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Game-theoretic centrality measures in networks and applications

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Introductoin

Relevance of thesis topic. The development of social networks has had a huge impact on the development of new methods of network analysis, including game-theoretic methods. Methods of social network analysis are applied in many fields, such as economics, physics, biology and information technology. At the qualitative level, a social network is understood as a social structure consisting of a set of agents (individual or collective subjects, e.g., individuals, families, groups or organisations) and a set of relations defined on it (a set of links between agents, e.g., acquaintance, cooperation, communication) [1]. To describe such a network, it is convenient to use a graph in which a finite set of vertices corresponds to a set of agents, and a set of edges describes the interaction of these agents.

The concept of modern network analysis was formulated on the basis of the works of J. Moreno [2] through maps of relations between agents (adjacency matrices) and visualisations of these maps (actually, graphs) [3].

Many system analysis problems are reduced to graph analysis. Structural graph analysis is an effective tool for investigating complex systems, discovering hidden patterns and predicting behaviour. It is an integral part of data analytics, science and product development, where the relationships between system elements play an important role. For example, in social networks, understanding the structure of social connections between people or organisations allows you to investigate information influence, identify key agents, define groups or communities, and analyse the spread of information or diseases. Vertex ranking can help to identify the most important community leaders who have the most influence on other participants [4]. In bioinformatics, structural analysis allows us to study molecular structures, gene networks and protein interactions. The methods of vertex ranking are used to find key proteins that control many biological processes [5], to identify key regions of the genome associated with various diseases [6; 7]. Vertex ranking methods can be used for solving machine learning problems, for example, for pattern recognition or disease risk prediction. Pre-ranking of graph vertices can improve the efficiency of solving some applied optimisation problems [8]. Peer-to-peer systems can also be represented as graphs. In them, the location of vertices and the ranking of vertices depending on their location in the system occupy an important place. For this purpose, centrality measures can be used [9—12].

In connection with the above mentioned, the research and development of new methods for ranking graph vertices can lead to the improvement of quality of solving different kinds of problems relevant in the modern world. In general, graphs are a powerful tool for representing complex relationships between objects or entities.

Centrality of a vertex is one of the key concepts when studying structural characteristics of a graph. Often in small systems the most central vertex can be determined intuitively. As D. Brass and M. Burkhardt [13] note, when describing a network structure, "most people would simply look at the graph and declare which of the central actors is the most powerful" . At the Massachusetts Institute of Technology in the 50s of the XX century, A. Bavelas [14] and G. Livitt [15] conducted a number of studies, as a result of which the researchers came to the conclusion that the central role is related to social status, power, and the agent's satisfaction with group activity [16].

The centrality measure demonstrates how important a vertex is to the graph as a whole, how well placed it is on the paths connecting the vertices of the graph, whether it is key to maintaining links between the others vertices. The centrality of a graph vertex is of great importance for analysing transport systems, social, electrical and other networks.

One of the first definitions of centrality was the concept of "betweenness centrality" [17], which was related to the shortest paths that pass through the considered vertex. However, considering only the shortest paths raises serious objections, since information may spread not necessarily along the shortest path. Therefore, centrality measures based on random walks [18; 19] and other centrality measures based on the computation of the inverse Laplace matrix have emerged. The latter have a good analogy with electrical networks and Kirchhoff's laws [20—22]. Recently, works related to game-theoretic centrality measures have appeared [23—28]. As a measure of centrality, one can consider the income of a player in a network from economic interaction with other related players, where the income is discounted as a function of the shortest distance between players [29]. Game-theoretic methods are also used for graph clustering. For this purpose, the notion of Nash-stable coalition partitioning is used, where players in their own coalition are disadvantaged to join players of other coalitions [30—33].

To determine the centrality of vertices in a graph, an approach related to the theory of cooperative games can be used. In this case, the solution of a cooperative game is used to rank vertices. R. Myerson in his 1976 paper [34] proposes to describe cooperation between players using an undirected graph; in this case, the presence of an edge between two vertices of the graph means that the pair of players corresponding to these vertices can mutually act directly. This interaction can describe familiarity or cooperation, but can also be interpreted as, for example, the existence of a motorway between two transport nodes, or the possibility of transferring information or resources between organisations. For this class of problems, Mayerson proposed to distribute the payoffs between the players according to the Shepley scheme [35]. This approach later became known as the Myerson vector [36]. Despite the large number of methods developed to distribute winnings between players in a network, such as [37—43], the Myerson vector is one of the most used.

The aim of this thesis is to develop new game-theoretic methods for analysing graph structure and their applications in transport systems, social and communication networks.

To achieve this goal it was necessary to solve the following **tasks**:

1. Development of a method for finding the centrality of vertices in directed graphs based on the number of vertices occurrences in paths of fixed length
2. Development of a method for finding the centrality of vertices in undirected graphs based on the number of occurrences of vertices in paths of fixed length.

3. Development of a method for the centrality measurement of the graph vertices based on the values of the absolute potentials of the nodes of an electric circuit calculated using the Laplace matrix.
4. Building a model of the Petrozavodsk transport system to test the proposed methods.

Methodology and methods of research. The methods of graph theory, linear algebra, cooperative game theory are used in this work.

Relation of the work to scientific programmes: the results of the work were partially obtained as part of research carried out with financial support of RSF No 22-11-20015, "Research and development of mathematical models and software for finding equilibrium traffic flows and optimization of a transportation network on the case of Petrozavodsk city", jointly with support of the authorities of the Republic of Karelia with funding from the Venture Investment Foundation of the Republic of Karelia.

Approbation of the work. The main results of the work were presented at the following seminars and conferences:

1. LIV International Conference of Postgraduates and Students "Control Processes and Stability" (CPS'23), St. Petersburg, 3 - 7 April 2023;
2. All-Russian Scientific Conference "Theory and Practice of System Dynamics", Apatity, 3 - 7 April 2023;
3. The Sixteenth International Conference on Game Theory and Management (GTM2023), St. Petersburg, 28-30 June 2023;

and at scientific seminars of the Institute of Applied Mathematical Research of the Karelian Centre of the Russian Academy of Sciences.

Publications. The dissertation materials were published in 7 scientific papers, including 1 article in a journal indexed in the Web of Science bibliographic database; 2 articles in a journal indexed in the bibliographic list Scopus; 1 article in a journal indexed in the RSCI bibliography list; 3 articles in journals indexed in the РИНЦ bibliographic database.

Scope and structure of the work. The thesis consists of an introduction, 3 chapters, conclusion and 1 appendix. The full volume of the thesis is 123 pages, including 52 figures and 24 tables. The list of literature contains 78 titles.

The **introduction** reflects the relevance, significance and objectives of the work; substantiates the scientific novelty, theoretical and practical value of the research; formulates the main provisions for defence.

The **first chapter** is devoted to the description of a game-theoretic approach to calculating the centrality of graph vertices based on the number of vertex appearances in paths. The use of the Myerson vector as a measure of centrality for oriented graphs is described. Formulas for calculating the number of occurrences of a vertex in simple paths of length 2 and 3 through the adjacency matrix are given. The notion of integral centrality as a definite integral of the sharing function in a cooperative game, where the characteristic function is defined using the number of total appearances of a graph vertex in simple paths is introduced. It is shown that the Boldi - Vigna axioms [44] are valid for such a centrality measure. Also, a modification of the Myerson value for vertices of an undirected graph is proposed for the case when cycles are included in the consideration. A formula for calculating the number of appearances of graph vertices in paths of fixed length with cycles is given. A number of special cases are considered.

Chapter 2 demonstrates an approach to ranking of graph vertices based on the values of absolute potentials of the electric circuit nodes. A two-stage procedure for ranking the vertices of a graph with weighted edges is proposed, where at the first stage the nodes are ranked on the basis of absolute potentials at consecutive current supply to all nodes of the circuit. At the second stage, a tournament table is constructed and a final ranking is performed based on the sum of the previously found ranks. The distributions of vertex ranks are obtained for a clique, a star graph, a double star graph, and a complete bipartite graph. For a graph with weighted edges and vertices, a ranking procedure based on the total work required to transfer the charge between the nodes of an electric circuit is proposed. Analytical formulas for calculating the values of the total work done in the vertices of the star, clique and full bipartite graphs are obtained.

The **third chapter** presents the results of applying the proposed methods to the graphs obtained by solving a number of applied problems. It describes the process of building a model of the transport network of Petrozavodsk city, where intersections of motorways are considered as vertices. In addition, the applicability of the proposed approaches to calculating the centrality of graph vertices for analysing biological populations and social networks is shown.

The **conclusion** lists the main results of the work carried out.

Appendix A contains a table with information on passenger flow at St. Petersburg Subway stations.

Main scientific results:

1. During the dissertation work we have proposed an approach to compute the centrality values of vertices of directed graphs as the Myerson value in a cooperative game, where the characteristic function is defined in terms of the number of simple paths of fixed length in the subgraph generated by the coalition. In a more general case, we introduce the notion of integral centrality as the value of the definite integral of the payoff function in a cooperative game, where the characteristic function is defined in terms of a polynomial by analogy with Jackson's scheme [45].
2. A method for determining the measure of node centrality in an undirected graph based on the modification of the Myerson value in a cooperative game, where the number of paths (including cycles) in the subgraph corresponding to the coalition is used as a characteristic function, is proposed.
3. A method for the centrality measurement of the graph vertices with weighted edges based on the values of the absolute potentials of the nodes of an electric circuit calculated using the Laplace matrix has been developed.
4. A method for the centrality measurement of the graph vertices with weighted vertices and edges based on the total work of charge transfer between nodes of an electric circuit is developed.

Practical significance. In the course of the work the application of the obtained results was demonstrated on the example of the transport network of Petrozavodsk. The most important sections as well as "bottlenecks" were determined. Analytical expressions for finding the centrality values of vertices using different approaches for solving cooperative games were obtained.

Results submitted for defense:

1. A method is proposed for determining the centrality measure of the directed graph vertices, as the Myerson value in a cooperative game, where the number of simple paths acts as a characteristic function in the subgraph corresponding to the coalition.
2. A method for ranking graph vertices is proposed, based on the introduced new concept of integral centrality, as the value of a definite integral of the payoff function in a cooperative game, where the characteristic function is determined through the total number of occurrences of vertices in the subgraph corresponding to the coalition. An axiomatic justification for this centrality measure is given.
3. A method for determining the measure of node centrality in a undirected graph based on the modification of the Myerson value in a cooperative game, where the number of paths (including cycles) in the subgraph corresponding to the coalition is used as a characteristic function, is proposed.
4. A method for the centrality measurement of the graph vertices based on the values of absolute potentials of the electric circuit nodes calculated using the Laplace matrix is proposed.
5. A method for the centrality measurement of the vertices of a graph with weighted vertices based on the total work of charge transfer between the nodes of an electric circuit is proposed.

Chapter 1. A game-theoretic approach to calculating centrality

1.1 Centrality measures in graph theory

Centrality of a vertex in a graph is an indicator defining the most important vertices in the graph. Centrality allows to identify the most important vertices, to evaluate how well they are located in the graph, how they influence the processes occurring in the network. There are different approaches to calculating centrality indices (values), each of which has its own interpretation. Let us describe the most common approaches. Let us consider a graph G with a set of vertices V, $|V| = n$ and a set of edges E.

Degree centrality

The simplest in terms of computation is degree centrality [46], which in general shows how many neighbours a vertex has.

$$c_D(v_i) = \sum_{\substack{v_j \in V \\ j \neq i}} a_{ij}, \tag{1.1}$$

where a_{ij} are the elements of the corresponding adjacency matrix. The vertex with the largest number of links is the most central. When analysing directed graphs, the numbers of incoming and outgoing connections can be considered (in-degree centrality, out-degree centrality) [47].

Betweenness centrality

Historically, one of the first approaches is betweenness centrality [48; 49]. This centrality measure is based on calculating the number of shortest paths connecting all pairs of vertices in a graph. In this case, the centrality of a vertex is determined by the number of paths passing through the vertex.

$$c_B(v) = \frac{1}{n_B} \sum_{s,t \in V} \frac{\sigma_{s,t}(v)}{\sigma_{s,t}}, \tag{1.2}$$

where $\sigma_{s,t}$ is the number of geodesic paths between vertices s and t, $\sigma_{s,t}(v)$ is the number of geodesic paths between vertices s and t that pass through vertex v. The coefficient n_B is chosen as follows: $n_B = (n-1)(n-2)$ if vertex v cannot be the initial (s) or final (t) vertex, and $n_B = n(n-1)$, otherwise.

Closeness centrality

Also, one of the common methods is closeness centrality [14; 50; 51], where the most central vertex is the closest to other vertices in the network. The closeness centrality was defined by A. Bavelas in 1950 as a value inverse to remoteness:

$$c_C(v) = \frac{1}{\sum\limits_{w \in V} d(v,w)}, \quad v,w \in V, \tag{1.3}$$

where $d(v,w)$ is the length of the shortest path in the graph between vertices v and w.

For directed graphs, shortest distances from or to a vertex can be considered.

A modification of this approach was proposed in [52], taking into account that in graphs with infinitely large distances between vertices, the harmonic mean, rather than the arithmetic mean, gives the best results. Thus, harmonic centrality was introduced:

$$c_H(v) = \sum_{w \in V} \frac{1}{d(v,w)}. \tag{1.4}$$

For directed graphs, the approach was reviewed in [44].

PageRank

As one of the most common approaches to calculate the centrality of graph vertices, we can consider the ranking of vertices using the PageRank method. This method can be related to the process of random walk [53]. On a set of n web pages, hyperlinks are clicked with some probability α, and with probability $1-\alpha$ the process can go to a random web page, then the stationary distribution of the process can be interpreted as the final probability of being in the vertices of the graph. The higher the probability, the more important the vertex is to the system, and hence the higher its centrality value.

The transition probability matrix $\widetilde{P}$ is calculated as follows

$$\widetilde{P} = \alpha P + (1-\alpha)(\frac{1}{n}\mathbb{E}), \tag{1.5}$$

where the value of α is chosen from (0,1), $\mathbb{E}_{n\times n}$ is an identity matrix, and $P_{n\times n}$ is a matrix whose elements are equal:

$$p_{ij} = \begin{cases} \frac{1}{k} & \text{vertex } i \text{ has } k > 0 \text{ outgoing links and } j \text{ is one of them,} \\ 0 & \text{vertex } j \text{ is not an outgoing link for } i, \\ \frac{1}{n} & \text{vertex } i \text{ has no outgoing links, } k = 0. \end{cases}$$

Also, the PageRank method can be used for a weighted graph with a weight matrix W and a vertex degree matrix D, then the matrix $P = D^{-1}W$.

Game-theoretic centrality

In recent years, game-theoretic centrality measures have become more and more widespread [24—27; 30; 54]. In many applications, centrality based on the role that a graph vertex plays by itself is not relevant. For example, if we consider the problem of stopping an epidemic, vaccinating a connected group is more efficient than vaccinating individuals. In this case, the vaccinated group can be viewed as a coalition of players in a cooperative game.

A cooperative game of n persons is a pair $\Gamma = \langle N,v\rangle$ where $N = \{1,2,\dots,n\}$ is the set of players, $v : 2^N \to \mathbb{R}$ is a mapping that assigns to each coalition $S \in 2^N$ (where 2^N is the set of all subsets of the set N) some numerical value: $v(\varnothing) = 0$. The function v is called the characteristic function of a cooperative game [55].

By the solution of a cooperative game we mean the sharing of the winnings of the grand coalition $v(N)$ among all players. A payoff in a cooperative game Γ is a vector $X = (X_1,X_2,\dots,X_n)$ which satisfies *the property of individual rationality*

$$X_i \geqslant v_i, \quad i = 1,\dots,n, \tag{1.6}$$

and *the property of collective rationality*

$$\sum_{i\in N} X_i = v(N). \tag{1.7}$$

One of the most popular solutions in the theory of cooperative games is the Shapley vector [35]. Let us recall its definition. We denote by $\sigma = (\sigma(1),..., \sigma(n))$ an arbitrary permutation of players of the set N 1,...,n. Let us assume that all permutations σ are equally probable, and then the probability of each permutation is $1/n!$.

Consider player i. Let us imagine that a coalition will be formed when he appears. Denote by $P_\sigma(i) = \{j \in N : \sigma^{-1}(j) < \sigma^{-1}(i)\}$ the set of players who arrived before his appearance in the permutation σ. The contribution of player i in this coalition is equal to

$$m_i(\sigma) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)). \tag{1.8}$$

Definition. *The Shapley value is the average contribution of each player in all possible permutations*

$$\varphi_i(v) = \frac{1}{n!}\sum_\sigma m_i(\sigma) = \frac{1}{n!}\sum_\sigma \left[v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))\right], i = 1,..., n. \tag{1.9}$$

There are games where players are connected by some relations, which can be conveniently described by a graph structure, where vertices are players and edges are interactions between them.

A cooperative game of n persons with network structure is called a triple $\langle N,v,G\rangle$ where N is the set of players, $v : 2^N \to \mathbb{R}$ is a characteristic function, G is the graph of connections of players from the set N. Players $i,j \in N$ can interact directly only if there exists an edge ij in the graph G.

For games defined on graphs, R. Myerson modified the Shapley vector. He suggested that it is natural to consider unconnected coalitions as a set of connected components. For each such component K, its payoff is $v(K)$, and it is this payoff that is distributed among the members of the coalition K. The distribution of the gain is done according to the Shapley scheme. R. Myerson [34] introduced axioms that uniquely define such a payoff distribution for each player $i \in N$, a given graph G and a characteristic function v.

A1. *Axiom of component efficiency.* If S is a graph component, then the sum of the payoffs of the coalition players is equal to the value of the whole coalition, i.e.

$$\sum_{i \in S} X_i(v,G) = v(S). \tag{1.10}$$

A2. *Fairness axiom.* For any pair of players $i,j \in N$, both players gain or lose benefits equally if edge ij is added or removed:

$$X_i(v,G) - X_i(v,G - ij) = X_j(v,G) - X_j(v,G - ij), \tag{1.11}$$

where $G - ij$ – is the graph G without edge ij.

1.2 Myerson vector as the centrality measure of a directed graph vertices

Let us define a cooperative game $\Gamma = \langle N,v \rangle$ on a directed graph $G = (N,E)$, where N is the set of vertices and E is the set of edges. In this game, N is the set of players on which is given a characteristic function $v(K), K \subset N$, $v(K)$ is equal to the number of simple paths of fixed length $k = 1,2,\ldots$ in the subgraph, generated by the set of players K. For ranking the vertices in the directed graph the solution of the cooperative game in the Shapley-Mierson form can be used.

Theorem 1. The Myerson value for player $i \in N$ in a cooperative game on a directed graph G, where the characteristic function $v(K)$ is defined as the number of directed simple paths of fixed length k in the subgraph generated by the set $K \subset N$, can be found by the formula

$$X_i = \frac{n_k(i)}{k+1}, \tag{1.12}$$

where $n_k(i)$ is the number of simple paths of length k passing through vertex i.

Proof. To prove the statement it is sufficient to prove the fulfilment of the Myerson axioms [34].

For simplicity, assume that the graph G is connected. In this case $v(N)$ is the number of directed paths of length k in the graph G. Let us renumber all paths

$l \in \{1,2,...,v(N)\}$. We define $\delta_l(i)$ as follows. We will assume $\delta_l(i) = 1$ if vertex i is in path l and 0 otherwise. Then

$$\sum_{i=1}^{n} X_i = \frac{1}{k+1}\sum_{i=1}^{n} n_k(i) = \frac{1}{k+1}\sum_{i=1}^{n}\sum_{l=1}^{v(N)} \delta_l(i) = \frac{1}{k+1}\sum_{l=1}^{v(N)}\sum_{i=1}^{n} \delta_l(i).$$

Each path consists of $k+1$ different vertices $(i_1,...i_{k+1})$. Hence $\sum\limits_{i=1}^{n} \delta_l(i) = k+1$. Therefore

$$\sum_{i=1}^{n} X_i = \frac{1}{k+1}\sum_{l=1}^{v(N)}\sum_{i=1}^{n} \delta_l(i) = v(N).$$

Thus, Axiom 1 is satisfied. Let us proceed to Axiom 2.

For example, let $ij \in E(G)$. Let's delete this edge, in this case, directed. Then all paths of length k that previously passed through edge ij will be subtracted when counting paths simultaneously from $n_k(i)$ and $n_k(j)$ in the new graph $G - ij$. Hence,

$$X_i(G) - X_i(G - ij) = X_j(G) - X_j(G - ij).$$

Thus axiom 2 is also true, which proves the theorem. $\square$

As follows from Theorem 1, the Myerson value is defined in terms of the number of simple paths of a given length. The problem of computing the number of simple paths passing through a vertex is non-trivial. However, if we restrict ourselves to directed graphs with no bidirectional edges, we can compute the number of simple paths of lengths 2 and 3 from the adjacency matrix. For the calculations we need the square and cube of the adjacency matrix.

Statement 1.1. Let A be the adjacency matrix of a directed graph G, and A^2 is its square. Then the number of appearances of vertex i in simple paths of length 2 $n_2(i)$ can be calculated by the formula

$$n_2(i) = \sum_{k=1}^{n}\left(a_{ik}^{(2)} + a_{ki}^{(2)}\right) + \sum_{k=1}^{n}\sum_{j=1}^{n} a_{ki}a_{ij}. \tag{1.13}$$

The first expression corresponds to the number of all simple paths of length 2 starting or ending at the vertex i. The second expression takes into account paths in which vertex i lies in the middle of the path.

Statement 1.2. Let A be the adjacency matrix of a directed graph G, and A^2, A^3 be the square and cube of the matrix A. Then the number of occurrences of vertex i in simple paths of length 3 $n_3(i)$ can be calculated by the formula

$$n_3(i) = \sum_{\substack{k=1 \\ k\neq i}}^{n} \left(a_{ik}^{(3)} + a_{ki}^{(3)}\right) + \sum_{k=1}^{n} a_{ki} \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{ij}^{(2)} + \sum_{k=1}^{n} a_{ki}^{(2)} \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{ij}. \tag{1.14}$$

Here, the first summand corresponds to the number of all simple paths of length 3 starting or ending at vertex i. The second summand takes into account the paths in which vertex i lies at the second position in the path, and the third summand takes into account the paths in which vertex i lies at the third position in the path.

Example 1.1. Let us illustrate the above formula on the example of an directed graph G_1 of 6 vertices (Fig. 1.1) with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us write out, for example, the paths of length $d = 3$ passing through vertex 3. There are 8 such paths in total:

3654 3612

1365 5361

6132 6134 6532 6534

The first line lists the paths starting at vertex 3. The second line lists paths with vertex 3 in the second place, and the third line lists vertex 3 in the third place. There are no other paths with vertex 3.

