changes in the competitive position of the firms in the market with their equilibrium behavior, for which we will use the data in Table 3.9.

Analyzing the data in Table, we can conclude that, unlike the short-term interaction, the change in the competitive position of firms in the market occurs faster with risky investment behavior and unilateral interaction. At the same time, the lowest rate of change in the competitive position of firms is observed with cautious investment behavior and bilateral interaction.

Table 3.9. **The relationship between competitiveness and the competitive position of firms in the market, with their equilibrium behavior in models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$.**

t		bilateral interaction					unilateral interaction				
		model $\bar{\Gamma}_0^{\text{en}}$									
	i	$\Delta c^*_{i1}(t)$	$\Delta c^*_{i2}(t)$	$\Delta c^*_{i3}(t)$	$\Delta c^*_{i4}(t)$	$\mathcal{D}^*_i(t)$	$\Delta c^*_{i1}(t)$	$\Delta c^*_{i2}(t)$	$\Delta c^*_{i3}(t)$	$\Delta c^*_{i4}(t)$	$\mathcal{D}^*_i(t)$
1	1	-	0	0	2.054	0.520	-	0	0	3.078	0.784
	2	0	-	0	2.054	0.520	0	-	0	3.078	0.784
	3	0	0	-	2.054	0.520	0	0	-	3.078	0.784
	4	-2.054	-2.054	-2.054	-	0.000	-3.078	-3.078	-3.078	-	0.000
2	1	-	0	0	4.078	1.040	-	0	0	6.113	1.583
	2	0	-	0	4.078	1.040	0	-	0	6.113	1.583
	3	0	0	-	4.078	1.040	0	0	-	6.113	1.583
	4	-4.078	-4.078	-4.078	-	0.000	-6.113	-6.113	-6.113	0	0.000
3	1	-	0	0	6.074	1.550	-	0	0	9.106	2.379
	2	0	-	0	6.074	1.550	0	-	0	9.106	2.379
	3	0	0	-	6.074	1.550	0	0	-	9.106	2.379
	4	-6.074	-6.074	-6.074	-	0.000	-9.106	-9.106	-9.106	-	0.000
		model $\bar{\bar{\Gamma}}_0^{\text{en}}$									
	i	$\Delta c^{**}_{i1}(t)$	$\Delta c^{**}_{i2}(t)$	$\Delta c^{**}_{i3}(t)$	$\Delta c^{**}_{i4}(t)$	$\mathcal{D}^{**}_i(t)$	$\Delta c^{**}_{i1}(t)$	$\Delta c^{**}_{i2}(t)$	$\Delta c^{**}_{i3}(t)$	$\Delta c^{**}_{i4}(t)$	$\mathcal{D}^{**}_i(t)$
1	1	-	0	0	1.887	0.475	-	0	0	2.829	0.717
	2	0	-	0	1.887	0.475	0	-	0	2.829	0.717
	3	0	0	-	1.887	0.475	0	0	-	2.829	0.717
	4	-1.887	-1.887	-1.887	-	0.000	-2.829	-2.829	-2.829	-	0.000
2	1	-	0	0	3.906	0.989	-	0	0	5.856	1.505
	2	0	-	0	3.906	0.989	0	-	0	5.856	1.505
	3	0	0	-	3.906	0.989	0	0	-	5.856	1.505
	4	-3.906	-3.906	-3.906	-	0.000	-5.856	-5.856	-5.856	0	0.000
3	1	-	0	0	6.066	1.547	-	0	0	9.095	2.375
	2	0	-	0	6.066	1.547	0	-	0	9.095	2.375
	3	0	0	-	6.066	1.547	0	0	-	9.095	2.375
	4	-6.066	-6.066	-6.066	-	0.000	-9.095	-9.095	-9.095	-	0.000

3.3. Comparative analysis of types of network interactions and some patterns of equilibria for models with constant output

In this Section we will focus on comparing the equilibrium behavior of firms and its performance (profit and competitiveness) for different duration of network interaction.