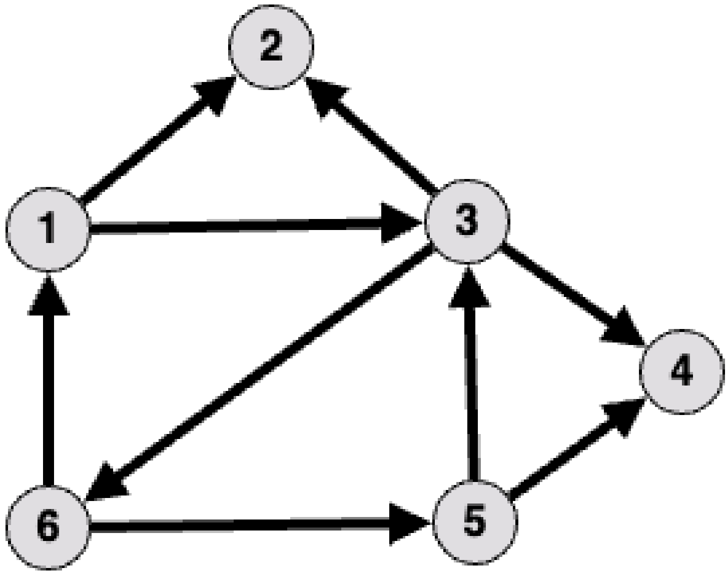


Figure 1.1 — Graph G_1

Now let's calculate the square and cube of the adjacency matrix.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

By the formula from statement 1.2 we find

$$n_3(3) = \sum_{\substack{k=1 \\ k\neq 3}}^{6} \left(a_{3k}^{(3)} + a_{k3}^{(3)}\right) + \sum_{k=1}^{6} a_{k3} \sum_{\substack{j=1 \\ j\neq k}}^{6} a_{3j}^{(2)} + \sum_{k=1}^{6} a_{k3}^{(2)} \sum_{\substack{j=1 \\ j\neq k}}^{6} a_{3j} =$$
$$= (a_{32}^{(3)} + a_{34}^{(3)}) + (a_{13}a_{35}^{(2)} + a_{53}a_{31}^{(2)}) + (a_{63}^{(2)}a_{32} + a_{63}^{(2)}a_{34}) = 2 + 2 + 4 = 8.$$

This coincides with the number of simple paths of length 3 passing through vertex 3.

1.3 Integral centrality

Above, on a directed graph $G = (N,E)$, $|N| = n$, we considered a cooperative game of n individuals $\Gamma = \langle N,v(K)\rangle$, where the characteristic function $v(K)$ is defined as the number of directed paths of fixed length d passing through the considered vertex in the subgraph generated by the coalition K. By varying the path length d, we can define the characteristic function more generally as a polynomial:

$$v(K) = \sum_{i\in K}\sum_{d=1}^{n-1} n_d(i)r^d,\ r \in [0,1], \tag{1.15}$$

where $n_d(i)$ is the number of simple paths of length d passing through vertex i.

The value of r can be determined by analogy to Jackson's approach [45], where players get r for creating a direct link, the coalition gets r^2 for creating a path of length 2, etc. Here the players forming the link get r for appearing a *pair* in paths of length 1, r^2 for the appearance of a *triple* in paths of length 2, etc.

Similarly to Section 1.2, it can be shown (see also [27]) that the distribution of coalition payoff among the players according to the Myerson value has the form:

$$X_i(r) = \frac{n_1(i)}{2}r + \frac{n_2(i)}{3}r^2 + \cdots + \frac{n_{n-1}(i)}{n}r^{n-1} = \sum_{d=1}^{n-1}\frac{n_d(i)}{d+1}r^d. \tag{1.16}$$

By choosing a particular value of r, one can obtain the value of the payoff function $X_i(r)$ for all players. To eliminate the step with the choice of the r value, the values of the definite integral of the payoff function on the variable r on the interval [0,1] can be used in the ranking. The payoff functions are polynomial functions, which makes it easy to write down expressions for determining centrality:

$$I_i = \int_0^1 X_i(r)dr = \int_0^1 \sum_{d=1}^{n-1}\frac{n_d(i)}{d+1}r^d\ dr = \sum_{d=1}^{n-1}\frac{n_d(i)}{(d+1)^2}. \tag{1.17}$$

Definition Value $I_i = \int_0^1 X_i(r)dr = \sum_{d=1}^{n-1}\frac{n_d(i)}{(d+1)^2}$, where $X_i(r)$ is the payoff function in the cooperative game Γ on the graph, is the integral centrality of vertex i.

Example 1.2. Let us consider as an example a fragment of the Math-Net citation graph (Fig. 1.2). In this case, the presence of directed edge ij means that the author i refers to the work of the author j.

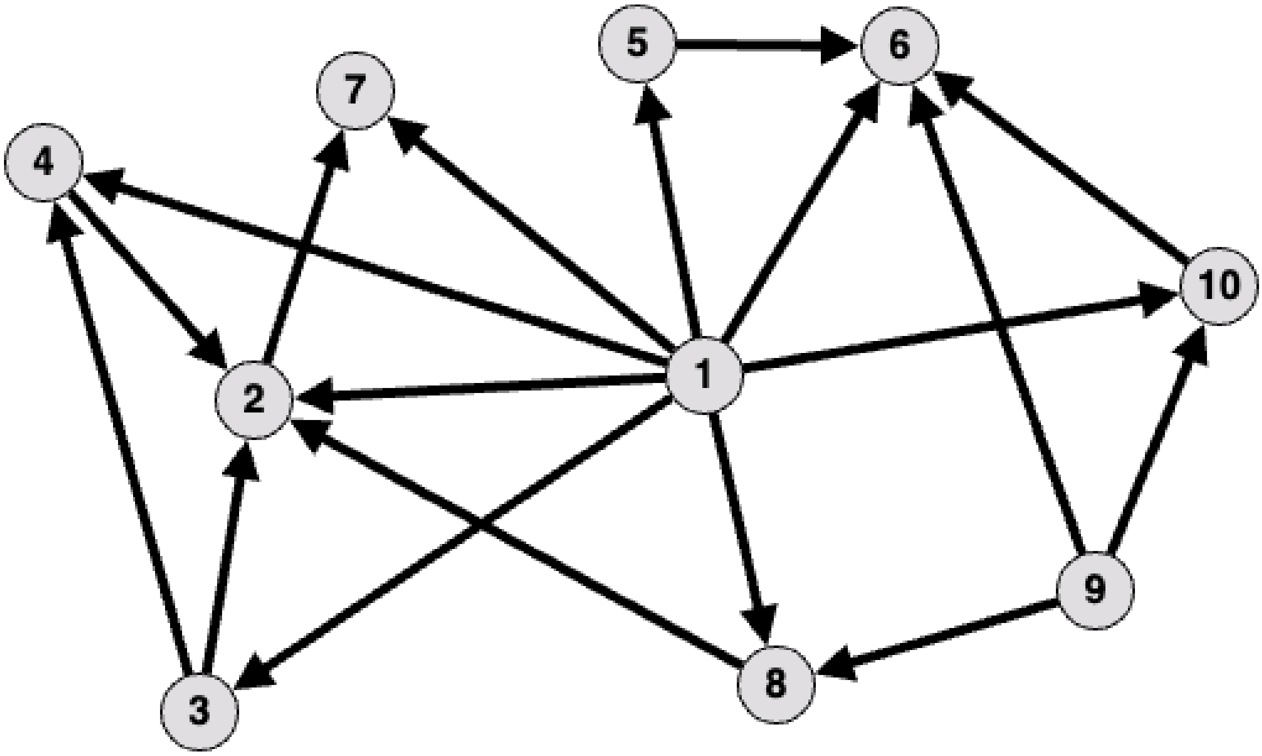


Figure 1.2 — Math-Net citation graph fragment

Let us find the integral centrality of the vertices of this graph. Let us denote by n_k the vector of the number of appearances of the graph vertices in paths of length k.

$$n_1 = (8, 5, 3, 3, 2, 4, 2, 3, 3, 3)\,,$$
$$n_2 = (7, 9, 4, 4, 1, 3, 4, 3, 2, 2)\,,$$
$$n_3 = (4, 6, 3, 3, 0, 0, 5, 2, 1, 0)\,,$$
$$n_4 = (1, 1, 1, 1, 0, 0, 1, 0, 0, 0)\,,$$

all other $n_k, k \geqslant 5$ are zero.

Then the values of integral centrality according to the definition are:

$$\begin{aligned} I &= \frac{n_1}{4} + \frac{n_2}{9} + \frac{n_3}{16} + \frac{n_4}{25} = \\ &= (3.067, 2.665, 1.421, 1.421, 0.611, 1.333, 1.296, 1.208, 1.034, 0.972). \end{aligned}$$

As a result, vertex 1 has the highest centrality, vertex 2 is also important.

1.3.1 Special cases: cycle and complete graph

Let us calculate the integral centrality of vertices for the cycle and the complete graph. Obviously, due to symmetry, the centrality of all vertices in this case is the same and it is enough to calculate the centrality of one of them.

For all vertices of the $p-$cycle the payoff is defined by the function:

$$X(r) = \sum_{d=1}^{p-1} \frac{n_d}{d+1} r^d = \sum_{d=1}^{p-1} \frac{d+1}{d+1} r^d = \sum_{d=1}^{p-1} r^d. \tag{1.18}$$

Then the value of integral centrality is written as follows:

$$I_p = \int_0^1 X(r)dr = \int_0^1 \sum_{d=1}^{p-1} r^d dr = \sum_{d=1}^{p-1} \frac{1}{d+1}. \tag{1.19}$$

For $k-$clique vertices the formula is as follows:

$$n_d(k) = (d+1)! \binom{k-1}{d}. \tag{1.20}$$

Indeed, in a path of length d: $l = (i_1,...,i_{d+1})$, vertex i can be at the first, second,..., $d+1$-th place. The remaining d of $k-1$ vertices can be selected in

$$A_{k-1}^d = \binom{k-1}{d} \cdot d!$$

ways. Hence,

$$X(r) = \sum_{d=1}^{k-1} \frac{n_d(k)}{d+1} r^d = \sum_{d=1}^{k-1} d! \binom{k-1}{d} r^d, \tag{1.21}$$

and then the integral centrality is

$$I_k = \int_0^1 X(r)dr = \int_0^1 \sum_{d=1}^{k-1} d! \binom{k-1}{d} r^d dr = \frac{1}{k} \sum_{d=1}^{k-1} d! \binom{k}{d+1} = \\ = \sum_{d=1}^{k-1} \frac{(k-1)!}{(d+1)(k-d-1)!}. \tag{1.22}$$

1.3.2 Axioms of centrality

Boldi - Vigna [44] describes a system of centrality axioms based on checking the change of centrality measure when studying cliques and directed cycles. The following should be checked: whether all vertices of a $k-$clique have the same centrality value; whether all vertices of a directed $p-$cycle have the same centrality value; whether vertices of a $k-$clique are more important than vertices of a directed $p-$cycle. Let us give the formulations of the axioms as they are presented in [44; 56].

A1 *(Size Axiom).* Consider the graph $S_{k,p}$ (Fig. 1.3) consisting of two components: a $k-$clique and a $p-$cycle. A centrality measure satisfies the size axiom if for each k there exists a number P_k, such that for all $p \geqslant P_k$ in the graph $S_{k,p}$, the centrality of a vertex in a $p-$cycle is strictly greater than the centrality of a vertex in a $k-$clique, and for each p there exists a number K_p, such that for all $k \geqslant K_p$, the centrality of a vertex in a $k-$clique is strictly greater than the centrality of a vertex in a $p-$cycle.

A2 *(Density Axiom).* Consider the graph $D_{k,p}$ (Fig. 1.4) consisting of a $k-$clique and a $p-$cycle (k,$p \geqslant 3$) connected by a bidirectional bridge $x \longleftrightarrow y$, where x is a vertex of the clique and y is a vertex of the cycle. A centrality measure satisfies the density axiom if for $k = p$ the centrality of x is strictly greater than the centrality of y.

A3 *(Score–Monotonicity Axiom).* A centrality measure satisfies the score-monotonicity axiom if for every graph G and every pair of nodes x and y such that

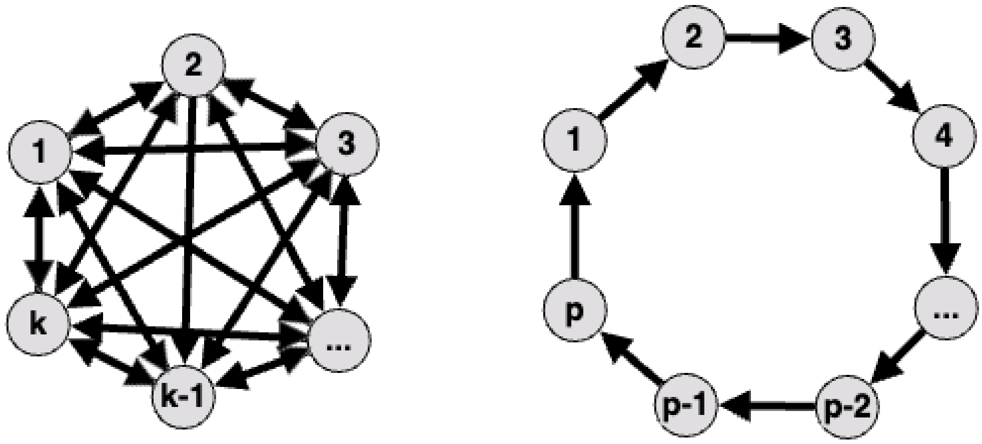


Figure 1.3 — Graph $S_{k,p}$

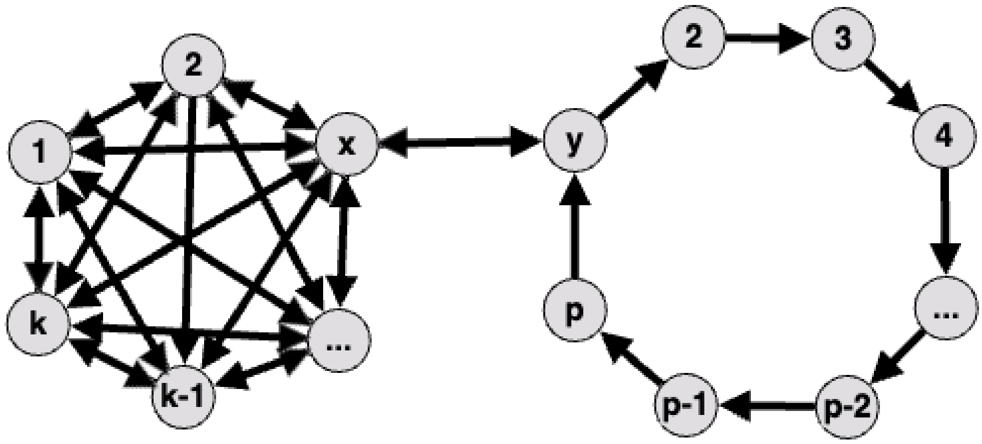


Figure 1.4 — Graph $D_{k,p}$

$x \to y$ does not belong to the set of edges E_G of graph G, if we add to G such an edge, then the centrality of y will increase.

Let us add to this system the axiom of connectivity.

A4 *(Connectivity Axiom).* A centrality measure satisfies the connectivity axiom if for any graph G and any two connectivity components G_1 and G_2 of graph G, and for every pair of vertices $x \in G_1$ and $y \in G_2$, the centrality of all vertices in G_2 does not decrease if an edge $x \to y$ is added.

Theorem 2. For integral centrality (1.17), axioms A1, A2, A3, A4 are true.

Proof. Let us prove that these axioms hold for the integral centrality measure.

A1 To prove this axiom, it suffices to show that the centrality of the vertices of both $p-$cycle and $k-$clique increases unboundedly with increasing p and k. In the previous section it was shown that for all vertices of the $p-$cycle the integral centrality is determined by the function:

$$I_p^c = \sum_{d=1}^{p-1} \frac{1}{d+1}.$$

The centrality value of the $p-$cycle vertices can be estimated as $I_p = S_p - 1$, where S_p is the sum of the first p terms of the harmonic series. L. Euler obtained an asymptotic expression for the sum of the first n terms of the harmonic series:

$$S_n = \ln n + \gamma + \varepsilon_n,$$

where $\gamma = 0{,}5772\ldots$ – Euler – Mascheroni constant [57], ε_n – error, $\varepsilon_n \to 0$, $n \to \infty$. Then $I_p^c \to \infty$, $p \to \infty$.

For $k-$clique vertices we have the formula

$$I_k^q = \sum_{d=1}^{k-1} \frac{(k-1)!}{(d+1)(k-d-1)!}.$$

It is easy to see that $I_k^q \geqslant I_k^c$, whence follows the unbounded growth of I_k^q, when $k \to \infty$.

A2. It was noted above that for $k = p$ $I_k^q(x) \geqslant I_k^c(y)$ i.e., centrality of vertex x in the clique is greater than the centrality of vertex y in the cycle. Thus, for any d, the number of directed paths of length d in the clique $n_d^q(x)$ is greater than the number of directed paths of length d in the cycle $n_d^c(y)$. Let us connect vertex x of the clique to vertex y in the cycle by a bidirectional bridge $x \longleftrightarrow y$. Then the number of paths of any length d in both the clique and the cycle increases by the same value. Therefore, the integral centrality of vertex x will still be greater than the centrality of vertex y. The validity of axiom $A2$ is proved.

A3. Axiom $A3$ is obviously true, because after adding the edge $x \to y$, the vertex y will occur in paths ending in it, which will increase the centrality value.

A4. Axiom A4 is true because after adding an edge $x \to y$, the number of directed paths passing through vertices from the graph G_2 can only increase. $\square$

It is worth noting that according to [44] the axioms $A1$, $A2$, $A3$ without any conditions are valid only for harmonic centrality [52].

1.4 Computing vertices centrality in directed graphs with cycles

In the proposed approach for determining the centrality of vertices in a graph, the main problem is to compute the number of simple paths (without cycles) of fixed length passing through a given vertex. In [54], a modification of Myerson's value is given for the case when cycles are also considered in addition to simple paths. In this case it is possible to obtain quite simple expressions for this characteristic using the elements of the adjacency matrix of the considered graph. For ranking in this case the values $\frac{s_i(k)}{k+1}$ is used, where $s_i(k)$ is the number of appearances of vertex i in paths of length k. Also in [54] proofs of theorems on the number of appearances of vertices in paths of fixed length, including cycles, in undirected graphs are given. Further investigation showed that the similar formula is also valid for the number of occurrences of a vertex in paths in directed graphs.

Theorem 3. Let A^d be the adjacency matrix of a directed graph G raised to the power d. Then the number of appearances of vertex i in paths of fixed length d (including cycles) $n_d(i)$ can be calculated as

$$n_d(i) = \sum_{k=1}^{n} \left(a_{ik}^{(d)} + a_{ki}^{(d)} \right) + \sum_{l=1}^{d-1} \left[\sum_{k=1}^{n} a_{ki}^{(l)} \cdot \sum_{j=1}^{n} a_{ij}^{(d-l)} \right]. \qquad (1.23)$$

Proof. The first sum takes into account the occurrences of vertex i at the beginning and end of paths of length d. The values $a_{ik}^{(d)}$ and $a_{ki}^{(d)}$ – the elements of the A^d matrix – correspond to the number of paths of length d starting in vertex i and ending in it. The second sum allows to take into account the occurrences of the considered vertex in the middle of paths of length d: $a_{ki}^{(l)}$ is an element of the matrix A^l, equal to the number of paths of length l ending in vertex i; $a_{ij}^{(d-l)}$ is an element of the matrix A^{d-l}, describing the number of paths of length $d-l$ starting in the same vertex. By adding their products for all admissible l, we obtain the number of occurrences of vertex i in the middle of paths of fixed length d. $\square$

Example 1.3. Let us illustrate the above formula on the example of an directed graph G_2 of 6 vertices (Fig. 1.5) with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

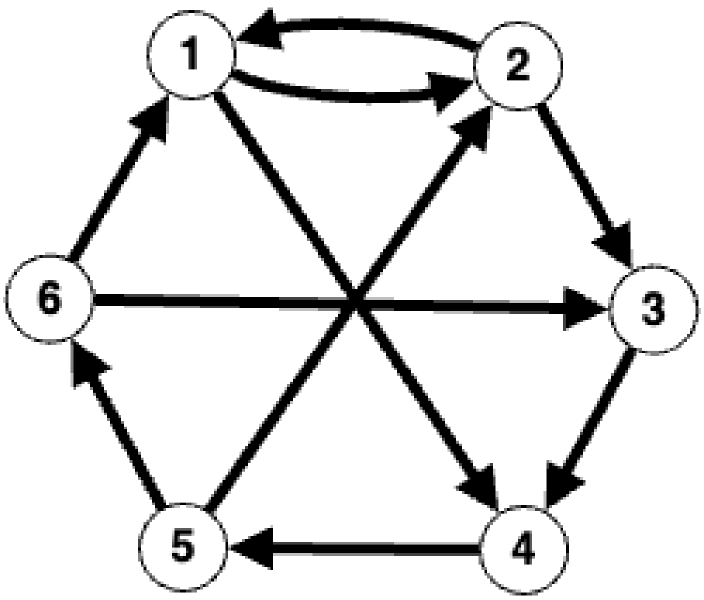


Figure 1.5 — Graph G_2

Let us write out the paths of length $d = 3$. There are 25 such paths in total:

Vertex 2 occurs in paths of length 3 21 times. Let us calculate the number of occurrences of vertex 2 by formula 1.23:

$$n_3(2) = \sum_{k=1}^{6} \left(a_{2k}^{(3)} + a_{k2}^{(3)} \right) + \sum_{l=1}^{2} \left[\sum_{k=1}^{6} a_{k2}{}^{(l)} \cdot \sum_{j=1}^{6} a_{2j}^{(3-l)} \right].$$

1212	2121	3456	4521	5212	6121
1234	2123	3452	4561	5612	6123
1452	2345		4523	5214	6345
1456	2145		4563	5614	6145
1214				5234	
				5634	

The computations require the square and cube of the adjacency matrix.

$$A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

$$n_3(2) = (2+2+1+1+2+1) + \left[\sum_{k=1}^{6} a_{k2} \cdot \sum_{j=1}^{6} a_{2j}^{(2)} + \sum_{k=1}^{6} a_{k2}^{(2)} \cdot \sum_{j=1}^{6} a_{2j} \right] =$$
$$= 9 + [2 \cdot 3 + 3 \cdot 2] = 9 + 6 + 6 = 21.$$

For directed acyclic graphs, i.e., directed graphs without directed cycles but allowing parallel paths, the number of occurrences of a vertex in paths of fixed length coincides with the number of simple paths passing through this vertex. Therefore, the value obtained with the help of formula (1.23) can be used to find the Myerson value in an directed graph by formula (1.12).