■ **Network behavior.** The conditions of bilateral network interaction of firms in equilibrium, described in (3.2), (3.10), (3.17) and (3.22), have a common structure: firm i offers interaction to firm j if her cost of interaction associated with this connection in the network structure does not exceed a certain amount, which depends on the volume of her investments. In the case of unilateral network interaction, the relevant conditions have a similar structure. Moreover, in the case of unilateral interaction of firms, the number of links in the equilibrium network is not less than in the corresponding network with bilateral interaction, and at the same time the upper limits of feasible costs turn out to be very close within the same type of investment behavior of firms. For time-independent values $\alpha_i(t)$ and $\varepsilon_i(t)$, the volume of investment $y_i^{\mathrm{N}}(t)$ of firm i in Nash equilibrium is a monotonically decreasing function of time at $u_i^{\mathrm{N}} > \eta(1-\rho\delta)$. Moreover, the time-independent parameters $\beta_{ij}(t)$ and $\gamma_{ij}(t)$ lead to a monotonous decrease of the upper bounds of the allowable costs of network interaction $\pi_{ij}^{\mathrm{N}}(t)$ in (3.5), below which firms are willing to form partnerships (collaborate). In other words, the latter means that the number of such links in the network does not increase over time.

■ **Production behavior.** With a constant production plan, the quantities of goods produced by each firm differ slightly when comparing the equilibrium behavior of firms in models $\bar{\Gamma}^{\mathrm{en}}$, $\bar{\bar{\Gamma}}^{\mathrm{en}}$, $\bar{\Gamma}_0^{\mathrm{en}}$, $\bar{\bar{\Gamma}}_0^{\mathrm{en}}$. At the same time, in equilibrium, these models share a common feature, which we formulate in the form of the following remark.

Remark 3.1 (equilibrium production behavior)**.** *The production behavior of firms in equilibrium has the same functional structure that does not depend on the duration of their network interaction and their investment behavior — cautious or risky.*

This remark is based on the expressions (3.3), (3.11), (3.18) and (3.23) of the equilibrium production behavior of firms in the compared models.

■ **Investment behavior.** In this chapter, two types of the investment behavior of firms in equilibrium (risky and cautious) have been considered. At the same time, as follows from Theorems 3.1– 3.4, within the same type of investment behavior of firms, the investment volumes have the same functional expression. With risky investment behavior, this follows from (3.4) and (3.19), and with caution — from (3.12) and (3.24). In addition, all four models $\bar{\Gamma}^{\mathrm{en}}$, $\bar{\bar{\Gamma}}^{\mathrm{en}}$, $\bar{\Gamma}_0^{\mathrm{en}}$, $\bar{\bar{\Gamma}}_0^{\mathrm{en}}$ regarding the investment behavior of firms can be combined with the following remark.

Remark 3.2 (on equilibrium production and investment behavior)**.** *Regardless of the type of the firms' investment behavior, the duration and type of network interaction, in the Nash equilibrium there is a linear relationship between the firm's current level of investment and its output.*

This remark is based on the equivalences (3.8) and (3.14), which, as already noted, retain their functional expression within the same type of investment behavior of firms. Note also that in Chapter 1 the proposition 1.1 was presented which, assuming the time-independent of network parameters, also allows us to prove the observation that the firm with the lowest unit cost in open-loop Nash equilibrium strategies produces more goods and invests more funds than the firm that has higher unit costs. For all models, this follows from the Corollary 3.1, which, as noted earlier, can be concluded with respect to any of the models $\bar{\Gamma}^{\text{en}}$, $\bar{\bar{\Gamma}}^{\text{en}}$, $\bar{\Gamma}_0^{\text{en}}$, $\bar{\bar{\Gamma}}_0^{\text{en}}$.