In the case of an arbitrary directed graph, we will define the vertex centrality in the following form

$$X_i(r) = \frac{n_1(i)}{2} r + \frac{n_2(i)}{3} r^2 + \cdots + \frac{n_{n-1}(i)}{n} r^{n-1} = \sum_{d=1}^{n-1} \frac{n_d(i)}{d+1} r^d. \tag{1.24}$$

The payoff X obviously satisfies the first Myerson axiom [34] (by the way of constructing the payoff and specifying the characteristic function), but it does not satisfy the second axiom (fairness axiom), which states that both players i and j must equally gain or lose benefits when creating or removing the link ij. This condition is not fulfilled due to the inclusion of cycles in the consideration. In general, it is possible that paths of the form $\dots ijijijiji \dots$ may appear, which leads to a different number of appearances of vertices i and j depending on the parity of the path length.

Let us illustrate it by means of a counterexample. For this purpose, let us return to the graph G_2 (Example 1.3). Let's remove the connection $1-4$ from this graph (Fig. 1.6).

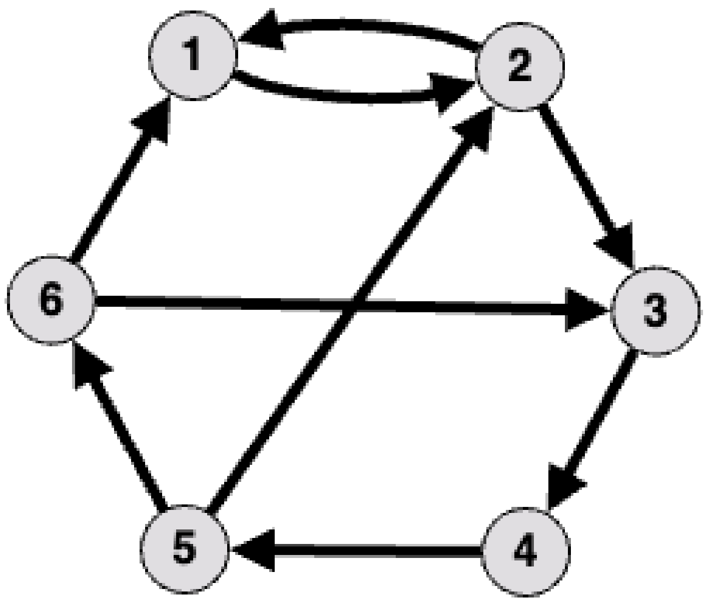


Figure 1.6 — Graph $G_2 - 14$

Let's write down the payoffs of players 1 and 4 before removing the link $(X_1,\ X_4)$ and afterwards $(X_1',\ X_4')$.

$$X_1 = \frac{4}{2}r + \frac{10}{3}r^2 + \frac{21}{4}r^3 + \frac{42}{5}r^4 + \frac{82}{6}r^5$$

$$X_4 = \frac{3}{2}r + \frac{8}{3}r^2 + \frac{18}{4}r^3 + \frac{36}{5}r^4 + \frac{70}{6}r^5$$

$$X_1' = \frac{3}{2}r + \frac{7}{3}r^2 + \frac{13}{4}r^3 + \frac{22}{5}r^4 + \frac{37}{6}r^5$$

$$X_4' = \frac{2}{2}r + \frac{5}{3}r^2 + \frac{11}{4}r^3 + \frac{19}{5}r^4 + \frac{34}{6}r^5$$

Starting from the summand corresponding to the number of occurrences of vertices in paths of length 3, the differences in the substractions $X_1 - X_1'$, $X_4 - X_4'$ appears, which violates the condition of Myerson's axiom.

$$X_1 - X_1' = \frac{1}{2}r + \frac{3}{3}r^2 + \frac{8}{4}r^3 + \frac{20}{5}r^4 + \frac{45}{6}r^5$$

$$X_4 - X_4' = \frac{1}{2}r + \frac{3}{3}r^2 + \frac{7}{4}r^3 + \frac{17}{5}r^4 + \frac{36}{6}r^5$$

By choosing a particular value of r, the value of the payoff function $X_i(r)$ for all players can be obtained. These values can then be used to rank the vertices of the graph, which allows us to introduce another approach to calculating centrality.

Fig. 1.7 shows the plots of the payoff functions for the players in the graph G_2. Let us choose the value $r = \frac{1}{2}$, then the payoff:

$$X = (3.44, 3.44, 2.09, 2.79, 2.79, 2.09)\,.$$

It can be seen that with this approach, vertices 1 and 2 still have the highest centrality values, while vertices 3, 6 have the lowest centrality.

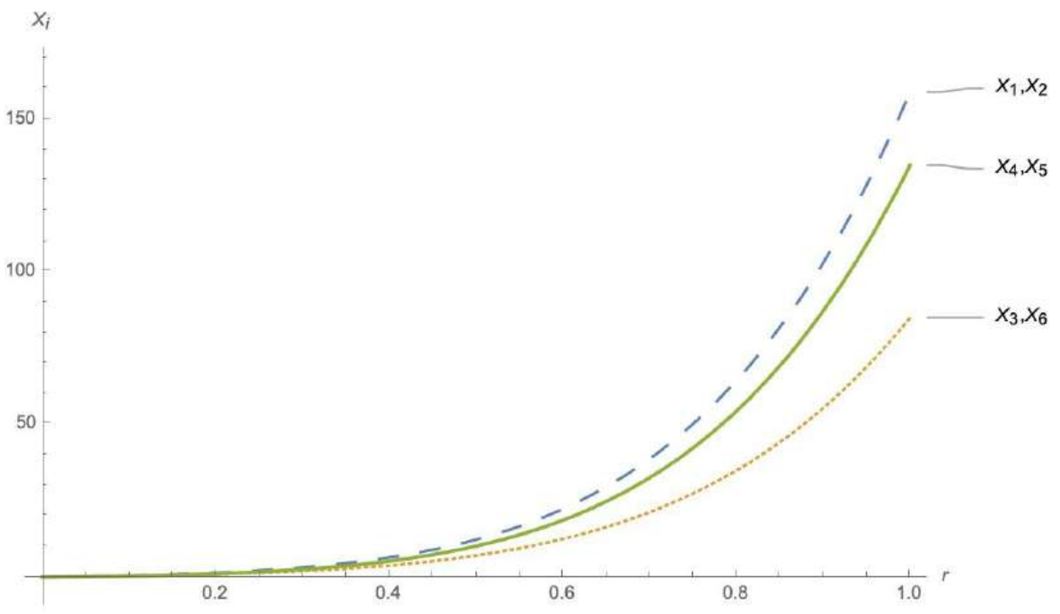


Figure 1.7 — Plots of the payoff functions

We can also write the I vector of integral centrality of the G_2 vertices:

$$I(G_2) = (7.38, 7.38, 4.17, 6.14, 6.14, 4.17).$$

The order of the vertices in the ranking is preserved.

1.5 Centrality of graph vertices based on tournament matrix

In [58], a two-stage procedure for ranking the vertices of a graph was proposed, where, at the first stage, the vertices are ranked based on the absolute potentials of the nodes of an electric circuit when current is supplied to all nodes in sequence. At the second stage, a tournament table is constructed and the final ranking is carried out based on the sum of previously found ranks, by analogy with the Borda rule [59].

In this case, the tournament table can be constructed for values $n_d(i)$ for different d. Let us make a tournament table of vertices of the graph G_2 (Table 1). Centrality estimation based on the total number of appearances of a vertex in paths of different lengths allows to draw a conclusion about the greatest significance of nodes 1 and 2. Vertices 3 and 6 have the lowest centrality.

Table 1 — Tournament table of graph G_2

Vertex	d					$\sum$
	1	2	3	4	5	
1	4	10	21	42	82	159
2	4	10	21	42	82	159
3	3	6	11	22	43	85
4	3	8	18	36	70	135
5	3	8	18	36	70	135
6	3	6	11	22	43	85

Table 2 — Estimation of degree centrality for G_2

Vertex	in-degree	out-degree
1	2	2
2	2	2
3	2	1
4	2	1
5	1	2
6	1	2

Let us compare these results with the values of degree centrality. Table 2 shows the values of the incoming and outgoing links number in the graph. The highest centrality is also possessed by vertices 1 and 2; when analysing the least centrality vertices, different interpretations are possible depending on the applied problem to be solved. If the directionality of edges is not taken into account, the ranks for degree centrality will coincide with the ranks of vertices of order 2. If the direction is taken into account, vertices 5 and 6 get the same ranks in terms of degree centrality, however, in terms of the involvement of vertices in creating paths in the graph, vertex 5 is considered more important.

1.6 Myerson vector as a centrality measure of an undirected graph vertices

We define a cooperative game $\Gamma = \langle N,v\rangle$, $|N| = n$ on the graph $G = (N,E)$, where N is the set of vertices and E is the set of edges. In this game, vertices are players and the characteristic function $v(K), K \subset N$ is defined as the number of simple paths of length m in the subgraph generated by the set of vertices K. The number $m = 1,2...$ is fixed. Obviously, the function $v(K)$ is monotone, i.e., $v(K_1) \leqslant v(K_2)$ if $K_1 \subset K_2$. Then we can use the solution of the cooperative game in the form of a Shapley-Myerson vector to rank the vertices in the graph.

In Section 1.2 (and in [60]) we give a proof of the Myerson value theorem in a cooperative game on a directed graph with a characteristic function defined in terms of the number of simple paths of fixed length in a subgraph generated by a coalition, based on the validity of the Myerson axiom. In [54] we give another proof of a similar statement for undirected graphs.

Theorem 1'. Let $G = (N,E)$ be a graph. We define the characteristic function $v(K), K \subset N$, as the number of simple paths of length m in the subgraph generated by a coalition K. Then the Myerson value for player i is

$$\varphi_i = \frac{a(i)}{(m+1)}, \tag{1.25}$$

where $a(i)$ is the number of simple paths of length m passing through vertex i.

Proof. Consider all permutations σ with non-zero contribution of player i. Then the permutation $P_\sigma(i) \cup \{i\}$ must necessarily contain vertices from a path of length m where vertex i is the last to enter. Then the contribution of player i is $+1$.

For a fixed path with exactly $n!/(m+1)$ permutations, vertex i comes after all vertices of this path. Hence, for vertex i it is true that the sum of contributions of vertex i over all permutations is

$$\frac{a(i)}{n!}\frac{n!}{m+1} = \frac{a(i)}{m+1}.$$

From the additivity of the Myerson vector follows the statement of the theorem. □

Corollary. Let the characteristic function $v(K), K \subset N$ defined as

$$v(K) = \sum a_m r^m \tag{1.26}$$

where a_m is the number of simple paths of length m in the subgraph generated by a coalition K.

Then the Myerson value for player i is

$$\varphi_i = \sum \frac{a_m(i)}{m+1} r^m, \tag{1.27}$$

where $a_m(i)$ is the number of simple paths of length m passing through vertex i.

Thus, as a measure of centrality of vertex i in a graph we can put the Myerson value $Y = (Y_1,...,Y_n)$, where $Y_i = \varphi_i, i = 1,...,n$. In [27; 30], the effectiveness of such an approach for estimating the vertices centrality for different kinds of graphs was shown. The interpretation of this representation of the characteristic function and the corresponding centrality measure is as follows.

In the cooperative game the characteristic function is defined by analogy with Jackson's scheme [45], where each direct link – a path of length 1 – brings players a revenue r, where $0 \leqslant r \leqslant 1$. In addition, the players also benefit from indirect links, but in a smaller way. For each path of length 2, the coalition receives r^2, for each path of length 3 it receives r^3, etc. Since any two vertices can be connected by several paths of different lengths, only the shortest paths are considered when constructing the characteristic function. And if two vertices are connected by several paths of the same length, then all such paths are taken into account, provided that there is no other shorter path between these vertices.

Thus, the distribution of payoffs between vertices characterises how efficiently a vertex is located in the graph in terms of information propagation in this network.

Example 1.4. Consider a graph G_3, consisting of five vertices (Fig. 1.8).

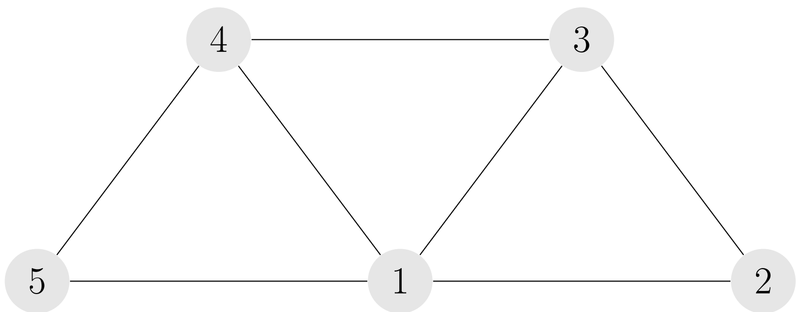


Figure 1.8 — Graph G_3

Consider paths of length 3. There are 17 of them:

1234, 1345, 1432, 1543, 2134, 2143, 2145, 2154, 2345,

2314, 2315, 3145, 3154, 3214, 3215, 3415, 4315.

Of these, 16 of them contain 1. Therefore it follows from Theorem 1'

$$\varphi_1 = 16/4 = 4.$$

If we count by definition through permutations $\sigma = (\sigma(1),\sigma(2),...,\sigma(5))$, then player 1 gets $+1$ if he enters the coalition fourth or fifth.

In the first case, in half of the cases its positive contribution will be in 6 permutations and in the other half in two. In the second case, its contribution will be in 16 permutations.

Thus,

$$Y_1 = \frac{1}{5}\left(\frac{1}{2}6 + \frac{1}{2}2\right) + \frac{1}{5}16 = 4.$$

1.7 Modification of the Myerson vector

Theorem 3 is also valid for undirected graphs [54]. We denote by $s_k(i)$ the number of occurrences of vertex i of an undirected graph in paths of fixed length k, including cycles.

Theorem 3'. Let $A^{(k)}$ be the adjacency matrix of an undirected graph raised to degree k. Then the number of appearances of vertex i in paths of length k starting at vertex l, including cycles, $s_k^{(l)}(i)$

$$s_k^{(l)}(i) = \delta_{\{l=i\}} \sum_{j=1}^{n} a_{lj}^{(k)} + a_{li} \sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)} \sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k)}, \qquad (1.28)$$

where $\delta_{\{l=i\}}$ – is the event $\{l = i\}$ indicator.

Proof. Consider vertex i and all paths of length k in the graph G. If a path starts at vertex i, then vertex i occurs at least once in this path. The number of paths of length k starting at vertex i can be calculated using the matrix A^k, by summing all elements in the row i. This corresponds to the first summand in formula 1.28.

A vertex i can occur several times in a path of length k. So if vertex i occurs in a path of length k at the next step, it will occur at least once in all paths of length $k-1$. This corresponds to the second summand in formula 1.28. Reasoning by induction, we obtain that if vertex i is encountered in a path of length k at step m, $m = 1,\ldots,k$, then the number of its occurrences in all paths further on is equal to the number of paths of length m from the initial vertex to vertex i ($a_{li}^{(m)}$) multiplied by the number of paths of length $k-m$ from vertex i ($\sum_{j=1}^{n} a_{ij}^{(k-m)}$). $\square$

Corollary. The total number of appearances of vertex i in paths of length k can be calculated as

$$s_k(i) = \sum_{j=1}^{n} a_{ij}^{(k)} + \sum_{l=1}^{n} \left[a_{li} \sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)} \sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k)} \right]. \tag{1.29}$$

Example 1.5. The validity of the formula for $s_k^{(l)}(i)$ is illustrated by the following example. Consider a graph G_4 of 8 vertices (Fig. 1.9) with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We will search all the possible paths of length 4 passing through vertex 1. Figure 1.10 shows a tree of paths starting at vertex 1.

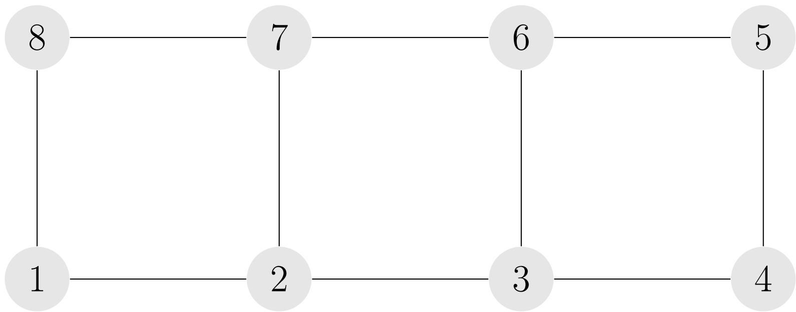


Figure 1.9 — Graph G_4 of 8 vertices

Since all paths starting at vertex 1 contain this vertex at least once, it is necessary to calculate the total number of paths of length 4 starting at vertex 1. This value can be calculated using the sum of the elements of the first row of the matrix

$$A^4 = \begin{pmatrix} 9 & 0 & 9 & 0 & 4 & 0 & 12 & 0 \\ 0 & 18 & 0 & 9 & 0 & 16 & 0 & 12 \\ 9 & 0 & 18 & 0 & 12 & 0 & 16 & 0 \\ 0 & 9 & 0 & 9 & 0 & 12 & 0 & 4 \\ 4 & 0 & 12 & 0 & 9 & 0 & 9 & 0 \\ 0 & 16 & 0 & 12 & 0 & 18 & 0 & 9 \\ 12 & 0 & 16 & 0 & 9 & 0 & 18 & 0 \\ 0 & 12 & 0 & 4 & 0 & 9 & 0 & 9 \end{pmatrix},$$

since it is known that the elements of the matrix $A^k = a_{ij}^{(k)}$ are equal to the number of paths of length k from vertex i to vertex j. Then $\sum\limits_{j=1}^{8} a_{1j}^{(4)} = 34$. At the next step, the vertices 2 and 8 may meet. At the second step, the vertex 1 may again be encountered; there are two such paths.

The number of occurrences of node 1 in the subtree at step 2 corresponds to the number of paths of length 2 starting at node 1. This value can be derived as the sum of the elements of the first row of the matrix A^2:

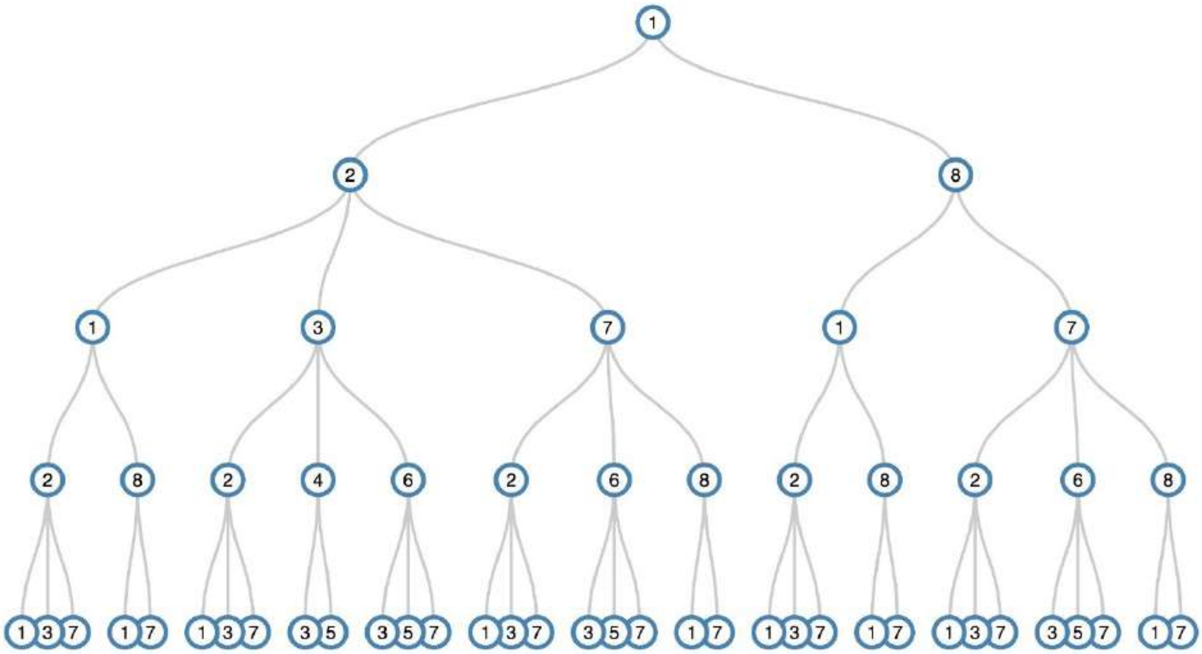


Figure 1.10 — Tree of paths starting at vertex 1

$$A^2 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 & 0 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 3 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

$\sum_{j=1}^{8} a_{1j}^{(4-2)} = 5$. Thus, vertex 1 will occur $2 \cdot 5 = 10$ more times. Finally, vertex 1 can be found at the end of a path of length 4. The number of such paths can be found as $a_{1,1}^{(4)} = 9$.

Adding according to the formula 1.28, we get $34 + 10 + 9 = 53$. This is the number of times vertex 1 will appear in paths of length 4 starting from vertex 1.

Example 1.6. Let the adjacency matrix for a graph G_5 of 4 vertices (Fig. 1.11) have the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

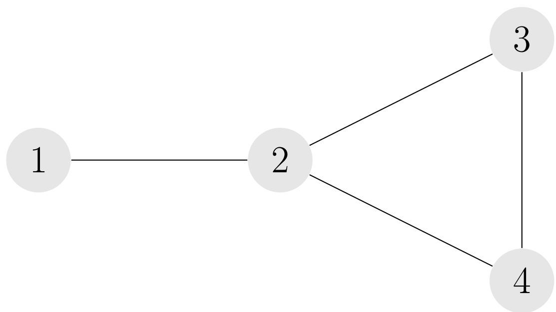


Figure 1.11 — Graph G_5 of 4 vertices

Let's write out all paths of length 2 contained in the given graph:

$$121, 123, 124, 212, 232, 234, 242, 243, 321$$

$$323, 324, 342, 343, 421, 423, 424, 423, 434$$

Let us calculate the number of appearances of vertices in paths of length 2. It is obvious that $s_2(1) = 7, s_2(2) = 19, s_2(3) = 14, s_2(4) = 14.$

Let us use the above formula. This requires the square of the adjacency matrix.

$$A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$s_2(1) = \sum_{j=1}^{4} a_{1j}^{(2)} + \sum_{l=1}^{4} \left[a_{l1} \sum_{j=1}^{4} a_{1j} + a_{l1}^{(2)} \right] = 3 + [1 + 1 \cdot 1 + 1 + 1] = 7$$

$$s_2(2) = \sum_{j=1}^{4} a_{2j}^{(2)} + \sum_{l=1}^{4} \left[a_{l2} \sum_{j=1}^{4} a_{2j} + a_{l2}^{(2)} \right] = 19$$

$$s_2(3) = \sum_{j=1}^{4} a_{3j}^{(2)} + \sum_{l=1}^{4} \left[a_{l3} \sum_{j=1}^{4} a_{3j} + a_{l3}^{(2)} \right] = 14$$

$$s_2(4) = \sum_{j=1}^{4} a_{4j}^{(2)} + \sum_{l=1}^{4} \left[a_{l4} \sum_{j=1}^{4} a_{4j} + a_{l4}^{(2)} \right] = 14$$

Vertex number 1 is a dangling vertex, so it is less likely to appear in paths, which is confirmed by the smallest value of $s_2(1)$. Vertex 2 connects a dangling vertex with a loop, which implies its occurrence in a large number of paths and hence the highest value of $s_2(2)$. Vertices 3 and 4 are symmetrically located and for them the values of $s_2(3)$ and $s_2(4)$ coincide.