■ **Equilibrium profits and changes in the competitive position of firms in the market.** With unilateral network interaction, in all the examples in this chapter, it is clear that some firms have the opportunity to obtain higher profits compared to profits with bilateral interaction in the corresponding model. However, the rate of change in the competitive position of firms in the market, according to the conclusions in Subsections 3.1.3 and 3.2.3, depends on the type of network interaction. Thus, in the case of short-term interaction, the largest changes are observed in the case of unilateral interaction and cautious investment, and in the case of long-term interaction — in the case of bilateral interaction and risky investment of firms. Comparing Tables 3.2 and 3.7, we conclude that with careful investment, the duration of interaction may not be fundamental, but from Tables 3.1 and 3.6, that with risky investment, long-term interaction is preferable for most firms — in terms of the rate of change in competitiveness.

3.4. A one-time investment model

In the models considered, the mutually beneficial network interaction of even one pair of firms makes other firms interested in network interaction and the implementation of non-zero investments. In the final part of the thesis, we will consider a model in which firms implement their investment behavior once, in the initial period, and choose a one-time investment volume. We assume that in this case firms make a one-time decision about their willingness to collaborate with other firms, bear one-time costs of interaction in the established network, and also have a one-time effect on the investment behavior of all firms. Thus, the feasible behavior of firm $i \in \mathcal{N}$ in the initial period will be a triple $s_i(0) = (g_i(0), u_i, y_i(0)) \in \mathbb{G}_i \times \mathbb{U}_i \times \mathbb{Y}_i$ and $s_i(t) = u_i \in \mathbb{U}_i$ at all subsequent times $t \neq T$. This allows us to describe the dynamics of the unit cost of firm i by the equation

$$c_i(t+1) = \\ = \begin{cases} \delta c_{i0} - \alpha_i(0)y_i(0) - \sum\limits_{j\neq i}\Big(\beta_{ij}(0)g_{ij}(0)g_{ji}(0) + \gamma_{ij}(1-g_{ij}(0)g_{ji}(0))\Big)y_j(0), & t = 0, \\ \delta c_i(t), & t \notin \{0, T\}, \end{cases}$$

or

$$c_i(t) = \delta^{t-1}\Big[\delta c_{i0} - \alpha_i(0)y_i(0) - \\ - \sum_{j\neq i}\Big(\beta_{ij}(0)g_{ij}(0)g_{ji}(0) + \gamma_{ij}(0)(1 - g_{ij}(0)g_{ji}(0))\Big)y_j(0)\Big], \quad (3.25)$$

where $c_i(0) = c_{i0}$ and $t \neq 0$, and the firm's profit is given by the following expression

$$J_i(c_0, g(0), u, y(0)) = \sum_{t=0}^{T-1}\rho^t\Big(p - c_i(t) - \sum_{j\in\mathcal{N}} u_j\Big)u_i - \frac{\varepsilon_i(0)}{2}\big(y_i(0)\big)^2 - \\ - \sum_{j\neq i}\pi_{ij}(0)g_{ij}(0)g_{ji}(0) + \rho^T(\eta_i - \eta c_i(T)).$$

Denote this dynamic model of competition with endogenous network formation of constant production volume and one-time investment of firms by $\widetilde{\Gamma}_0^{\text{en}}$. The open-loop Nash equilibrium with bilateral interaction of firms for this model is characterized by the following theorem.

Theorem 3.5. *The Nash equilibrium in the $\widetilde{\Gamma}_0^{\text{en}}$ model is a set of strategies $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$, whose components are*

$$\tilde{s}_i(t) = \begin{cases} (\tilde{g}_i(0), \tilde{u}_i, \tilde{y}_i(0)), & t = 0, \\ \tilde{u}_i, & t \notin \{0, T\}, \end{cases}$$

for $i \in \mathcal{N}$, have form:

$$\tilde{g}_{ij}(0) = \begin{cases} 1, & \pi_{ij}(0) < \tilde{\pi}_{ij},\, \pi_{ji}(0) < \tilde{\pi}_{ji}, \quad j \neq i, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.26}$$

$$\tilde{u}_i = \frac{p - \sum\limits_{\tau=0}^{T-1} \rho^\tau \Big((n+1)\tilde{c}_i(\tau) - \sum\limits_{j \in \mathcal{N}} \tilde{c}_j(\tau) \Big) \cdot \Big(\sum\limits_{\tau=0}^{T-1} \rho^\tau \Big)^{-1}}{n+1}, \tag{3.27}$$