Theorem 4. The number of paths of length k (including cycles) R_k in a graph is

$$\frac{\sum\limits_{i=1}^{n} s_k(i)}{k+1}, \tag{1.30}$$

where $s_k(i)$ is the total number of occurrences of vertex i in paths of length k, n is the number of vertices in the graph.

Proof.

$$\sum_{i=1}^{n} s_k(i) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^{(k)} + \sum_{l=1}^{n} \left[a_{li} \sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)} \sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k)} \right] \right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(k)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{li} \sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)} \sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k)} \right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(k)} + \sum_{i=1}^{n} \sum_{l=1}^{n} a_{li}^{(k)} +$$

$$+ \sum_{i=1}^{n} \sum_{l=1}^{n} \left(a_{li} \sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)} \sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k-1)} \sum_{j=1}^{n} a_{ij} \right)$$

$\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}^{(k)}$ — sum of elements of matrix $A^{(k)}$, i.e., the number of paths of length k, including cycles, in the graph G. Then the obtained expression can be rewritten as

$$2R_k + \sum_{i=1}^{n}\sum_{l=1}^{n}\left(a_{li}\sum_{j=1}^{n} a_{ij}^{(k-1)} + a_{li}^{(2)}\sum_{j=1}^{n} a_{ij}^{(k-2)} + \ldots + a_{li}^{(k-1)}\sum_{j=1}^{n} a_{ij}\right).$$

Since the property $A^m \cdot A^n = A^{m+n}$ is satisfied for the degrees of matrices, the expression in brackets is the sum of $(k-1)$ summands equal to R_k, so

$$\sum_{i=1}^{n} s_k(i) = (k+1) \cdot R_k$$

The theorem is proved. $\square$

Thus, if we consider the number of paths of length k in a subgraph, including cycles, as the coalition gain, i.e. the value of

$$\frac{\sum\limits_{i=1}^{n} s_k(i)}{k+1},$$

then it is natural to distribute the payoffs between the players in the form

$$\left(\frac{s_k(1)}{k+1}, \quad \frac{s_k(2)}{k+1}, \ldots, \quad \frac{s_k(n)}{k+1}\right).$$

Definition. The k-th order centrality of vertex i is the number of appearances of vertex i in paths of length k, including cycles.

Let us introduce a vector $\sigma(k)$ — a vector of vertices centrality in the graph G, the i-th component of which is equal to $\sigma_i(k) = \frac{s_k(i)}{k+1}, i = 1,...,n$. For the graph from Example 3 the following vectors were obtained for paths of lengths 2, 3, 4 and 5:

$$\sigma(2) = \{2.3;\ 6.3;\ 4.67;\ 4.67\}$$

$$\sigma(3) = \{4;\ 14;\ 10;\ 10\}$$

$$\sigma(4) = \{9;\ 30.6;\ 22.2;\ 22.2\}$$

$$\sigma(5) = \{17.67;\ 66.3;\ 48;\ 48\}$$

The obtained results can be compared with the Myerson vectors $\varphi(k)$, where the i-th component is the contribution of the player – vertex i – to the grand-coalition payoff, when the characteristic function is the number of paths of length k

in the coalition [30].

$$\varphi(2) = \{2.3;\ 6.3;\ 4.67;\ 4.67\}$$

$$\varphi(3) = \{4.67;\ 12.67;\ 10.3;\ 10.3\}$$

$$\varphi(4) = \{10.8;\ 26.8;\ 23.17;\ 23.17\}$$

$$\varphi(5) = \{23.5;\ 55.5;\ 50.5;\ 50.5\}$$

It can be seen that the obtained values differ insignificantly (and for paths of length 2 they coincide at all), while the order of vertices is preserved, which allows us to use the obtained vector to estimate the centrality of vertices, without calculating the number of paths in the graph, as it is required by the Myerson vector. Moreover, the standard approach to its computation implies a rather labour-intensive process of enumerating all possible coalitions.

The proposed centrality values can be used to rank vertices in graphs when solving applied problems. An example of such tasks is the problem of ranking a corpus of texts [61].

1.8 Special cases

1.8.1 Star

Let S be a star with n vertices, in the centre of which player 1 is located (Fig. 1.12). Then the adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

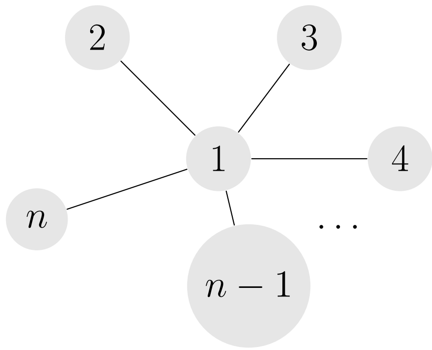


Figure 1.12 — Star S

In order to use formula 1.29, we need to raise the matrix A to power k. Depending on the parity of k, we obtain two kinds of matrices. Raising the matrix to an even power, we have:

$$A^k = \begin{pmatrix} (n-1)^{\frac{k}{2}} & 0 & 0 & \dots & 0 \\ 0 & (n-1)^{\frac{k-2}{2}} & (n-1)^{\frac{k-2}{2}} & \dots & (n-1)^{\frac{k-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (n-1)^{\frac{k-2}{2}} & (n-1)^{\frac{k-2}{2}} & \dots & (n-1)^{\frac{k-2}{2}} \end{pmatrix}.$$

In the case of odd power the matrix is:

$$A^k = \begin{pmatrix} 0 & (n-1)^{\frac{k-1}{2}} & (n-1)^{\frac{k-1}{2}} & \dots & (n-1)^{\frac{k-1}{2}} \\ (n-1)^{\frac{k-1}{2}} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-1)^{\frac{k-1}{2}} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Applying formula 1.29, we obtain expressions for centrality of vertices of graph G. For even values of k:

$$s_k(1) = (n-1)^{\frac{k}{2}} \left(\frac{k}{2}n + 1\right) \tag{1.31}$$

$$s_k(i) = (n-1)^{\frac{k-2}{2}} \left(\frac{k+2}{2}n - 1\right) \tag{1.32}$$

for odd values:

$$s_k(1) = (k+1)\,(n-1)^{\frac{k+1}{2}} \tag{1.33}$$

$$s_k(i) = (k+1)\,(n-1)^{\frac{k-1}{2}} \tag{1.34}$$

It should be noted that for odd path lengths the following relation holds

$$\frac{s_k(1)}{s_k(i)} = n - 1,$$

the same as for the Myerson vector.

1.8.2 Chain

Let P be a chain (path) of n vertices (Fig. 1.13).

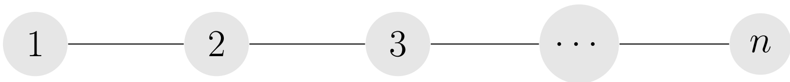


Figure 1.13 — Chain P

The corresponding adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

If the path length k is less than $\frac{n}{2}$, starting from the $(k+1)-$th vertex in the chain and up to the $(n-k)-$th vertex, a stable centrality value of order k is established. To calculate this value we can use the formula

$$s_k(i) = 2^k(k+1), \quad i = k+1..n-k. \tag{1.35}$$

Chapter 2. Graph vertex ranking using circuit node absolute potentials

Various metrics can be used to estimate the centrality of graph vertices. A number of works use methods based on the electric circuit model and Kirchhoff's laws [18; 22]. The graph is considered as an electric circuit with ideal elements, the vertices of the graph are nodes of the electric circuit, the edges are conductors of electric current with known electrical conductivity. An electric current is passed through the circuit grounded at some vertex. In [20—22], the calculations are based on the currents flowing through the vertex under consideration. Here we propose to use the ranks of the graph vertices based on the values of the absolute potentials of the nodes of the electric circuit calculated with the Kirchhoff's rules to estimate centrality. To rank the vertices, one can use the methods of voting theory, when the ranking is based on the tournament matrix [62], and in the simplest case, after the calculation of absolute potentials, the Borda rule [59] can be used.

2.1 Graph vertex ranking based on Laplace matrix

Let $G = (V,E,W)$, where V is the set of n vertices of the graph, E is the set of edges of the graph, W is the edge weight matrix:

$$W = \begin{pmatrix} 0 & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & 0 & w_{23} & \dots & w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \dots & 0 \end{pmatrix}.$$

The graph G is not directed, i.e. $w_{ij} = w_{ji}$. The weights of the edges are interpreted as the edge conductivity.

Based on the matrix of weights, we can construct a diagonal matrix D of degrees of a vertex, where $d_i = \sum\limits_{j=1}^{n} w_{ij}$. The matrix $L = D - W$ is called the

Laplace matrix of the graph G and has the form

$$L(G) = \begin{pmatrix} d_1 & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & d_2 & -w_{23} & \dots & -w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & d_n \end{pmatrix}.$$

Next, consider the graph G' obtained by adding an artificial vertex with number $n+1$ connected to all vertices of the original graph by edges with the same conductivity δ (Fig. 2.1).

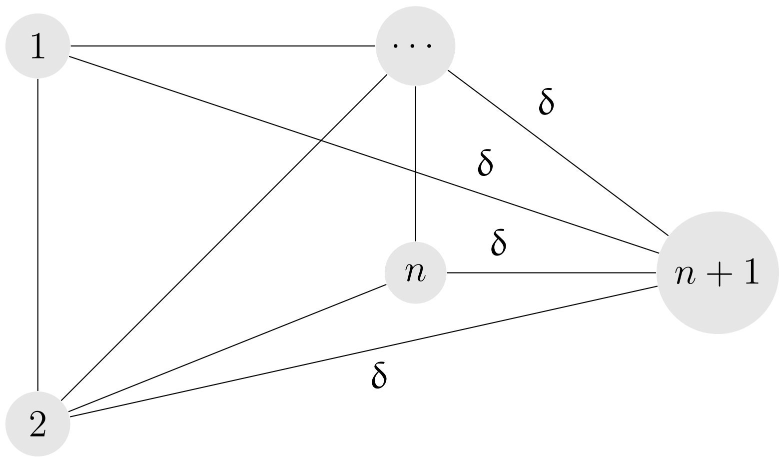


Figure 2.1 — Graph G'

Then the Laplace matrix for the graph G' takes the following form:

$$L(G') = \begin{pmatrix} d_1+\delta & -w_{12} & -w_{13} & \dots & -w_{1n} & -\delta \\ -w_{21} & d_2+\delta & -w_{23} & \dots & -w_{2n} & -\delta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & d_n+\delta & -\delta \\ -\delta & -\delta & -\delta & \dots & -\delta & \delta n \end{pmatrix}.$$

Since this matrix is degenerate, it is necessary to remove the row and column corresponding to the artificially added vertex in order to calculate the potentials:

$$\widetilde{L}(G') = \begin{pmatrix} d_1+\delta & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & d_2+\delta & -w_{23} & \dots & -w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & d_n+\delta \end{pmatrix}.$$

Let a unit of electric current be supplied to some node v_k $(k = 1,\dots,n)$ of an electric circuit, which is grounded at the vertex v_{n+1}. Then, applying Kirchhoff's rules enables one to calculate the absolute potentials of the nodes in the circuit by solving the system of equations.

$$\varphi^k = \widetilde{L}^{-1}(G')b_k, \tag{2.1}$$

where $\varphi^k = (\varphi_1^k,\varphi_2^k,\dots,\varphi_n^k)^T$ — absolute potential vector, b_k — vector-column, whose elements are defined by the rule $b_k(k) = 1$, $b_k(i \neq k) = 0$. The absolute potential at the vertex v_{n+1} is assumed to be zero.

In the paper [21], the graph nodes were ranked based on the currents flowing through them when the current was randomly supplied to the graph nodes. In our current research, we suggest ranking the graph vertices, which correspond to the nodes of an electric circuit, by calculating the potentials in all nodes of the circuit when current is supplied to node v_k. Thus, applying current sequentially to all vertices of the graph G, it is possible to obtain a tournament table of vertices based on absolute potentials. The higher the potential value obtained by a vertex, the higher its position in the tournament table.

Example 2.1. As an example, let us consider the graph shown in Fig. 2.2, with the weight matrix W:

$$W = \begin{pmatrix} 0 & 300 & 100 & 0 & 0 & 0 \\ 300 & 0 & 100 & 0 & 0 & 0 \\ 100 & 100 & 0 & 500 & 0 & 0 \\ 0 & 0 & 500 & 0 & 100 & 100 \\ 0 & 0 & 0 & 100 & 0 & 300 \\ 0 & 0 & 0 & 100 & 300 & 0 \end{pmatrix}$$

Assuming that a vertex is added to the graph, connected with other vertices of the graph by edges with carrying conductivity $\delta = 0.1$, in which the electric circuit is grounded, let's supply current to vertex 1. The following values of absolute potentials will be obtained:

$$\varphi^1 = (1.670,\ 1.669,\ 1.666,\ 1.665,\ 1.663,\ 1.663)^T.$$

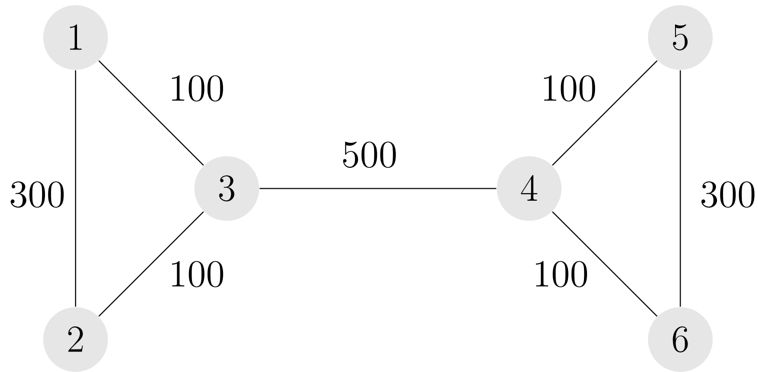


Figure 2.2 — Graph with vertices 1 – 6; 100, 300, 500 — edge weights

The vertex with the highest potential is always the vertex to which the electric current is applied, it will receive rank 1. The other vertices will have ranks 2, 3, 4, 5, 5 respectively. Due to the symmetry of the considered graph, when current is supplied to vertex 2, the vector of potentials will be different only for vertices 1 and 2:

$$\varphi^2 = (1.669, 1.670, 1.666, 1.665, 1.663, 1.663)^T.$$

If the current is supplied to vertex 3, we obtain equal values for equidistant vertices 1 and 2 and for symmetric vertices 5 and 6.

$$\varphi^3 = (1.666, 1.666, 1.668, 1.667, 1.665, 1.665)^T.$$

The values of the absolute potentials, can be written as a matrix Φ, where the column k contains the values corresponding to the vector φ^k.

$$\Phi = \begin{pmatrix} 1.670 & 1.669 & 1.666 & 1.665 & 1.663 & 1.663 \\ 1.669 & 1.670 & 1.666 & 1.665 & 1.663 & 1.663 \\ 1.666 & 1.666 & 1.668 & 1.667 & 1.665 & 1.665 \\ 1.665 & 1.665 & 1.667 & 1.668 & 1.666 & 1.666 \\ 1.663 & 1.663 & 1.665 & 1.666 & 1.670 & 1.669 \\ 1.663 & 1.663 & 1.665 & 1.666 & 1.669 & 1.670 \end{pmatrix}.$$

Table 3 shows the ranks of the vertices for each k value. It is worth noting that the introduction of ranks is necessary because the sum of matrix elements in each row and each column is the same and depends only on the choice of δ.

Table 3 — Tournament table

Vertex	k						$\sum$
	1	2	3	4	5	6	
1	1	2	3	4	5	5	20
2	2	1	3	4	5	5	20
3	3	3	1	2	4	4	17
4	4	4	2	1	3	3	17
5	5	5	4	3	1	2	20
6	5	5	4	3	2	1	20

Summing up all places for different k for each vertex, we find the values on the basis of which we can perform ranking again. The smaller the value of the sum of ranks a vertex has, the more favourable position it occupies in the graph. Thus, vertices numbered 3 and 4 are the most important for this graph, which is logical, since the edge (3,4) is a link for two groups of vertices. This approach in voting theory is known as the Borda rule [59].

We can compare our results with PageRank values [19]. The vector of PageRank values for this graph is as follows: (0.14, 0.14, 0.21, 0.21, 0.14, 0.14), with the higher the value obtained, the more important the vertex in the graph, i.e. vertices 3 and 4 also get the highest rank.

A tournament matrix can also be constructed from the tournament table, which can be used to construct a cooperative game [63].

2.2 Special cases

2.2.1 Clique

Statement 2.1. For n – clique C_n with unit weights of edges, all vertices are equal.

Laplace matrix for a clique consisting of n vertices:

$$\widetilde{L}(C'_n) = \begin{pmatrix} n-1+\delta & -1 & -1 & \dots & -1 \\ -1 & n-1+\delta & -1 & \dots & -1 \\ -1 & -1 & n-1+\delta & \dots & -1 \\ \vdots & -1 & -1 & \ddots & -1 \\ -1 & -1 & -1 & \dots & n-1+\delta \end{pmatrix}.$$

Then the inverse matrix $\widetilde{L}^{-1}(C'_n)$:

$$\widetilde{L}^{-1}(C'_n) = \frac{1}{\delta(n+\delta)} \begin{pmatrix} 1+\delta & 1 & 1 & \dots & 1 \\ 1 & 1+\delta & 1 & \dots & 1 \\ 1 & 1 & 1+\delta & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1+\delta \end{pmatrix}.$$

Obviously, the sums of ranks for all vertices will be the same.

2.2.2 Star

Statement 2.2. For a star graph S_1 (Fig. 2.3) with n vertices ($n > 3$) with unit edge weights, the rank distribution does not depend on the value of δ. The rank of the centre vertex is 1, the ranks of the other vertices are 2.

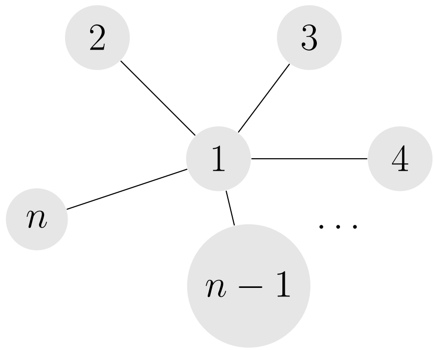


Figure 2.3 — Star graph S_1

For such a graph the Laplace matrix will take the following form:

$$\widetilde{L}(S_1') = \begin{pmatrix} n-1+\delta & -1 & -1 & \dots & -1 \\ -1 & 1+\delta & 0 & \dots & 0 \\ -1 & 0 & 1+\delta & \dots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ -1 & 0 & 0 & \dots & 1+\delta \end{pmatrix}.$$

Let's calculate the inverse matrix.

$$\widetilde{L}^{-1}(S_1') = D \begin{pmatrix} (1+\delta)^2 & (1+\delta) & (1+\delta) & \dots & (1+\delta) \\ (1+\delta) & (1+n\delta+\delta^2) & 1 & \dots & 1 \\ (1+\delta) & 1 & (1+n\delta+\delta^2) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1+\delta) & 1 & 1 & \dots & (1+n\delta+\delta^2) \end{pmatrix},$$

where

$$D = \frac{1}{\delta(1+\delta)(n+\delta)}.$$

The view of the tournament table for any value of δ is given in Table 4. For the central vertex of the graph the sum of ranks is $2n-1$, for the other vertices — $3(n-3)$. If $n > 3$ $2n-1 < 3(n-1)$, then the central vertex will always have the best rank.

Table 4 — Tournament table for graph S_1

Vertex	k					$\sum$
	1	2	3	…	n	
1	1	2	2	…	2	$2n-1$
2	2	1	3	…	3	$3(n-1)$
3	2	3	1	…	3	$3(n-1)$
…	…	…	…	…	…	…
$n-1$	2	3	3	…	3	$3(n-1)$
n	2	3	3	…	1	$3(n-1)$

Statement 2.3. For a star graph S_m (Fig. 2.4) of n vertices ($n > 3$) with edge weights $w_{1j} = w_{j1} = 1$ ($j = 2,\dots,n-1$), $w_{1n} = w_{n1} = m$ and Laplace matrix

$$\widetilde{L}(S'_m) = \begin{pmatrix} n-2+m+\delta & -1 & -1 & \dots & -1 & -m \\ -1 & 1+\delta & 0 & \dots & 0 & 0 \\ -1 & 0 & 1+\delta & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & 1+\delta & 0 \\ -m & 0 & 0 & \dots & 0 & m+\delta \end{pmatrix}$$

the ranks of vertices will be distributed depending on the weight m. For $m \in (0,1)$ we obtain the rank distribution $(1,2,2,\dots,2,3)$, for $m > 1$ — the rank distribution $(1,3,3,\dots,3,2)$, the central vertex always has rank 1.