$$\tilde{y}_i(0) = \frac{\rho \alpha_i(0)}{\varepsilon_i(0)} \Big(\tilde{u}_i \sum_{t=0}^{T-2} (\rho\delta)^t + \eta (\rho\delta)^{T-1} \Big), \tag{3.28}$$

where

$$\tilde{\pi}_{ij} = \frac{\varepsilon_i(0)}{\alpha_i(0)} (\beta_{ij}(0) - \gamma_{ij}(0))\, \tilde{y}_i(0)\, \tilde{y}_j(0).$$

The current unit cost of $\tilde{c}_i(t)$ is determined according to (3.25),

Proof. First, we assume that each firm i chooses as its network behavior, instead of a set of $g_i(0)$, an n-dimensional vector $z_i(0)$ with components $z_{ij}(0) \in [0,1]$. Let $z(0) = (z_1(0), \ldots, z_n(0))$; σ_i — the strategy of the firm i, and $\sigma = (\sigma_1, \ldots, \sigma_n)$ — a set of strategies. The best response of firm i to the fixed strategies of its competitors is a strategy whose components satisfy the following system (taking into account the linearity of J_i in the variables $z_{ij}(t)$):

$$z_{ij}(0) = \begin{cases} 1, & \big(\pi_{ij}(0) - \dfrac{\varepsilon_i(0)}{\alpha_i(0)} (\beta_{ij}(0) - \gamma_{ij}(0)) y_i(0) y_j(0)\big) z_{ji}(0) < 0, \\ & \text{и } j \neq i, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_i = \frac{1}{2} \Bigg(p - \sum_{j \neq i} u_j - \sum_{\tau=0}^{T-1} \rho^\tau c_i(\tau) \cdot \Big(\sum_{\tau=0}^{T-1} \rho^\tau \Big)^{-1} \Bigg),$$

$$y_i(0) = \frac{\rho \alpha_i(0)}{\varepsilon_i(0)} \Bigg(u_i \sum_{t=0}^{T-2} (\rho\delta)^t + \eta (\rho\delta)^{T-1} \Bigg).$$

Thus, if $\tilde{\sigma}$ is a Nash equilibrium, then $\tilde{s} = \tilde{\sigma}$ and $\tilde{g}_i(0) = \tilde{z}_i(0)$. The Nash equilibrium requires that firms i and j establish a connection in the initial period, while satisfying the inequalities

$$\pi_{ij}(0) < \frac{\varepsilon_i(0)(\beta_{ij}(0) - \gamma_{ij}(0))}{\alpha_i(0)}\tilde{y}_i(0)\tilde{y}_j(0), \;\; \pi_{ji}(0) < \frac{\varepsilon_j(0)(\beta_{ji}(0) - \gamma_{ji}(0))}{\alpha_j(0)}\tilde{y}_i(0)\tilde{y}_j(0).$$

Thus we arrive at the expressions (3.26), (3.27) and (3.28), the sufficiency of which is given by the negative-definiteness of the Hessian of the Lagrange function

$$-2\tilde{u}_i\left(\sum_{t=1}^{T-1}\rho^t\tilde{c}_i(t) + \tilde{u}_i\sum_{t=0}^{T-1}\rho^t\right) - \varepsilon_i(0)\tilde{y}_i^2(0) < 0,$$

which means that $\tilde{s}$ will be a Nash equilibrium in $\widetilde{\Gamma}_0^{\mathrm{en}}$. □

Despite the fact that firms implement their investment behavior on a one-time basis, we note that Corollary 3.1 presented in Subsection 3.1.1, with constant network parameters, retains its validity for equilibrium in model $\widetilde{\Gamma}_0^{\mathrm{en}}$. We will present this corollary further and without proof, which is similar to the proof of the Corollary 3.1.

Corollary 3.7. *Let $\alpha_i(0) = \alpha$ and $\varepsilon_i(0) = \varepsilon$ for arbitrary firms. Then, in Nash equilibrium, for any firms i and j, the following three conditions are equivalent:*

$$\frac{\sum\limits_{\tau=0}^{T-1}\rho^\tau\tilde{c}_i(\tau)}{\sum\limits_{\tau=0}^{T-1}\rho^\tau} < \frac{\sum\limits_{\tau=0}^{T-1}\rho^\tau\tilde{c}_j(\tau)}{\sum\limits_{\tau=0}^{T-1}\rho^\tau} \quad\Leftrightarrow\quad \tilde{u}_i > \tilde{u}_j \quad\Leftrightarrow\quad \tilde{y}_i(0) > \tilde{y}_j(0).$$

Next, we consider a variant of unilateral network interaction of firms in equilibrium for model $\widetilde{\Gamma}_0^{\mathrm{en}}$. As for the previous models, it can be obtained from the condition of bilateral network interaction in equilibrium.

Remark 3.3 (from Theorem 3.5). *If we assume that the interaction of the firms is one-sided, then the equilibrium behavior of the firms in terms of production and investment behavior is preserved and determined according to* (3.27) – (3.28)*, and the network behavior of the firm in Nash equilibrium is $i \in \mathcal{N}$:*

$$\tilde{g}_{ij}(0) = \begin{cases} 1, & \pi_{ij}(0) < \dfrac{\varepsilon_i(0)}{\alpha_i(0)}(\beta_{ij}(0) - \gamma_{ij}(0))\, \tilde{y}_i(0)\, \tilde{y}_j(0), \\ 0, & otherwise, \end{cases} \quad j \in \mathcal{N} \setminus \{i\}.$$

The relationship between the production and investment behavior of each firm in Nash equilibrium is characterized by the following remark, which can be inferred from (3.28).

Remark 3.4. *The investment behavior of the firm $i \in \mathcal{N}$ in Nash equilibrium for model $\widetilde{\Gamma}_0^{\text{en}}$ has a linear relationship with its production behavior.*

Let us move on to the example with the results of numerical simulations of the equilibrium in both bilateral and unilateral interaction in model $\widetilde{\Gamma}_0^{\text{en}}$ according to the conditions of Theorem 3.5 and Remark 3.3. Keeping the same modeling input parameters as before, we will only change the cost of interaction for firm 4: $\pi_{4j}(0) = 1200, j = 1, 2, 3$ and present the result in Table 3.10, rounding the results to the third decimal place.

Based on the data in Tables 3.10 and 3.11, it is still possible to explain the equilibrium network behavior of firms, which is why it is omitted here.

If, when numerically modeling the equilibrium in model $\widetilde{\Gamma}_0^{\text{en}}$, one adheres to the input parameters presented on the page 61, then the upper bounds of the allowable costs of network interaction between firms would be jointly $\tilde{\pi}_{ij}(0) = 1158.862$, where $i, j \in \mathcal{N}$, such that $i \neq j$. In this case, all firms would be interested in network interaction with all their competitors, regardless of the nature of the formation of interaction in the network $\tilde{\mathbf{g}}(0)$. At the same time, taking into account the data in Table 3.8, it can be noted that in case of one-time investments the upper limits of the allowable costs of network interaction of firms are significantly lower than the corresponding limits in case of regular investments (models $\bar{\Gamma}_0^{\text{en}}$ and $\bar{\bar{\Gamma}}_0^{\text{en}}$). This allows us to make the following observation.