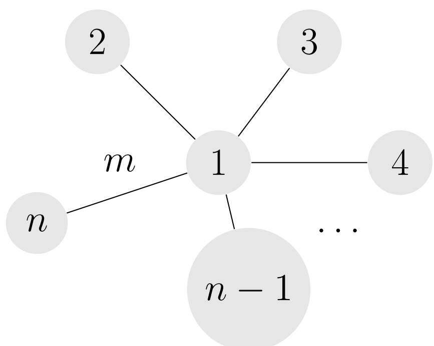


Figure 2.4 — Star graph S_m

The inverse matrix containing the values of absolute potentials of the graph vertices will be constructed according to the following scheme:

$$\widetilde{L}^{-1}(S'_m) = D_m \begin{pmatrix} l_1 & l_2 & l_2 & \dots & l_2 & l_3 \\ l_2 & l_d & l_5 & \dots & l_5 & l_4 \\ l_2 & l_5 & l_d & \dots & l_5 & l_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_2 & l_5 & l_5 & \dots & l_d & l_4 \\ l_3 & l_4 & l_4 & \dots & l_4 & l_6 \end{pmatrix},$$

where

$$D_m = \frac{1}{\delta(1+\delta)^{n-3}(nm+(n-1)\delta+2m\delta+\delta^2)},$$

$$l_1 = (m+\delta)(1+\delta)^{n-2},$$

$$l_2 = (m+\delta)(1+\delta)^{n-3},$$

$$l_3 = m(1+\delta)^{n-2},$$

$$l_4 = m(1+\delta)^{n-3},$$

$$l_5 = (m+\delta)(1+\delta)^{n-4},$$

$$l_6 = (1+\delta)^{n-3}(m+(n-1)\delta+m\delta+\delta^2),$$

$$l_d = (1+\delta)^{n-4}(m+\delta+nm\delta+(n-1)\delta^2+2m\delta^2+\delta^3).$$

Table 5 shows the rank distribution for the case when $m \in (0,1)$. For $n > 3$ the sums of ranks satisfy the relation $2n-1 < 3n-3 < 4n-4$, respectively, the central vertex of the star has the best rank; the vertex connected to the central edge with weight m gets the worst rank.

For $m > 1$ and $n > 3$, the sums of ranks follow the relation $2n-1 < 3n-3 < 4n-5$. The central vertex has the highest rank while the vertices connected to it by edges of unit weight have the lowest ranks (Table 6).

Table 5 — Tournament table for the graph S_m, $m \in (0,1)$

Vertex	k						$\sum$
	1	2	3	...	$n-1$	n	
1	1	2	2	...	2	2	$2n-1$
2	2	1	3	...	3	3	$3n-3$
3	2	3	1	...	3	3	$3n-3$
...	...	...	...	...	...	...	...
$n-1$	2	3	3	...	1	3	$3n-3$
n	3	4	4	...	4	1	$4n-4$

Table 6 — Tournament table for the graph S_m, $m > 1$

Vertex	k						$\sum$
	1	2	3	...	$n-1$	n	
1	1	2	2	...	2	2	$2n-1$
2	3	1	4	...	4	3	$4n-5$
3	3	4	1	...	4	3	$4n-5$
...	...	...	...	...	...	...	...
$n-1$	3	4	4	...	1	3	$4n-5$
n	2	3	3	...	3	1	$3n-3$

2.2.3 Double star

By a double star we will understand a graph $S_{p,q}$ obtained by joining two star graphs S_p and S_q ($p,q > 2$) by a common edge (Fig. 2.5). The Laplace matrix of such a graph has the form

$$\widetilde{L}(S'_{p,q}) = \left(\begin{array}{c|c} A_{p\times p} & B_{p\times q} \\ \hline C_{q\times p} & D_{q\times q} \end{array}\right),$$

$$A_{p\times p} = \begin{pmatrix} p+\delta & -1 & -1 & \dots & -1 \\ -1 & 1+\delta & 0 & \dots & 0 \\ -1 & 0 & 1+\delta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1+\delta \end{pmatrix},$$

$$D_{q\times q} = \begin{pmatrix} q+\delta & -1 & -1 & \dots & -1 \\ -1 & 1+\delta & 0 & \dots & 0 \\ -1 & 0 & 1+\delta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1+\delta \end{pmatrix},$$

elements of matrices B,C b_{11}, $c_{11} = -1$, the other elements are zero.

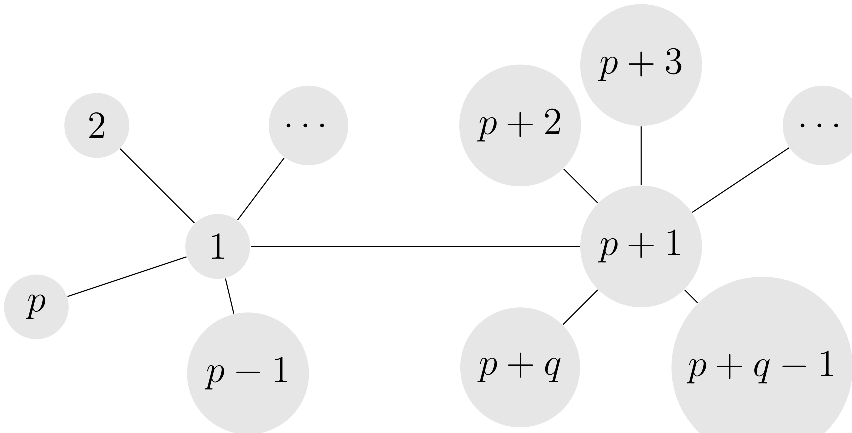


Figure 2.5 — Double star $S_{p,q}$

Statement 2.4. For a double star $S_{p,q}$ in the case $p < q$, the rank of the vertex v_1 having p neighbours is always lower than the rank of the vertex v_{p+1} with q neighbours.

The matrix of absolute potentials is also represented as a block matrix

$$\widetilde{L}^{-1}(S'_{p,q}) = D_{p,q} \left(\begin{array}{c|c} A'_{p\times p} & B'_{p\times q} \\ \hline C'_{q\times p} & D'_{q\times q} \end{array} \right),$$

$$D_{p,q} = \frac{1}{\delta(1+\delta)^{p+q-4}\big(\big(1+(p+1)(q+1)\big)\delta + (p+q+2)\delta^2 + \delta^3\big)},$$

$$A'_{p\times p} = a \begin{pmatrix} (1+\delta)^2 & 1+\delta & 1+\delta & \dots & 1+\delta \\ 1+\delta & a_d & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+\delta & 1 & 1 & \dots & a_d \end{pmatrix},$$

$$a = (1+\delta)(1+(q+1)\delta+\delta^2),$$

$$a_d = \delta^2 + (p+1)\delta + 1 + \frac{(q-1)\delta}{1+((1+q)\delta+\delta^2)},$$

Table 7 — Tournament table for the graph $S_{p,q}$

Vertex	k									$\sum$
	1	2	...	$p-1$	p	$p+1$	...	$p+q-1$	$p+q$	
1	1	2	...	2	3	4	...	4	4	$2p+4q-2$
2	2	1	...	3	4	5	...	5	5	$3p+5q-4$
...	...	...	...	...	...	...	...	...	...	...
$p-1$	2	3	...	1	4	5	...	5	5	$3p+5q-4$
p	3	4	...	4	1	2	...	2	2	$4p+2q-2$
$p+1$	4	5	...	5	2	1	...	3	3	$5p+3q-4$
...	...	...	...	...	...	...	...	...	...	...
$p+q-1$	4	5	...	5	2	3	...	1	3	$5p+3q-4$
$p+q$	4	5	...	5	2	3	...	3	1	$5p+3q-4$

$$B'_{p\times q} = (1+\delta)^{p+q-4}\begin{pmatrix} (1+\delta)^2 & 1+\delta & 1+\delta & \dots & 1+\delta \\ 1+\delta & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+\delta & 1 & 1 & \dots & 1 \end{pmatrix}.$$

The matrix $C'_{q\times p}$ can be constructed in the similar way.

$$D'_{q\times q} = d\begin{pmatrix} (1+\delta)^2 & 1+\delta & 1+\delta & \dots & 1+\delta \\ 1+\delta & d_d & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+\delta & 1 & 1 & \dots & d_d \end{pmatrix},$$

$$d = (1+\delta)^{p+q-5}(1+(p+1)\delta+\delta^2),$$

$$d_d = \delta^2+(q+1)\delta+1+\frac{(p-1)\delta}{1+((1+p)\delta+\delta^2)}.$$

The table 7 presents the distribution of vertex ranks. The sums of ranks of vertices located in the centres of stars satisfy the relation $4p+2q-2 < 2p+4q-2$, which is always true for $p < q$, i.e. a vertex — the centre of a star with q neighbours will always have rank 1. The relation $2p+4q-2 < 5p+3q-4$ holds for $q+2 < 3p$, so rank 2 can be assigned both to vertex v_1 with p neighbours and to the hanging vertices of star S_q. The rank of hanging vertices of the graph S_p is always the worst, since $3p + 5q - 4$ takes the largest values for $p < q, p,q > 2$.

2.2.4 Bipartite graph

Statement 2.5. For a complete bipartite graph $B_{2,n-2}$ (Fig. 2.6) with n vertices ($n > 4$), where the vertices are divided into two non-overlapping sets V_1 and V_2 such that $|V_1| = 2$, $|V_2| = n - 2$, and edge weights equal to one, the rank distribution does not depend on the value of δ. Vertices from set V_1 have ranks 1, vertices from set V_2 have ranks 2.

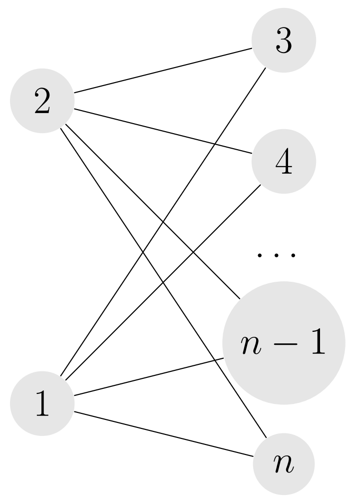


Figure 2.6 — Complete bipartite graph $B_{2,n-2}$

The Laplace matrix has the form

$$\widetilde{L}(B'_{2,n-2}) = \begin{pmatrix} n-2+\delta & 0 & -1 & -1 & \dots & -1 \\ 0 & n-2+\delta & -1 & -1 & \dots & -1 \\ -1 & -1 & 2+\delta & 0 & \dots & 0 \\ -1 & -1 & 0 & 2+\delta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & 0 & \dots & 2+\delta \end{pmatrix}$$

Table 8 — Tournament table for the graph $B_{2,n-2}$

Vertex	k							$\sum$
	1	2	3	4	$\dots$	$n-1$	n	
1	1	3	2	2	$\dots$	2	2	$2n$
2	3	1	2	2	$\dots$	2	2	$2n$
3	2	2	1	3	$\dots$	3	3	$3n-4$
4	2	2	3	1	$\dots$	3	3	$3n-4$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$n-1$	2	2	3	3	$\dots$	1	3	$3n-4$
n	2	2	3	3	$\dots$	3	1	$3n-4$

The inverse Laplace matrix is formed as follows:

$$\widetilde{L}^{-1}(B'_{2,n-2}) = D_{2,n-2}\begin{pmatrix} l_1 & l_2 & l_3 & l_3 & \dots & l_3 \\ l_2 & l_1 & l_3 & l_3 & \dots & l_3 \\ l_3 & l_3 & l_5 & l_4 & \dots & l_4 \\ l_3 & l_3 & l_4 & l_5 & \dots & l_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_3 & l_3 & l_4 & l_4 & \dots & l_5 \end{pmatrix},$$

where

$$D_{2,n-2} = \frac{1}{\delta(2+\delta)^{n-3}(n-2+\delta)(n+\delta)}$$

$$l_1 = (2+\delta)^{n-3}(n-2+n\delta+\delta^2)$$

$$l_2 = (n-2)(2+\delta)^{n-3}$$

$$l_3 = (2+\delta)^{n-3}(n-2+\delta)$$

$$l_4 = 2(2+\delta)^{n-4}(n-2+\delta)$$

$$l_5 = 2(2+\delta)^{n-4}(n-2+\delta)(2+n\delta+\delta^2)$$

Table 8 shows the ranks of the vertices of the graph independent of δ. When $n > 4$, the values $2n < 3n-4$, so the vertices of set V_1 will always have rank 1, and the vertices from V_2 – 2.

Statement 2.6. For a complete bipartite graph $B_{3,n-3}$ (Fig. 2.7) with n vertices ($n > 6$), where the vertices are divided into two non-overlapping sets V_1 and V_2 such that $|V_1| = 3$, $|V_2| = n-3$, and unit edge weights, the rank distribution does not depend on the value of δ. Vertices from set V_1 have ranks 1, vertices from set V_2 have ranks equal to 2.

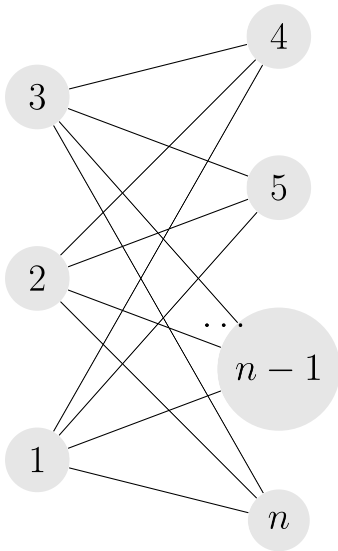


Figure 2.7 — Complete bipartite graph $B_{3,n-3}$

The Laplace matrix has the form

$$\widetilde{L}(B'_{3,n-3}) = \begin{pmatrix} n-3+\delta & 0 & 0 & -1 & -1 & \dots & -1 \\ 0 & n-3+\delta & 0 & -1 & -1 & \dots & -1 \\ 0 & 0 & n-3+\delta & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & 3+\delta & 0 & \dots & 0 \\ -1 & -1 & -1 & 0 & 3+\delta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & 0 & 0 & \dots & 3+\delta \end{pmatrix}$$

Table 9 — Tournament table for the graph $B_{3,n-3}$

Vertex	k							$\sum$
	1	2	3	4	…	$n-1$	n	
1	1	3	3	2	…	2	2	$2n+1$
2	3	1	3	2	…	2	2	$2n+1$
3	3	3	1	2	…	2	2	$2n+1$
4	2	2	2	1	…	3	3	$3n-5$
…	…	…	…	…	…	…	…	…
$n-1$	2	2	2	3	…	1	3	$3n-5$
n	2	2	2	3	…	3	1	$3n-5$

The inverse matrix

$$\widetilde{L}^{-1}(B'_{3,n-3}) = D_{3,n-3}\begin{pmatrix} l_1 & l_2 & l_2 & l_3 & l_3 & \dots & l_3 \\ l_2 & l_1 & l_2 & l_3 & l_3 & \dots & l_3 \\ l_2 & l_2 & l_1 & l_3 & l_3 & \dots & l_3 \\ l_3 & l_3 & l_3 & l_4 & l_5 & \dots & l_5 \\ l_3 & l_3 & l_3 & l_5 & l_4 & \dots & l_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_3 & l_3 & l_3 & l_5 & l_5 & \dots & l_4 \end{pmatrix}$$

$$D_{3,n-3} = \frac{1}{\delta(3+\delta)^{n-4}(n-3+\delta)^2(n+\delta)}$$

$$l_1 = (3+\delta)^{n-4}(n-3+\delta)(n-3+n\delta+\delta^2)$$

$$l_2 = (n-3)(3+\delta)^{n-4}(n-3+\delta)$$

$$l_3 = (3+\delta)^{n-4}(n-3+\delta)^2$$

$$l_4 = (3+\delta)^{n-5}(n-3+\delta)^2(3+n\delta+\delta^2)$$

$$l_5 = 3(3+\delta)^{n-5}(n-3+\delta)^2$$

By analogy, a statement can be formulated for the graph $B_{m,n-m}$.

Statement 2.7. For a complete bipartite graph $B_{m,n-m}$ (Fig. 2.8) with n vertices ($n > 2m$), which are divided into two non-overlapping sets V_1 and V_2 such that $|V_1| = m$, $|V_2| = n - m$, and with unit edge weights, the rank distribution does not depend on the value of δ. Vertices from set V_1 have ranks 1, vertices from set V_2 — ranks 2.

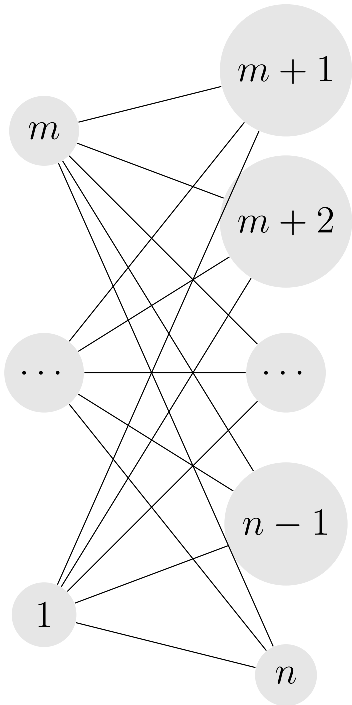


Figure 2.8 — Complete bipartite graph $B_{m,n-m}$

The Laplace matrix is represented in block form

$$\widetilde{L}(B'_{m,n-m}) = \left(\begin{array}{c|c} (n-m+\delta)\mathbb{E}_m & -\mathbb{1}_{m\times n-m} \\ \hline -\mathbb{1}_{n-m\times m} & (m+\delta)\mathbb{E}_{n-m} \end{array}\right),$$

where $\mathbb{E}$ — unit matrix; $\mathbb{1}$ — matrix of ones. In this case the inverse matrix can be found using the Frobenius formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BH^{-1}CA^{-1} & -A^{-1}BH^{-1} \\ -H^{-1}CA^{-1} & H^{-1} \end{pmatrix},$$

where $H = D - CA^{-1}B$. Thus

$$H = (m+\delta)\mathbb{E}_{n-m} - \frac{m}{n-m+\delta}\mathbb{1}_{n-m\times n-m},$$

Table 10 — Tournament table for the graph $B_{m,n-m}$

Vertex	k								$\sum$
	1	2	...	m	$m+1$	...	$n-1$	n	
1	1	3	...	3	2	...	2	2	$2n-2+m$
2	3	1	...	3	2	...	2	2	$2n-2+m$
...	...	...	...	...	...	...	...	...	...
$m-1$	3	3	...	3	2	...	2	2	$2n-2+m$
m	2	2	...	1	3	...	3	3	$2n-2+m$
$m+1$	2	2	...	2	1	...	3	3	$3n-2-m$
...	...	...	...	...	...	...	...	...	...
$n-1$	2	2	...	2	3	...	1	3	$3n-2-m$
n	2	2	...	2	3	...	3	1	$3n-2-m$

$$\widetilde{L}^{-1}(B'_{m,n-m}) = D_{m,n-m}\begin{pmatrix} (l_1-l_2)\mathbb{E}_m + l_2\mathbb{1}_{m\times m} & l_3\mathbb{1}_{m\times n-m} \\ l_3\mathbb{1}_{m\times n-m} & (l_4-l_5)\mathbb{E}_{n-m} + l_2\mathbb{1}_{n-m\times n-m} \end{pmatrix},$$

$$D_{m,n-m} = \frac{1}{\delta(m+\delta)(n-m+\delta)(n+\delta)},$$

$$l_1 = (m+\delta)(n-m+n\delta+\delta^2),$$

$$l_2 = (n-m)(m+\delta),$$

$$l_3 = (m+\delta)(n-m+\delta),$$

$$l_4 = (n-m+\delta)(m+n\delta+\delta^2),$$

$$l_5 = m(n-m+\delta).$$

Consider the tournament matrix for the graph $B_{m,n-m}$ (Table 10). The value $2n-2+m < 3n-2-m$ at $n > 2m$, respectively, vertices from set V_1 have ranks 1, vertices from set V_2 — ranks 2.

The above results were published in the paper [58].

2.3 Ranking taking into account vertex weights

When examining graph models of actual systems, it is not unusual for the ranking of graph vertices, which only considers the graph topology, to yield incorrect outcomes. For instance, when examining the graph of a transportation system, if there are numerous closely located vertices connected by edges of tiny length, these vertices obtain higher ranks. However, such vertices may correspond to minimally inhabited areas of the so-called "private sector". In this regard, it is useful to consider additional characteristics of graph vertices, such as the weight of a vertex, which can be interpreted as the number of inhabitants living in the neighbourhood of a graph vertex.

Let us consider the graph $G_T = (V,E,W,P)$, where V is the set of vertices, E is the set of edges, W is the matrix of edge weights, P is the diagonal matrix of vertex weights. According to Ohm's law, as the magnitude of the current applied to the system increases, the values of the potentials will increase proportionally. Thus, if an electric current is supplied to some node of the electric circuit v_k, the value of which depends on the weight of a vertex, the following expression can be used to calculate the absolute potentials of the graph vertices:

$$\Phi = \widetilde{L}^{-1}(G'_T)P, \tag{2.2}$$

where Φ represents the matrix of absolute potentials of the circuit nodes (graph vertices), in the k-column of which are the potentials of the nodes obtained by applying an electric current of magnitude p_k to node v_k, $\widetilde{L}^{-1}(G'_T)$ is the inverse Laplace matrix for the graph G_T to which node v_{n+1} was artificially added.

The potential difference of two points of the electric field multiplied by the magnitude of the charge is equal to the work required to move the charge between these points. Since the artificially added vertex v_{n+1} has zero potential, the values obtained in the matrix Φ can be interpreted as work to move the charge to vertex v_{n+1}.

By multiplying the matrix of absolute potentials with the unit vector-column, we can obtain the vector of sums of potentials of the graph vertices, whose i-th component is equal to $\sum_{k=1}^{n} \varphi_i^k$. Denote it by a.

The value a_i represents the total work of moving electric charges from vertex v_i to vertex v_{n+1} at sequential supply of current to the vertices of the graph v_k, $k = 1,\ldots,n$, where the value of the supplied current is equal to the weight of vertex p_k.

Calculating the potentials in this way, we can avoid the mandatory use of ranks in the previously proposed approach, since in this case the sums of the elements of the matrix Φ on the rows will be different.

Example 2.2. Consider the star graph S_p, shown in Fig. 2.9, with an edge weight matrix W whose elements are the inverse of the lengths of the edges between the corresponding vertices.

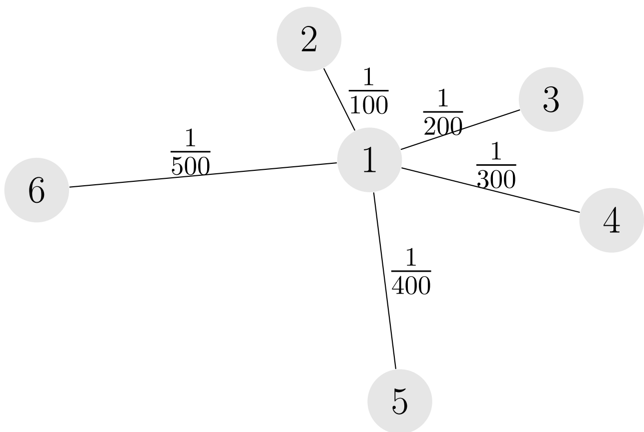


Figure 2.9 — Weighted star graph S_p

$$W = \begin{pmatrix} 0 & \frac{1}{100} & \frac{1}{200} & \frac{1}{300} & \frac{1}{400} & \frac{1}{500} \\ \frac{1}{100} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{200} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{300} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{400} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{500} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Calculating the values of potentials without taking into account the weights of vertices, based only on the graph topology, we obtain the tournament table (table 11) at $\delta = 0{,}0002$.