Observation 3.1. *The values of the upper limits of the allowable costs of network interaction depend on the duration of the firms' investments.*

Table 3.10. **The Nash equilibrium $\tilde{s}$ and the corresponding profits and unit costs of the firms, as well as the market price in the model $\widetilde{\Gamma}_0^{\mathrm{en}}$**

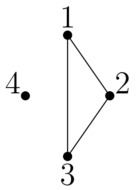
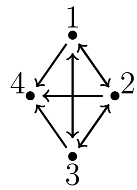
	bilateral interaction				unilateral interaction			
$t:$	$t=0$	$t=1$	$t=2$	$t=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\tilde{\mathbf{g}}_0$				–				
$\tilde{g}_1(0)$	(0,1,1,0)			–	(0,1,1,1)	–	–	–
$\tilde{g}_2(0)$	(1,0,1,0)			–	(1,0,1,1)	–	–	–
$\tilde{g}_3(0)$	(1,1,0,0)			–	(1,1,0,1)	–	–	–
$\tilde{g}_4(0)$	(0,0,0,0)			–	(0,0,0,0)	–	–	–
$\tilde{u}_1$	80.072	80.072	80.072	–	80.346	80.346	80.346	–
$\tilde{u}_2$	80.072	80.072	80.072	–	80.346	80.346	80.346	–
$\tilde{u}_3$	80.072	80.072	80.072	–	80.346	80.346	80.346	–
$\tilde{u}_4$	78.696	78.696	78.696	–	78.283	78.283	78.283	–
$\tilde{y}_1(0)$	2.043			–	2.044			–
$\tilde{y}_2(0)$	2.043			–	2.044			–
$\tilde{y}_3(0)$	2.043			–	2.044			–
$\tilde{y}_4(0)$	2.038			–	2.037			–
$\tilde{c}_1(t)$	100.000	98.218	105.093	112.449	100.000	97.196	104.000	111.280
$\tilde{c}_2(t)$	100.000	98.218	105.093	112.449	100.000	97.196	104.000	111.280
$\tilde{c}_3(t)$	100.000	98.218	105.093	112.449	100.000	97.196	104.000	111.280
$\tilde{c}_4(t)$	100.000	100.267	107.285	114.795	100.000	100.268	107.287	114.797
$\tilde{P}$	181.088	181.088	181.088	–	180.679	180.679	180.679	–
$\tilde{J}_1$	3 928.200				4 254.398			
$\tilde{J}_2$	3 928.200				4 254.398			
$\tilde{J}_3$	3 728.200				3 954.398			
$\tilde{J}_4$	2 903.154				2 720.382			

Table 3.11. **Upper limits of feasible costs of interaction in the model $\widetilde{\Gamma}_0^{\text{en}}$**

$\widetilde{\Gamma}_0^{\text{en}}$	bilateral interaction				unilateral interaction			
$i \setminus j$	1	2	3	4	1	2	3	4
1	0	1159.403	1159.403	1156.710	0	1160.477	1160.477	1156.439
2	1159.403	0	1159.403	1156.710	1160.477	0	1160.477	1156.439
3	1159.403	1159.403	0	1156.710	1160.477	1160.477	0	1156.439
4	1156.710	1156.710	1156.710	0	1156.439	1156.439	1156.439	0

3.5. Conclusions to Chapter 3

The presented results of equilibrium modeling in model $\widetilde{\Gamma}_0^{\text{en}}$ (with one-time investments by firms) allow us to make a simple but important remark. This remark helps to stabilize and maintain the balance of trade and consumer interaction in the sales market of a certain product: the absence of investments by firms from period $t = 1$ entails a monotonous increase in the unit costs of firms ($\delta = 1.07$) and a monotonous decrease in the intermediate profits of firms, as a result — a decrease in profitability and liquidity each firm has its own production facilities.

Under the considered conditions, the only possible way to maintain the liquidity of production and to increase the profits of firms is to invest in the modernization of their own production technologies and other components of unit costs. The rationality of interaction with other firms at the same time contributes to saving investments and increasing intermediate profits, and for consumers of goods — to reducing the market value of a unit of goods. This means that the positive investment behavior of competing firms is beneficial to all market participants.