Table 11 — Tournament table of graph S_p

№	$k=1$	$k=2$	$k=3$	$k=4$	$k=6$	$k=7$	$\sum$
1	1	2	2	2	2	2	11
2	2	1	3	3	3	3	15
3	3	3	1	4	4	4	19
4	4	4	4	1	5	5	23
5	5	5	5	5	1	6	27
6	6	6	6	6	6	1	31

The vertex v_1, located in the centre of the star, expectedly received the smallest sum of ranks, respectively, it is the most central. The worst ranks were obtained by the vertex v_6, the most distant from the centre of the star.

If we additionally introduce the matrix of vertex weights P:

$$P = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 800 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{pmatrix}$$

so that the central vertex has the smallest weight and the vertex v_6 furthest from the centre has the largest weight, we obtain the matrix Φ.

$$\Phi = \begin{pmatrix} 87407.34 & 171386.93 & 336182.06 & 494758.5 & 647461.75 & 794612.14 \\ 85693.47 & 187634.25 & 329590.26 & 485057.36 & 634766.42 & 779031.51 \\ 84045.52 & 164795.13 & 400175.06 & 475729.33 & 622559.37 & 764050.14 \\ 82459.75 & 161685.79 & 317152.89 & 636564.63 & 610812.97 & 749634.1 \\ 80932.72 & 158691.6 & 311279.69 & 458109.73 & 895797.91 & 735751.98 \\ 79461.21 & 155806.3 & 305620.05 & 449780.46 & 588601.59 & 1176920.13 \end{pmatrix}$$

For a star graph S_p, the total work vector a will take the following form:

$$a = (2531808.72, 2501773.25, 2511354.54, 2558310.11, 2640563.63, 2756189.75),$$

which corresponds to ranks (4,6,5,3,2,1). The largest sum of potentials is observed at vertex v_6, despite its remoteness, and vertex v_1 does not get the worst rank, its favourable location in the graph compensates for the small weight of the vertex.

2.3.1 Special cases: star, clique, complete bipartite graph

Statement 2.8. Elements of the total work vector for a star graph with edges of unit weight and vertex weight matrix P

$$P = \begin{pmatrix} p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & p_n \end{pmatrix},$$

can be calculated as follows

$$a_1 = \frac{1}{\delta(n+\delta)}\big(p_1(1+\delta) + \sum_{j=2}^{n} p_j\big), \tag{2.3}$$

$$a_i = \frac{1}{\delta(1+\delta)(n+\delta)}\big(p_1(1+\delta) + p_i(1+n\delta+\delta^2) + \sum_{j\neq 1,i} p_j\big). \tag{2.4}$$

In particular, for a matrix P of the form

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & p \end{pmatrix},$$

expressions for the elements of vector a:

$$a_1 = \frac{n-1+p+\delta}{\delta(n+\delta)}, \tag{2.5}$$

$$a_i = \frac{\delta^2+\delta(n+1)+p+n-1}{\delta(1+\delta)(n+\delta)}, i = 2,...,n-1, \tag{2.6}$$

$$a_n = \frac{p\delta^2+\delta(pn+1)+k+n-1}{\delta(1+\delta)(n+\delta)}. \tag{2.7}$$

Statement 2.9. For n – clique C_n with vertex weight matrix P, the elements of vector a are computed as

$$a_i = \frac{1}{\delta(n+\delta)}\left[p_i(1+\delta)+\sum_{j\neq i} p_j\right]. \tag{2.8}$$

Statement 2.10. For a complete bipartite graph $B_{m,n-m}$ with n vertices ($n > 2m$), where the vertices are partitioned into two non-overlapping subsets V_1 and V_2 such that $|V_1| = m$ and $|V_2| = n-m$, with unit edge weights, the elements of the total work vector can be computed as

$$a_i = \frac{(n-m)\mathrm{tr}(P)+(n\delta+\delta^2)p_i+\delta\sum\limits_{j=m+1}^{n} p_j}{\delta(n-m+\delta)(n+\delta)}, \quad i \in [1,m]; \tag{2.9}$$

$$a_i = \frac{m\mathrm{tr}(P)+(n\delta+\delta^2)p_i+\delta\sum\limits_{j=1}^{n} p_j}{\delta(n-m+\delta)(n+\delta)}, \quad i \in [m+1,n], \tag{2.10}$$

where $\mathrm{tr}(P)$ is the trace of P matrix.

Chapter 3. Application of the proposed methods to graph vertex ranking problems

3.1 Transport system of Petrozavodsk

3.1.1 Transport network model building

The condition of the transportation infrastructure plays a significant role in the socio-economic development of the region. It supports the interdependence of the city's resources, including businesses, residential areas, goods movement, public transport, and retail.Enhancements to the transport system enhance residents' quality of life, reduce cargo transportation expenses, minimize road accidents, and foster regional economic efficacy.

There are various ways to improve the transport system. This may involve improving the road surface, building interchanges, new bypasses, bridges, pedestrian crossings, increasing traffic lanes, introducing new traffic lights. In addition, it may be related to changes in traffic rules, restriction of entry to the city centre, introduction of one-way traffic on some streets, introduction of a special lane for public transport, rational definition of public transport routes.

To solve the latter type of problems, mathematical modelling of the city's transport system is necessary [64—69]. Transport system modelling involves several steps. At the first stage, the road network graph is constructed. Next, the flows on the road segments of the transport network are determined. This scheme is based on the correspondence matrix, which contains information about the movements of people from one vertex of the graph to another.

The correspondence matrix makes it possible to find equilibrium traffic flows [70], make an optimal public transport schedule, determine the optimal location of stops for public transport, determine rational routes for public transport, including bicycle lanes [71], estimate the number of transported passengers and the revenue of public transport companies. Information on correspondences can be obtained from population surveys or from information obtained from video cameras installed on roads. However, this information is usually difficult to obtain or may be unreliable. Traditionally, gravity and entropy models[67; 72; 73] are used to construct the correspondence matrix. Obviously, they should be combined with the results of the population survey.

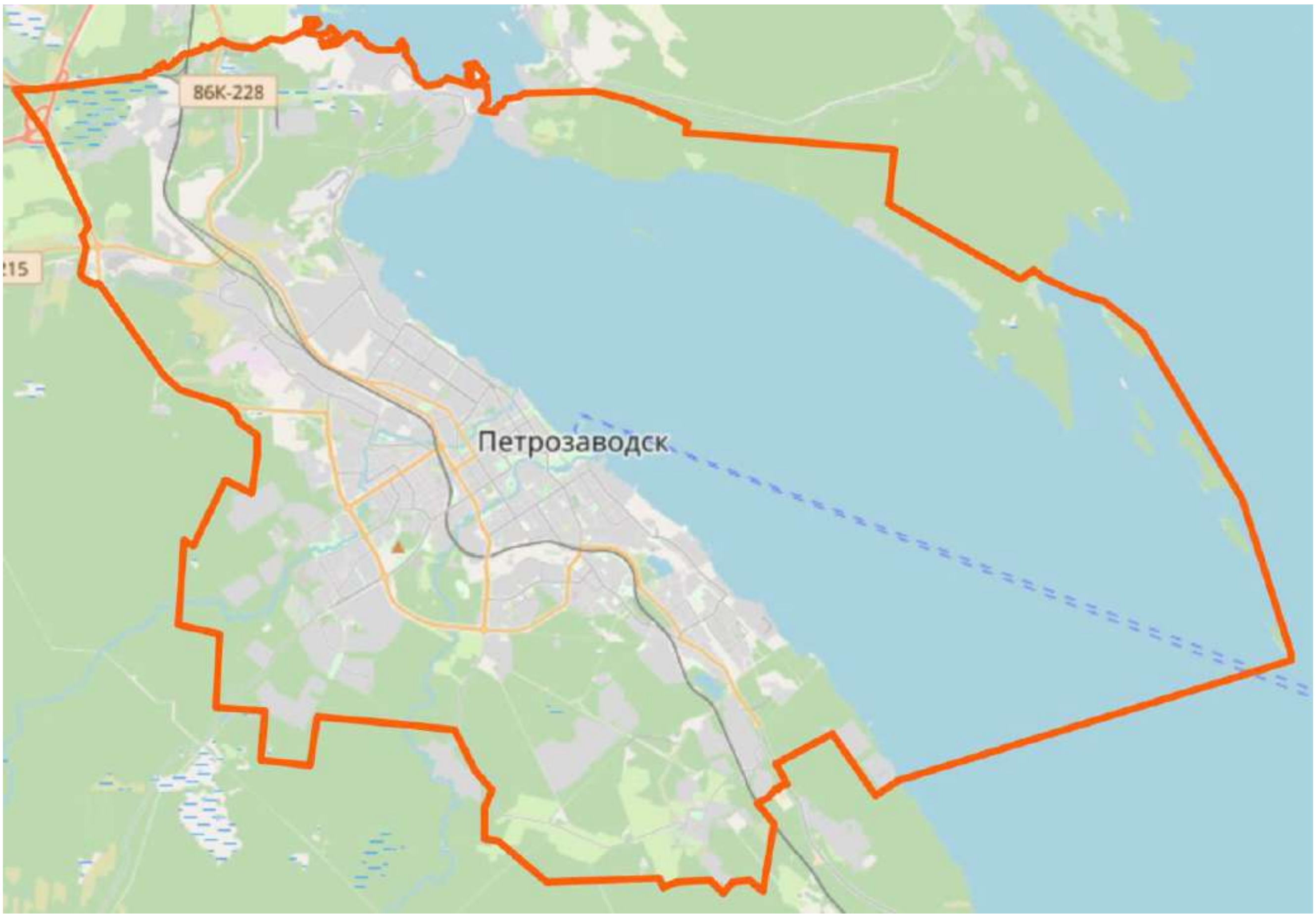


Figure 3.1 — Petrozavodsk urban district

The basis of the Petrozavodsk transport network model was built using the road graph represented in OpenStreetMap [74]. This is a non-commercial project aimed at creating a free geographical map of the world by Internet users.

The road class "roads with possibility of motorised traffic" was chosen as significant. When building the model, a graph was obtained, the edges of which correspond to roads with the possibility of road traffic, the vertices of the graph correspond to intersections of motorways. Thus, the graph of the road network of Petrozavodsk city was obtained, containing 1531 vertices and 2081 edges (Fig. 3.2).

Figure 3.2 — Petrozavodsk transport graph

According to the summary report on the results of the monitoring of the performance efficacy of the local government bodies of the urban districts and municipal areas of the Republic of Karelia in 2021[1], permanent average population of the Petrozavodsk urban district was 280 801. To distribute the city's residents according to the nodes of the graph, information was required on the number of living quarters located on the territory of Petrozavodsk. Web portal "Housing reform (Reforma ZHKKH)" [2] presents information about the houses of the Republic of Karelia

[1] https://gov.karelia.ru/upload/medialibrary/7d2/inatqynw9blv2ib412inu0u9lwcijshk/SVODNYY-DOKLAD.pdf

[2] http://reformagkh.ru

that is provided under government decree #731 of 23 September 2010. The exported data contains the address of each house according to the federal information address system, as well as the technical specifications of the building.

If we take as a valid approximation that an equal number of people live in an average dwelling unit, such data can be used to approximate the population density in a particular part of the city.

In the first stage, residents were distributed evenly over all living quarters, based on information from open sources. If the number of living quarters for the house was not specified, the parameter “total number of rooms” was chosen.

$$\frac{\text{total\_ppl}}{\text{total\_quarters}} = \text{residents\_per\_quarter},$$

where total_ppl is the total city population,
total_quarters is the total number of living quarters,
residents_per_quarter is the number of residents per quarter.

The number of the house residents is calculated as the product of the number of residents per quarter and the number of living quarters in the house:

$$\text{residents\_per\_quarter} \cdot \text{living\_quarters\_count},$$

where living_quarters_count is the number of living quarters in the house.

Next, for each house by the address received on the portal, geographic coordinates were obtained using the Yandex Maps service. These coordinates are used to find the nearest node of the graph to which the house was attached. According to this binding, the weight of each vertex represents the total number of residents living in houses in the vicinity of the graph node, i.e. the intersection of urban roads.

Table 12 shows the number of residents living in houses attached to the graph nodes belonging to the city districts.

Fig. 3.3 shows the distribution of city residents by vertices of the graph. The larger vertex size corresponds to a larger weight value. The largest vertex size corresponds to the dormitory districts of the city and places with new dense building.

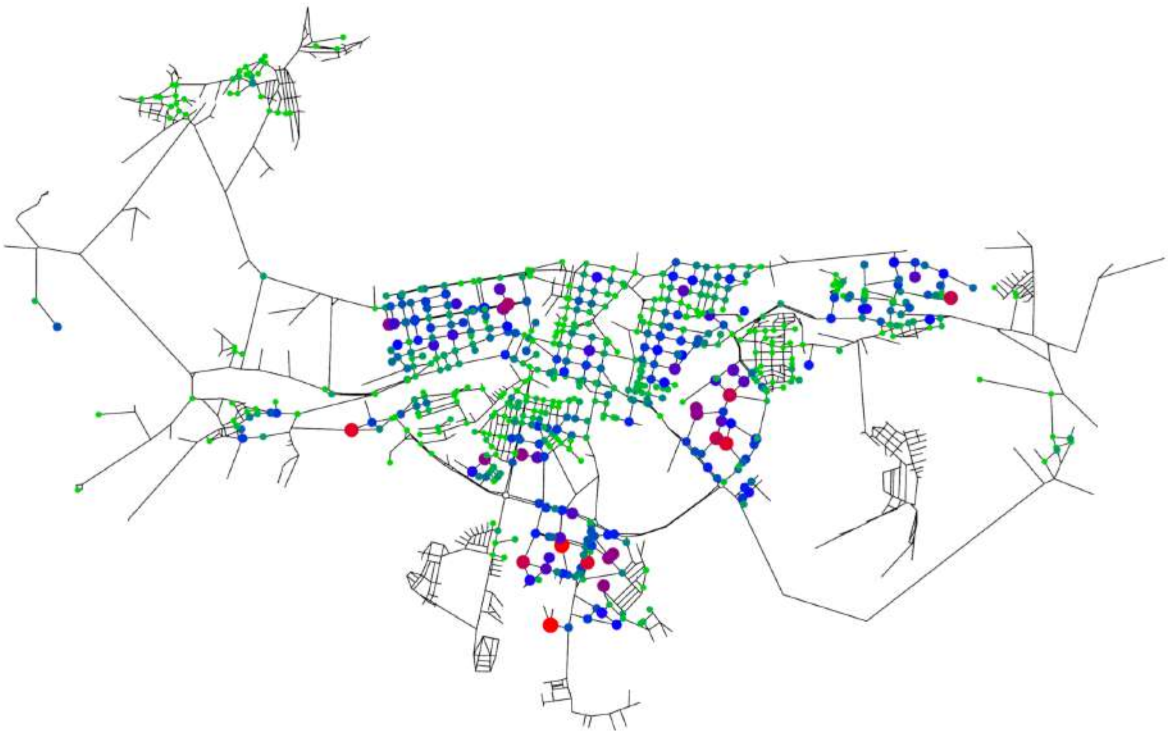


Figure 3.3 — Heat map of resident distribution

The next stage of work was to search for data on organizations located in the territory of the Petrozavodsk City District. Data from the OpenStreetMap project were used first. All objects described as, for instance, an office, an educational institution, a shopping center, etc. have been selected. For each type of facility, a weight was chosen based on both the approximate number of employees typically employed by that type of establishment and the attendance of such establishments by visitors. Thus, the number of staff and the visitor flow to the city clinic or school will exceed the same for the trade organization. The weight value was chosen from the half-interval (0,1].

Organizations were bound to the graph nodes according to the geographical coordinates, just like the residential buildings before. Each node was assigned an additional characteristic "weight of organizations", which is a type-weighted sum of the number of organizations associated with this vertex.

Fig. 3.4 visualizes the distribution of organizations over the vertices of the transport graph. The larger vertex size corresponds to the larger weighted sum of organizations. The figure shows that a large number of organizations are concentrated in the city center.

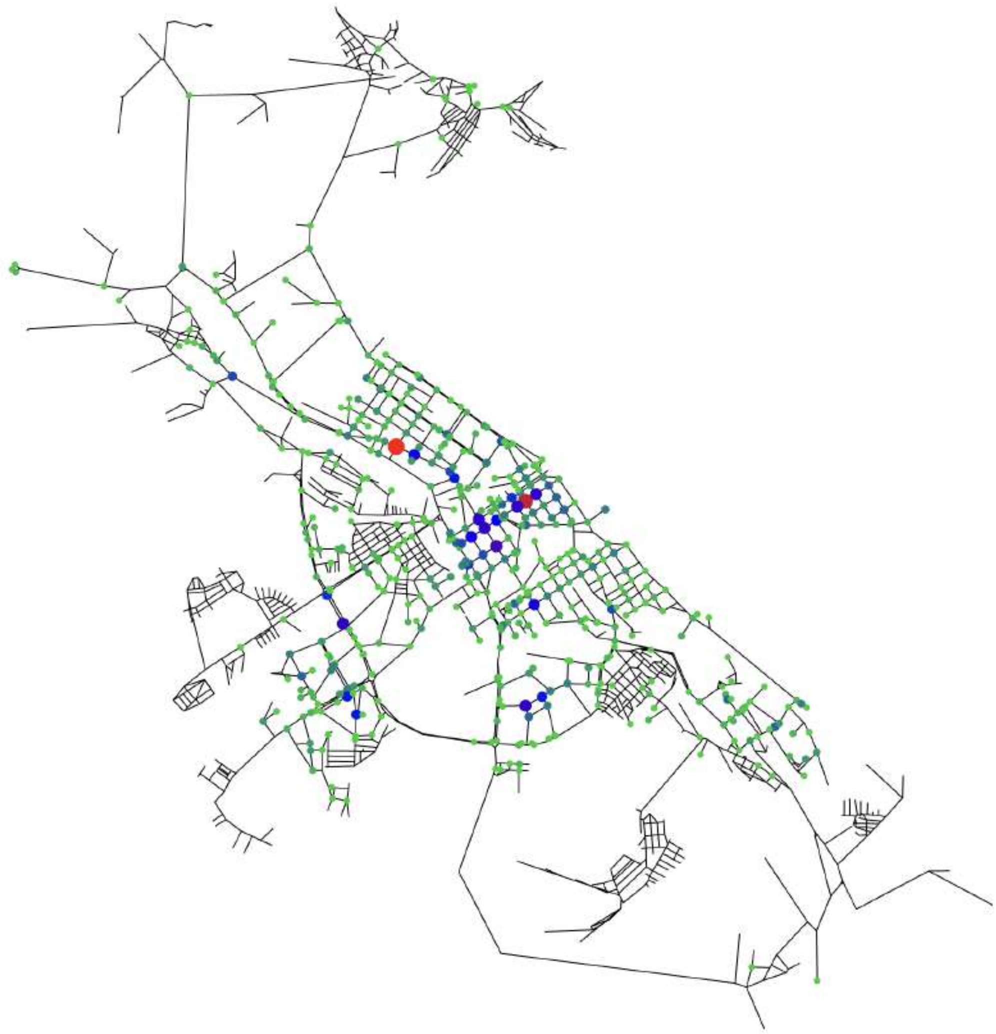


Figure 3.4 — Distribution of organisations by vertices of the graph

To visualise the results obtained, a web service was created, the process of creation of which is presented in [75].

Table 12 — Districts of Petrozavodsk

№	City district	Residents number	Organisations weight	Vertices number
1	Golikovka	25723.8	69.3	60
2	Drevlyanka	51134.7	123.7	98
3	Zheleznodorozhny	3414.4	6.9	14
4	Zareka	15482.9	36.1	44
5	Kamenny bor	5057.4	10.0	16
6	Kirpichnny zavod	1006.1	0.5	21
7	Klyuchevaya	26689.4	46.2	64
8	Kukkovka	36887.5	77.5	86
9	Oktyabrsky	35685.6	80.1	44
10	Pervomaisky	17684.1	91.0	42
11	Perevalka	26275.2	41.5	113
12	Peski	0.0	0.8	7
13	Ptizefabrika	850.8	0.7	16
14	Rybka	4017.6	4.6	37
15	Sainavolok	582.9	0.0	7
16	Severnaya promzona	747.2	19.4	18
17	Solomennoe	2419.6	5.3	27
18	Sulazhgora	4665.8	18.7	44
19	Teplichny	326.4	4.0	12
20	Tomici	47.3	0.6	15
21	Center	21526.2	264.1	89

3.1.2 Districts of Petrozavodsk

Let us consider a graph with the following structure: the vertices of the graph based on the Petrozavodsk transport system [76] were divided into non-overlapping subsets corresponding to the districts of the city (Fig. 3.5). Here, the vertex labels correspond to the numbering in Table 12. If two districts are directly connected by motorways, there is an edge between their corresponding vertices. The lengths of the edges are proportional to the lengths of the corresponding shortest paths in the original graph. For example, districts 16 and 12 are connected by a longer path than districts 16 and 19.