In this Chapter an attempt has been made to response the fundamental questions of equilibrium behavior and the corresponding conditions for rational interaction of each pair of firms. The assumptions proposed for the models in this Chapter made it possible to find and analyze the equilibrium behavior of firms in market competition, which makes it possible to distinguish these models as practical. On the basis of these models one can possible to find and analyze the equilibrium behavior of firms, to compare and evaluate the prospects of their long-term and short-term interaction with different types of investment behavior. It is shown that the difference in the profits of the firms in the considered models does not cause a significant difference in a short period of time, which cannot be said about the dynamics of their competitiveness. Thus, in Section 3.3 it is noted that in the case of short-term interaction the largest changes in competitiveness are observed under unilateral interaction and cautious investment behavior of firms, and in the case of long-term interaction - under unilateral interaction and risky investment behavior.

It is important to note that the results of the comparative analysis presented in Subsections 3.1.3, 3.2.3, as well as in Section 3.3 are based on the assumption that network parameters are constant over time and the same for different types of the duration of interaction. However, it is obvious that if the cost of interaction includes, for example, renting a warehouse to store goods, then renting for a longer period may be much cheaper than renting for a short period. Therefore, the theorems formulated in the Chapter on the equilibrium behavior of firms are recommended to be applied in the analysis of the prospects of potential interaction for each possible type of model parameters separately — so that it would be objectively possible to conclude the advantage of any type of duration of interaction for the considered set of model parameters.

The main results of the thesis described in this Chapter are presented in the article [29].

Conclusion

The thesis is devoted to the characterization and analysis of the equilibrium behavior of competing firms in dynamic models with network interaction and is related to the theory of dynamic network games. Within the framework of the recharacterization conducted, an investment and network modification of the Cournot oligopoly was built, in which firms competing in the common sales market are endowed with the ability to implement multicomponent strategic behavior in a dynamic manner. At the same time, the concept under consideration was applied to two types of the formation of network interaction of firms – exogenous (Chapter 1) and endogenous (Chapter 2). The adaptation of models with endogenous formation of network interaction to the characterization and analysis of the equilibrium behavior of firms in market conditions similar in practice is proposed (Chapter 3).
The main results of the thesis are as follows

1. An network modification of the Cournot oligopoly is constructed, for which the two-component Nash equilibrium behavior of firms with dynamic exogenous network formation is obtained. The equilibrium is presented and characterized for a open-loop strategies. In addition, a feedback Nash equilibrium has been obtained and the «proximity» of the two equilibria found has been established [27].

2. The influence of the network structure and the associated coefficients of the model on the behavior of firms in equilibrium, and how the structure of interaction of firms affects changes in their unit costs, competitiveness in the market, profits, as well as the price of a unit of goods in the market are analysed [27].

3. A feasible behavior of each firm is complemented by a component that characterizes the attitude toward network interaction with its competitors and is responsible for its network behavior. A functional structure of the Nash equilibrium behavior of firms with dynamic endogenous network formation obtained [28].

4. A Nash equilibrium is obtained for two types of network interaction — with the formation of a constant and a variable network structure. At the same

time, the costs associated with the networking of firms are also considered in two types — one-time and regular. A comparative analysis of the obtained results is carried out [28].

5. Assumptions are proposed and justified that serve to adapt the studied game theoretic models to the practical interaction of competing firms in the market. The conditions for choosing business partners and options for the duration of interaction between firms in the Nash equilibrium are considered. A comparative analysis of the Nash equilibrium is given for two types of the investment behavior of firms common in real conditions - risky (variable) and cautious (constant), taking into account the duration of their interaction [29].

6. A functional expression of the equilibrium behavior of firms under their one-time investment in their production is obtained. The relation between the changes of the upper limits of the allowable costs of network interaction and the duration of the investments of the firms is shown [29].

7. For each model of endogenous network formation studied in the thesis, the equilibrium network behavior of competitors are obtained, the fulfillment of which makes firms interested in network interaction with their competitors. At the same time, two types for the formation of network interaction are considered, represented by undirected or directed links between firms. The thesis notes that in network structures, which are formed when firms implement their equilibrium network behavior, it is unprofitable for any firm to unilaterally break any of its existing connections, as well as to strive to create a new one, for which the condition of equilibrium network behavior is not fulfilled [28, 29].

It remains to conclude that all the tasks formulated within the framework of the thesis have been completed and the objective set has been fully achieved.

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