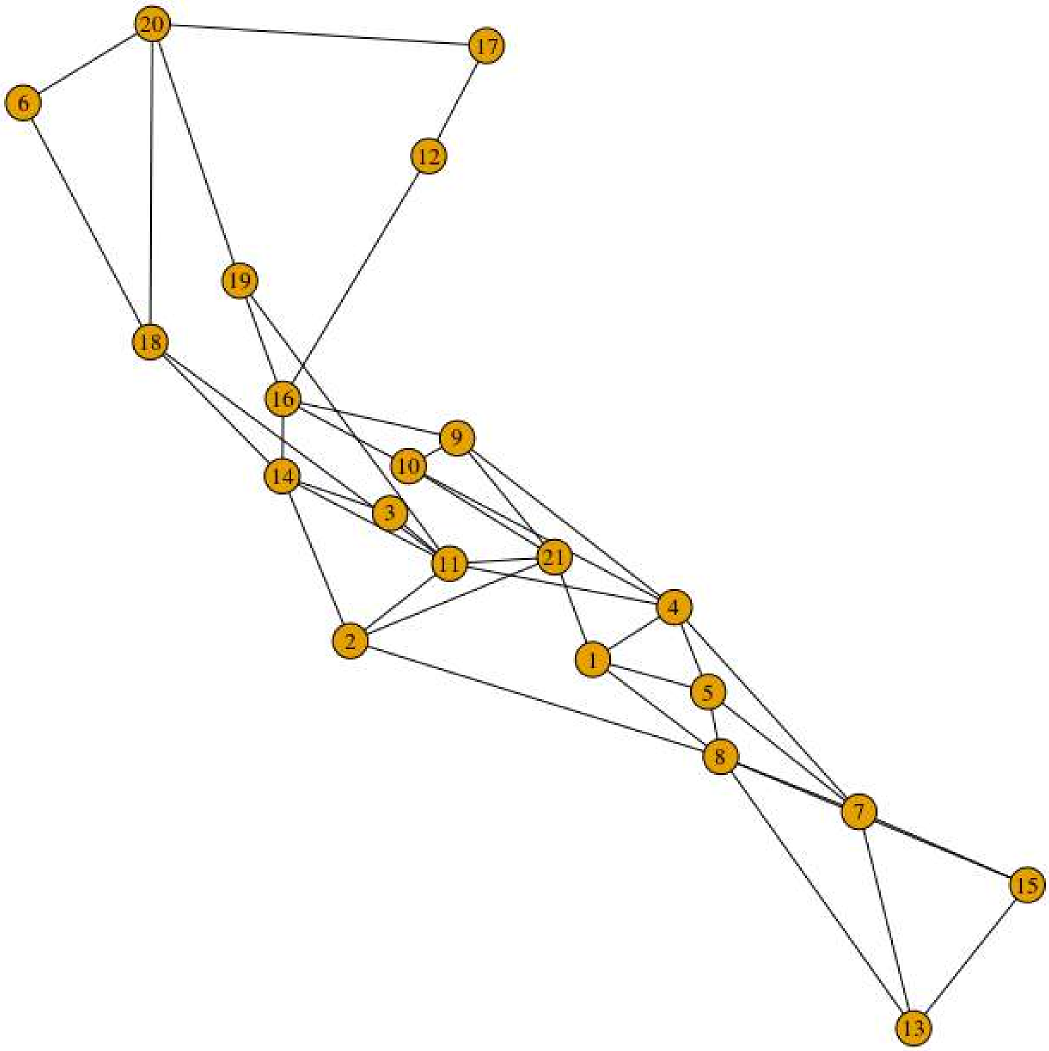


Figure 3.5 — Transport graph of the city districts (see Table 12)

The weights of the edges for calculation are chosen equal to the inverse of the lengths of the roads connecting the city districts. The value $\delta = 0{,}1$. Absolute potentials were calculated for cases when current is supplied sequentially to the graph vertices from 1 to 21.

Table 13 shows the sums of ranks calculated on the basis of Kirchhoff's rules and by the PageRank method. The most important vertices of the graph have the lowest rank value according to Kirchhoff's rules, but the highest one calculated by the PageRank method. For the convenience of comparing the results, the data were

Table 13 — Ranking the vertices of the neighbourhood graph

№	City district	Rank sum	PageRank
1	Golikovka	147	0.057171
2	Drevlyanka	155	0.038292
3	Zheleznodorozhny	171	0.037208
4	Zareka	140	0.064489
5	Kamenny bor	160	0.054556
6	Kirpichnny zavod	212	0.021275
7	Klyuchevaya	175	0.052078
8	Kukkovka	166	0.057698
9	Oktyabrsky	156	0.05312
10	Pervomaysky	151	0.054764
11	Perevalka	131	0.074412
12	Peski	202	0.037077
13	Ptizefabrika	205	0.028448
14	Rybka	154	0.060789
15	Sainavolok	195	0.032919
16	Severnaya promzona	156	0.059534
17	Solomennoe	213	0.035842
18	Sulazhgora	174	0.043686
19	Teplichny	169	0.034296
20	Tomici	193	0.040282
21	Centre	133	0.062062

normalised, and PageRank values were considered with opposite sign and ordered (Fig. 3.6). The best ranks were obtained for the districts of Perevalka, Centre, Zareka, Golikovka, which are well integrated into the transport system of the city, and the worst — for remote districts of the city, such as Kirpichny Zavod, Ptizefabrika, Solomennoe and Sainavolok.

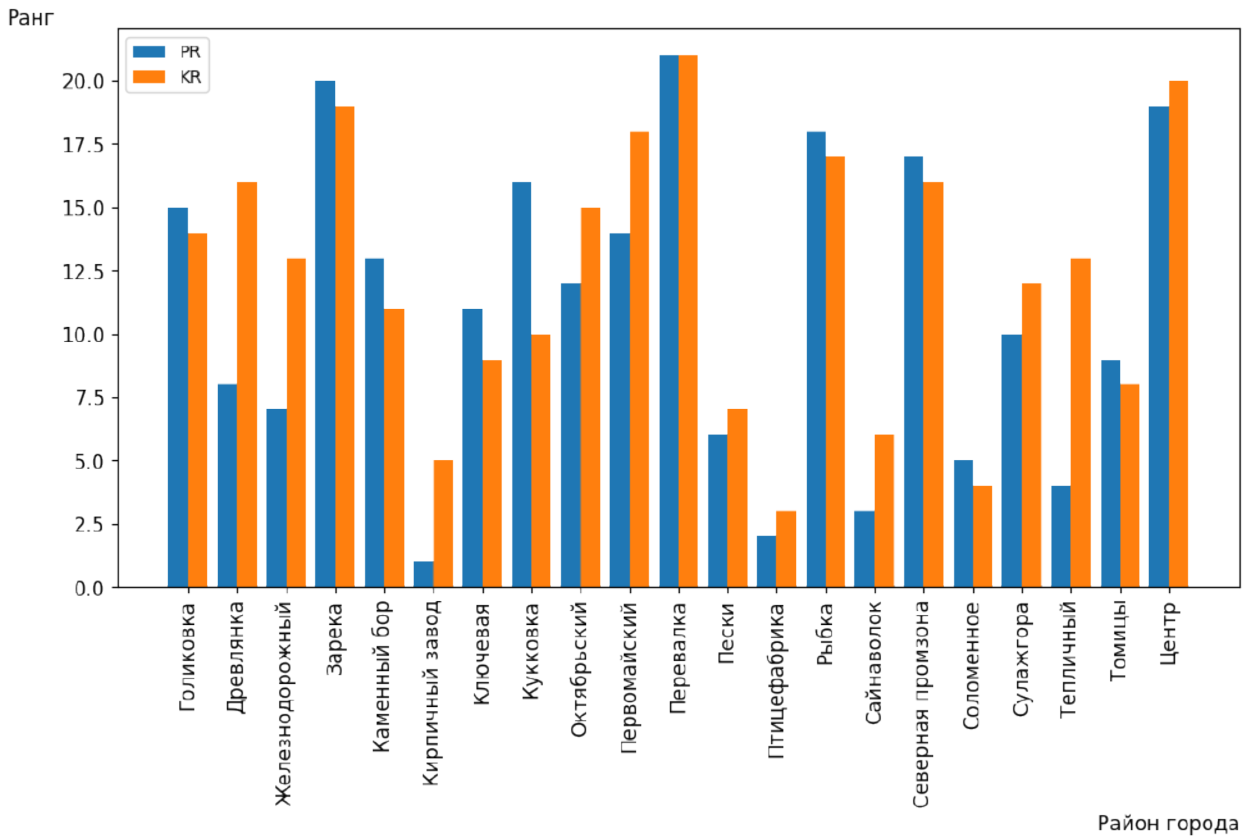


Figure 3.6 — Comparison of values obtained using Kirchhoff's rule (KR) and PageRank(PR) method

Let us introduce a weight matrix P for the graph of city districts, the diagonal elements of which correspond to the values of the number of residents in the districts of Petrozavodsk given in the table 14.

The figure 3.7 shows the curves obtained based on a vectors for different values of δ, as well as the curve of distribution of residents by vertices of the graph. The use of vertex weights based on the number of residents allows us to obtain more meaningful ranks for dormitory areas of the city (e.g., the Kliuchevaya district) compared to the original method. Fig. 3.8 shows comparison of the results of ranking of the graph nodes at $\delta = 0.00061$, R_1 corresponds to ranking without taking into account the node weights, R_2 takes into account the number of inhabitants in the graph nodes. Such districts as Drevlyanka, Kliuchevaya, Kukkovka and Oktyabrsky are the most densely populated and loaded, their ranks have increased compared to the unweighted approach, while there is a decrease in ranks for sparsely populated districts.

Table 14 — Number of residents in the city districts

№	City district	Number of residents
1	Golikovka	25723.8
2	Drevlyanka	51134.7
3	Zheleznodorozhny	3414.4
4	Zareka	15482.9
5	Kamenny bor	5057.4
6	Kirpichnny zavod	1006.1
7	Klyuchevaya	26689.4
8	Kukkovka	36887.5
9	Oktyabrsky	35685.6
10	Pervomaysky	17684.1
11	Perevalka	26275.2
12	Peski	0.0
13	Ptizefabrika	850.8
14	Rybka	4017.6
15	Sainavolok	582.9
16	Severnaya promzona	747.2
17	Solomennoe	2419.6
18	Sulazhgora	4665.8
19	Teplichny	326.4
20	Tomici	47.3
21	Centre	21526.2

3.1.3 City transport network graph vertex ranking

Next, let us consider the graph of the city transport system (Fig. 3.2). Above (and in [76]) the construction of an undirected graph of the transport network of Petrozavodsk is described. This graph can be considered as an directed graph if we take into account the direction of traffic on the road sections corresponding to the edges of the graph. It consists of 1530 vertices and 3781 edges. The values in

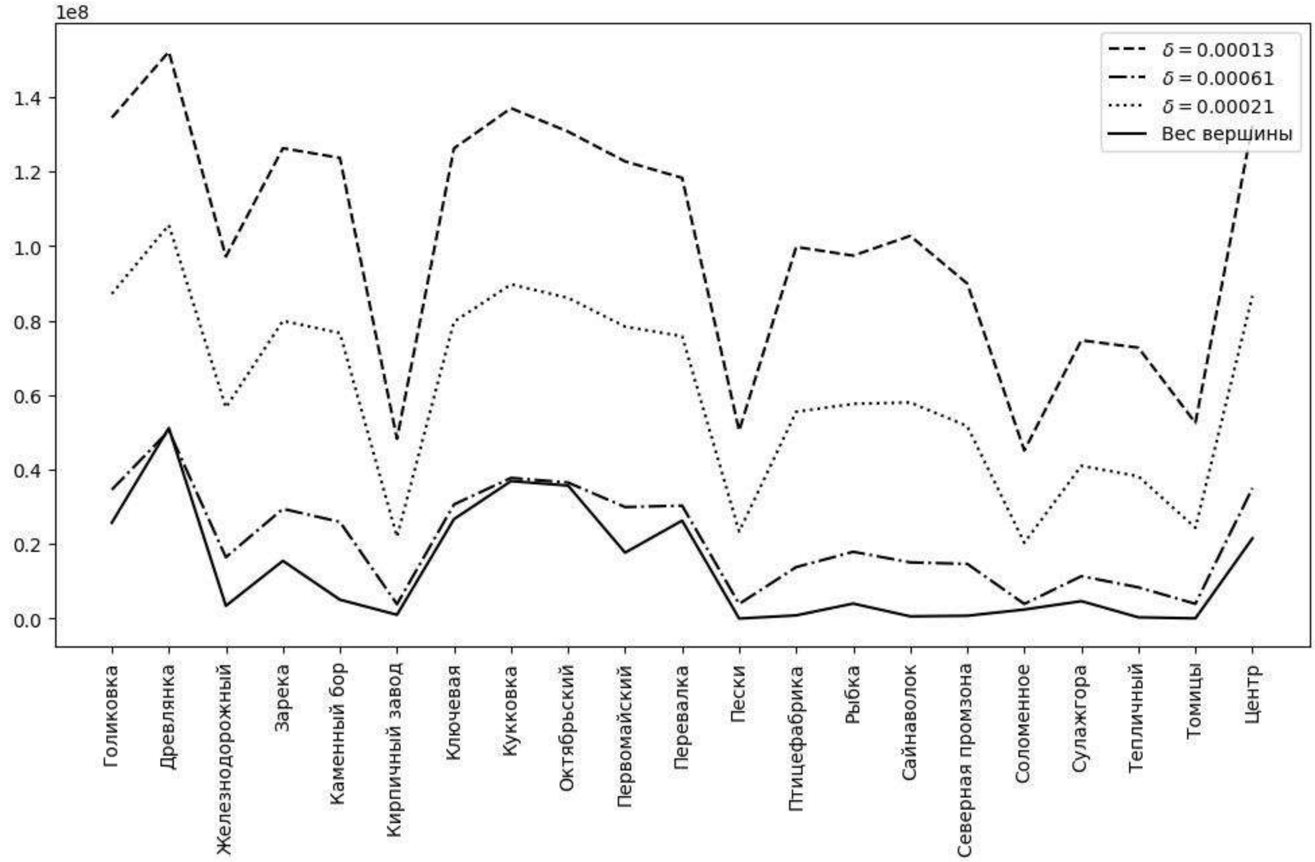


Figure 3.7 — Ranking curves depending on the choice δ

the graph adjacency matrix are equal to the inverse of the lengths of road segments between the corresponding pairs of vertices.

Let us compute the number of simple paths of length 3 passing through the vertices of the graph, according to statement 1.2. Figure 3.9 shows a heat map of n_3 values. Larger and darker coloured nodes correspond to the more frequent occurrence of a vertex in simple paths of length 3.

It can be seen that there is an accumulation of high ranking vertices in the districts of Kukkovka and Perevalka, as well as in the remote district of Solomennoye. This result can be explained by the fact that these areas are mainly low-rise buildings of the so-called private sector. The length of road segments there is usually small, which generates a large number of shortcuts.

Let us calculate the values of integral centrality of vertices. Since the calculation of the number of occurrences of a vertex in the paths requires raising the adjacency matrix to power d, to simplify the calculation process in graphs with a large number of vertices, we can restrict ourselves to considering paths of length

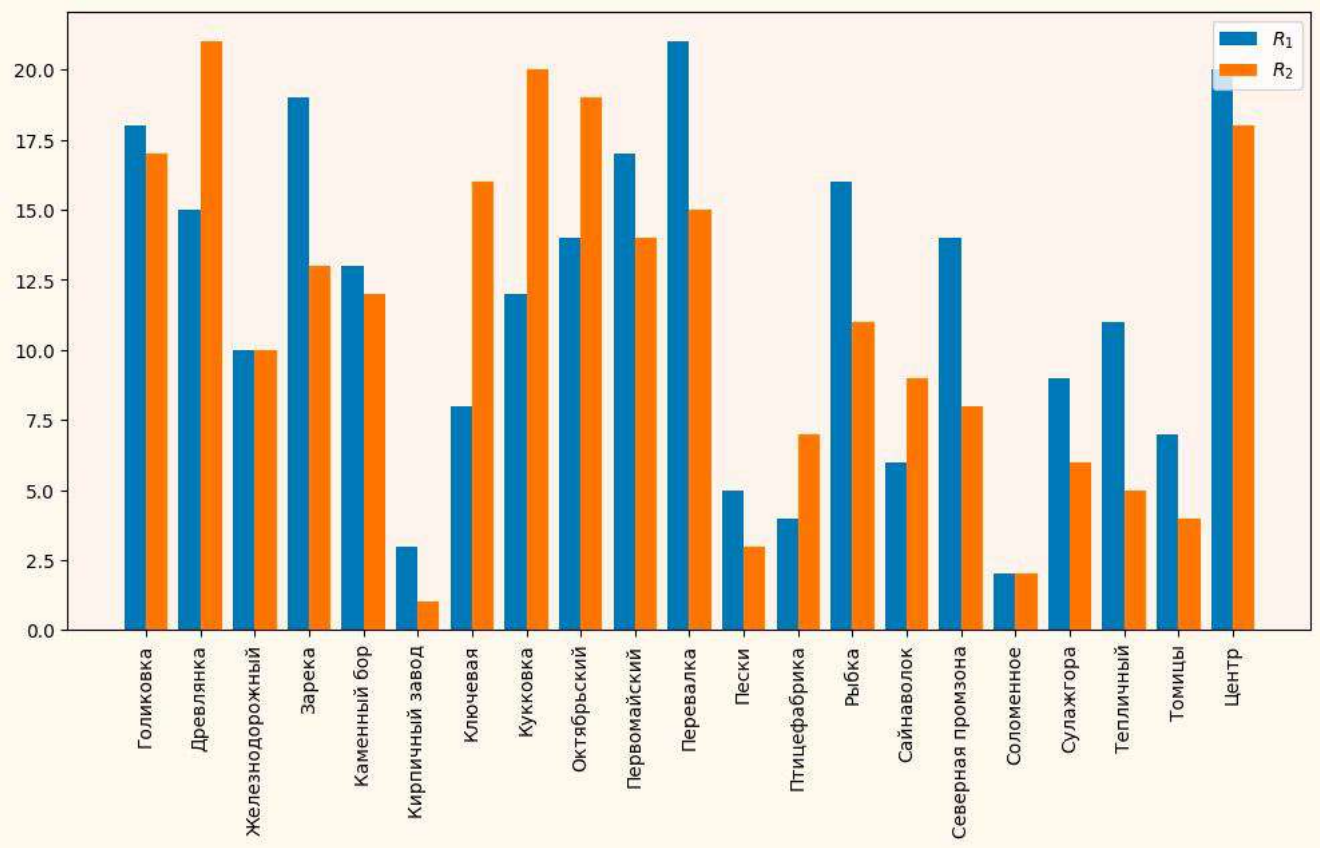


Figure 3.8 — Comparison of vertex ranking results

less than $n - 1$ to estimate centrality. Fig. 3.10 shows a heat map of the vertices of the transport graph for which the integral centrality values for path lengths up to $d = 100$ have been computed.

This approach also allows to obtain a cluster of high rank vertices at Kukkovka, Perevalka and Solomennoe. In this case, a large number of short cycles appear in the transport graph, which significantly increases the total number of vertex appearances in the paths.

Further we will consider an undirected graph of the transport network of Petrozavodsk. Fig. 3.11 shows the visual representation of centrality values of vertices of the road network graph. The larger vertex size corresponds to the larger Myerson centrality value. The approach that takes into account the number of occurrences of vertices in paths of length 10, including cycles, also indicates the importance of vertices located in Kukkovka and Perevalka districts. In addition, it can be seen that large nodes are located in the city centre.

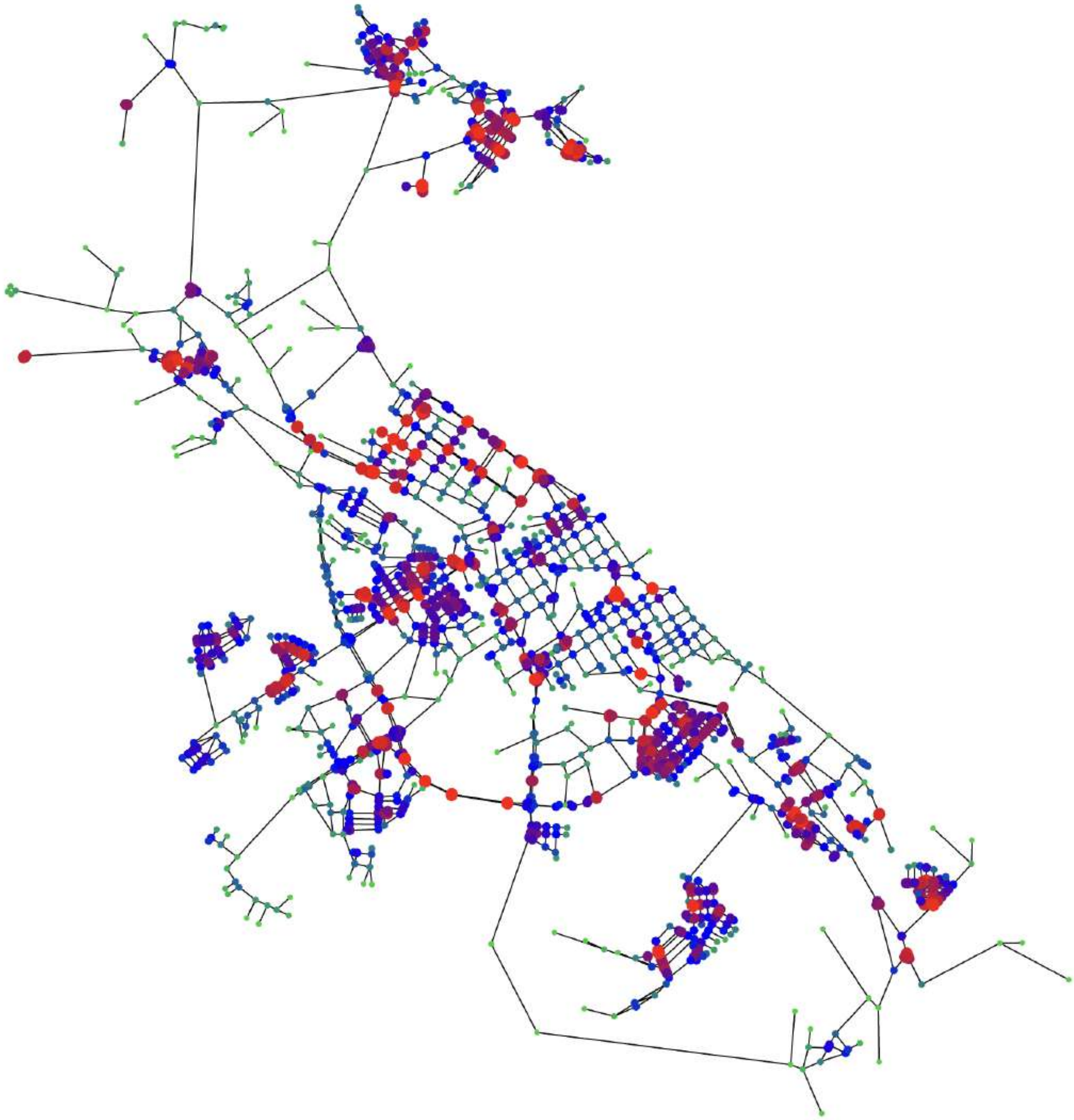


Figure 3.9 — Heat map of n_3 nodes of the Petrozavodsk transport network graph.

Let us calculate the sums of the ranks of the graph nodes based on the values of the absolute potentials of the nodes. A heat map was constructed for the rank vector (Fig. 3.12).

Similar results for these areas were obtained above when calculating the centrality values of vertices using the modified Myerson method, integral centrality and finding the number of occurrences of vertices in simple paths of length 3. Such results can be explained by the accumulation of a significant number of small streets

Figure 3.10 — Heat map of integral centrality values of the Petrozavodsk transport network graph vertices

with a small length of roads, which in turn leads to the appearance in the graph of edges with significant carrying capacity (the weight of the edge is chosen equal to the value inverse of the road length), as well as a large number of short cycles.

Let us compare the obtained results with the ranking results using the PageRank method. Fig. 3.13 shows the heat map of the nodes of the unweighted graph of the transport network; Fig. 3.14 – heat map of the vertices of the graph with

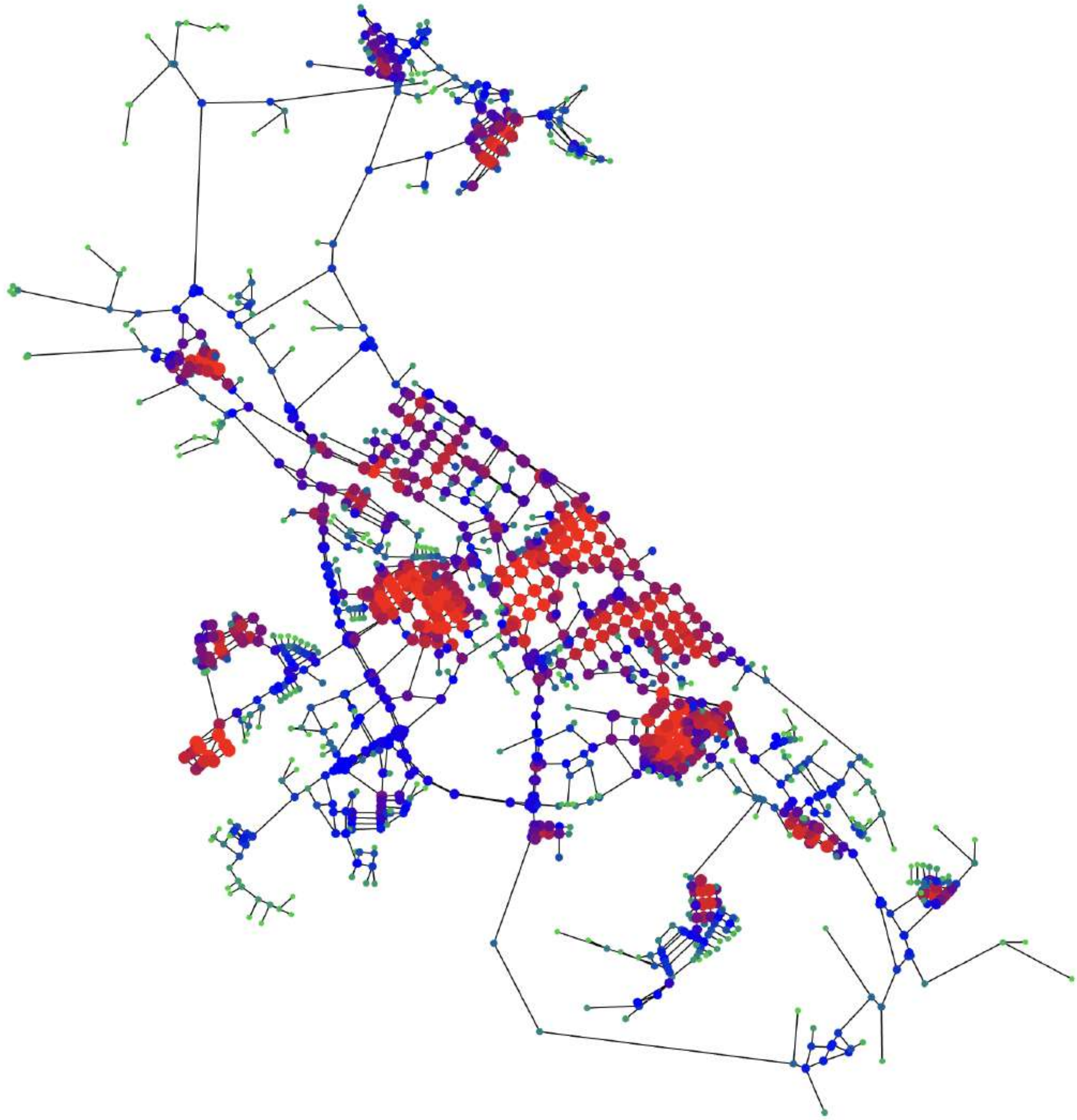


Figure 3.11 — Centrality of vertices by Myerson at $k = 10$

weighted edges. The weights of the edges are equal to the inverse of the lengths of the corresponding road segments. For PageRank, vertex ranks are distributed more evenly, the heat map has a large number of vertices with average ranks. But adding edge weights "lightens" the heatmap.

Applying the method of ranking the vertices of the graph based on Kirchhoff's rules, taking into account the weights of vertices equal to the number of residents living in close proximity to the road intersection corresponding to the vertex, to

Figure 3.12 — Heat map of transport graph vertex ranks based on absolute potential values (weighted edges)

the transport graph, we obtain a heat map of ranks (Fig. 3.15), where there are no clusters of vertices with high ranks. The darkest vertices of the graph on the heat map correspond to new districts with dense building.

Above, when estimating the centrality of the graph vertices, the weights of the edges were considered as values inverse to the lengths of the corresponding road segments. Based on the data obtained during the preparation of the [76] paper, the

Figure 3.13 — Heatmap of vertex ranks of unweighted transport graph, PageRank method

traffic flows on the edges of the transport graph were calculated. These values can also be used as weights of the edges.

Fig. 3.16 shows the heat map of graph vertex ranks, where the weights of edges are equal to the values of transport flows obtained on the basis of integral centrality, Fig. 3.17 – heat map of ranks obtained on the basis of absolute potential values, Fig. 3.18 – ranks calculated using the total work of charge transfer between

Figure 3.14 — Heatmap of vertex ranks of weighted transport graph, PageRank method

the nodes of the electric circuit. Here, the nodes with the highest centrality are coloured red, those with the lowest are coloured green.

The results obtained with the integral centrality in this case match well with the real traffic situation in the city, for example, the vertices corresponding to the intersections with Komsomolsky Avenue received the highest centrality values; this street is one of the most loaded in the city.

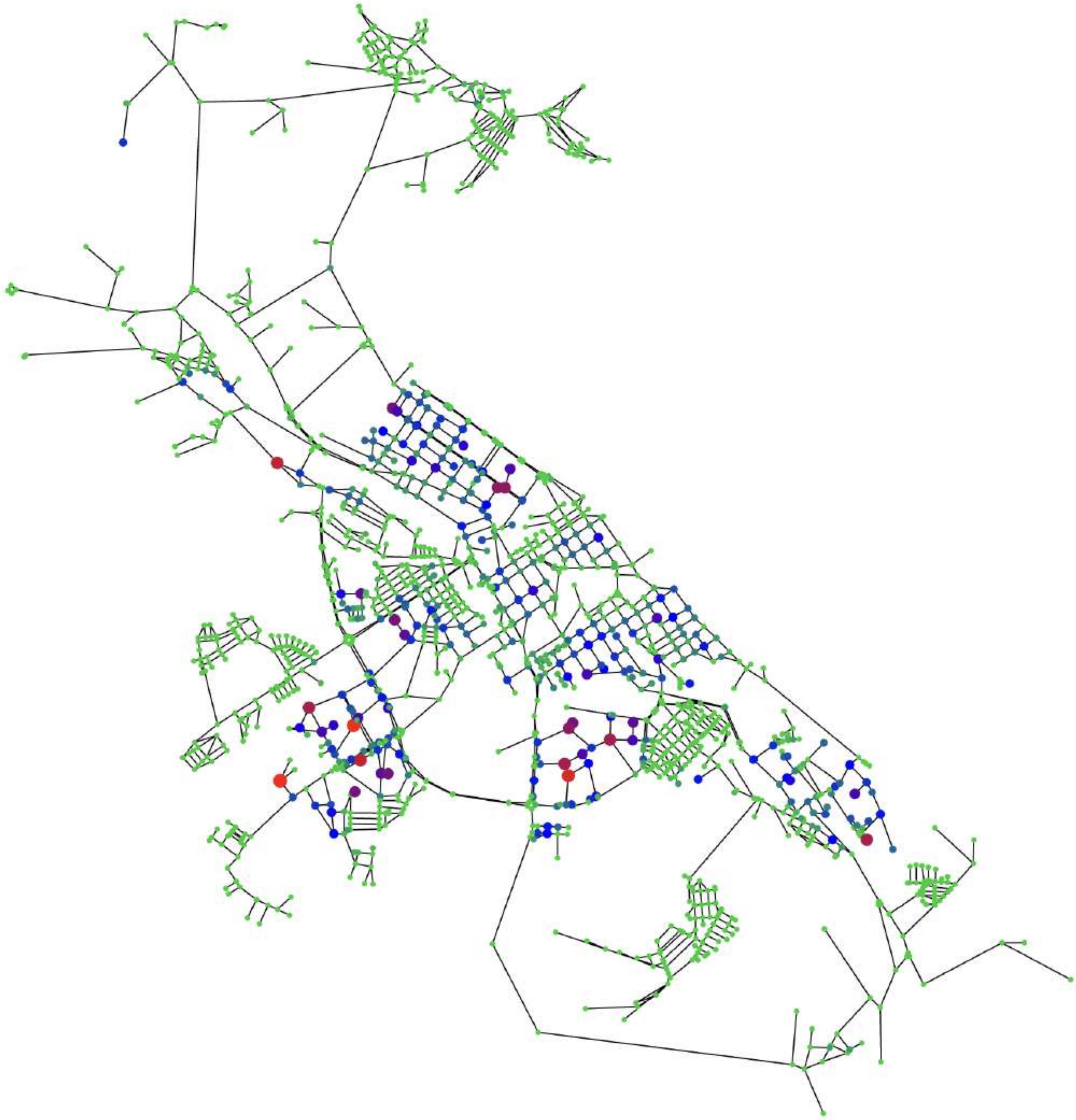


Figure 3.15 — Heat map of graph vertex ranks (weighted vertices and edges) based on values of total charge transfer work

3.2 Ant colony

Data from Swiss biologists obtained during the [77] study were used as initial data. A group of scientists recorded the interaction time between pairs of ants within a colony, provided that at a certain moment several ants are exposed to an infection. This resulted in interaction graphs (Fig. 3.19, Fig. 3.22), where the vertices are ants,

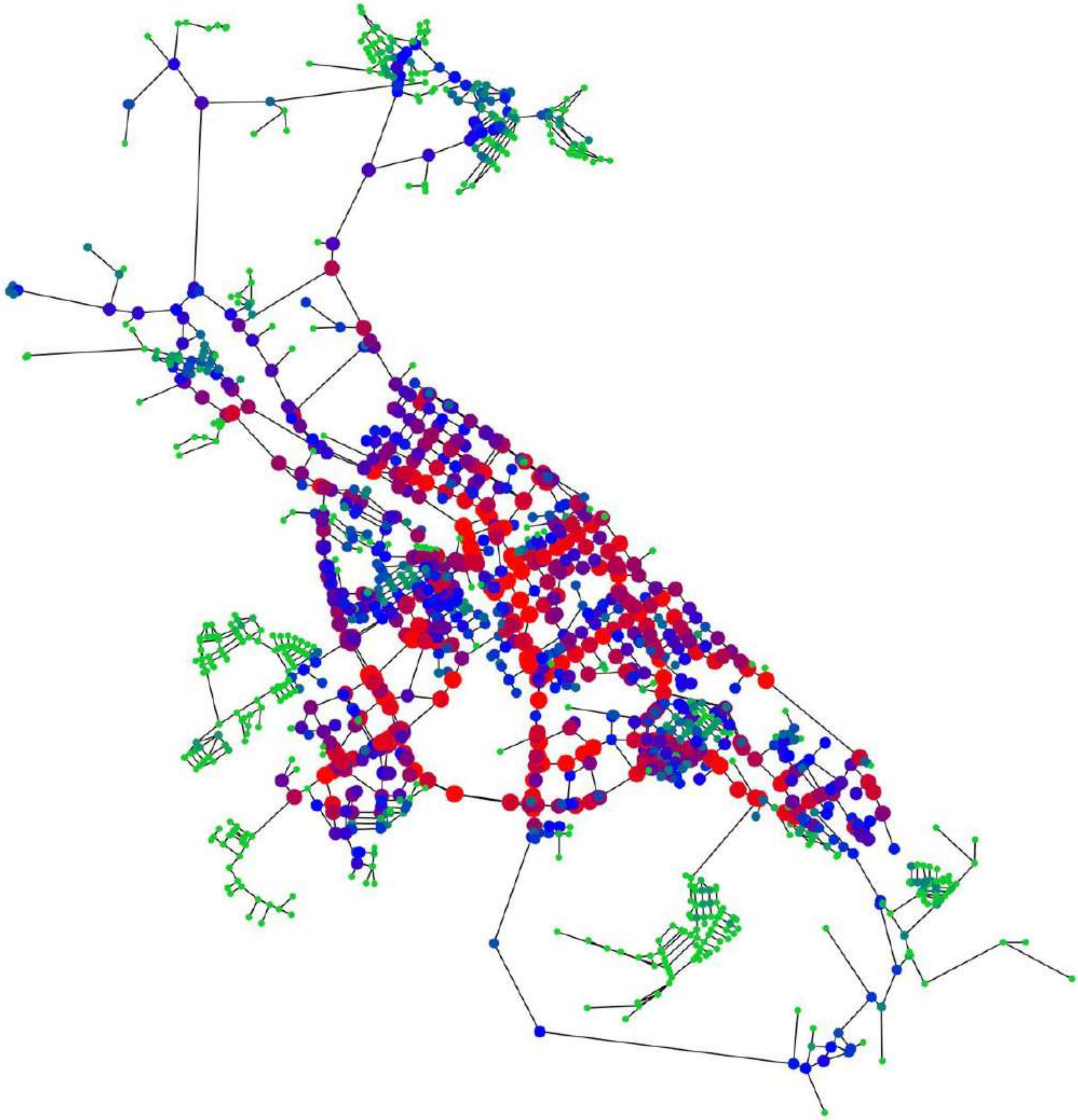


Figure 3.16 — Heat map of graph vertex rankings where the weight of edges is equal to the traffic flow (integral centrality)

and the weights of the edges between these vertices are equal to the total contact time during the experiment. For convenience of data presentation, edges describing interactions between ants that lasted less than 15 % of the maximum possible within a colony were removed. The colony under consideration consists of 105 ants.

Within a colony, ants are divided into groups "nurses" (n) – nurses, "foragers" (f) – foragers, and "queen" (q) – the colony's queen.

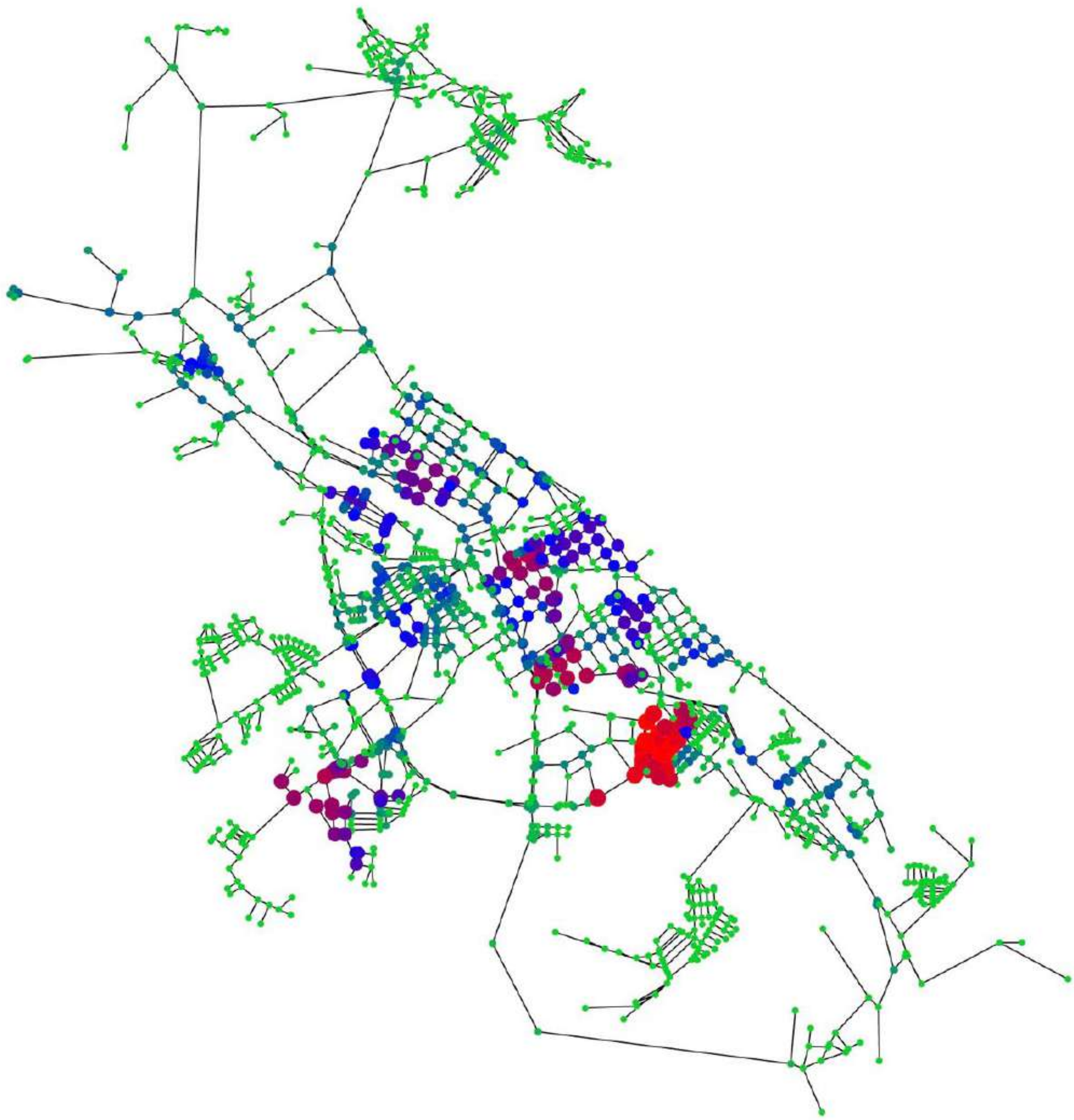


Figure 3.17 — Heat map of graph vertex ranks, where edge weights are equal to the transport flow (absolute potentials)

Fig. 3.19 shows the graph of colony interactions before infection. The vertex corresponding to the mate ("queen") is shown as a triangle. We rank the vertices of the pre-infestation interaction graph using the modified Myerson value and based on the values of absolute potentials that take into account the weights of edges. Since the method of calculating the vector of centrality using the modified Myerson value involves working with the adjacency matrix, we chose a threshold value of

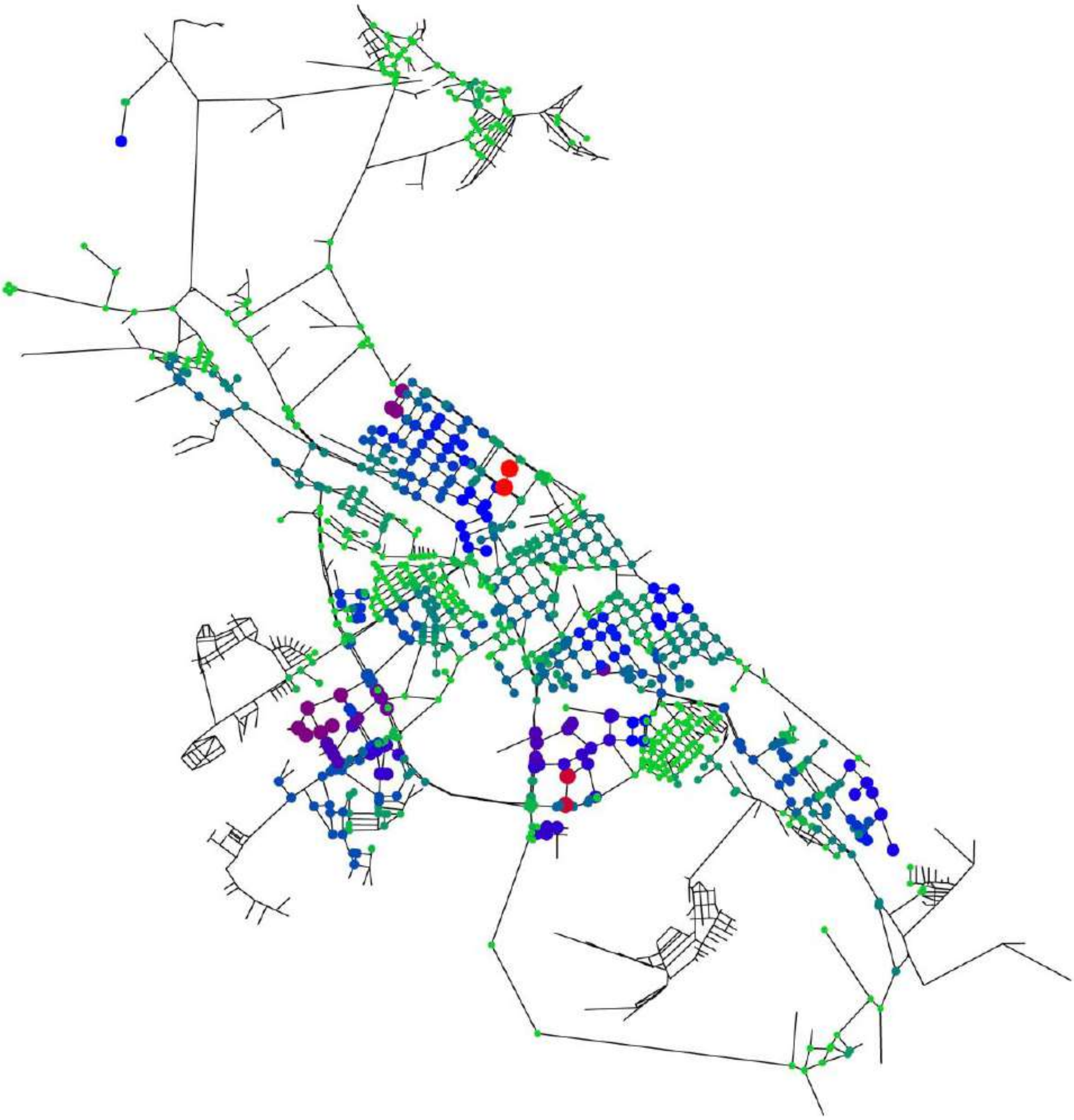


Figure 3.18 — Heat map of graph vertex ranks, where edge weights are equal to the transport flow (charge transfer work)

interaction time equal to $0.15 \cdot T_{max}$, where T_{max} is the maximum interaction time between ants in the colony. All edges whose weights were less than the threshold were labelled 0 in the adjacency matrix, while the rest were labelled 1.

The highest ranks were obtained by ants from the "nurses" group (nodes 19, 45, 59); on the heat maps (Fig. 3.20, Fig. 3.21) the corresponding nodes are shaded darkest. Table 15 shows the ranks of the vertices that will be subsequently infected.

After infestation, the structure of the graph changes markedly (Fig. 3.22). Healthy ants minimise communication with infected ants. Meanwhile, isolated

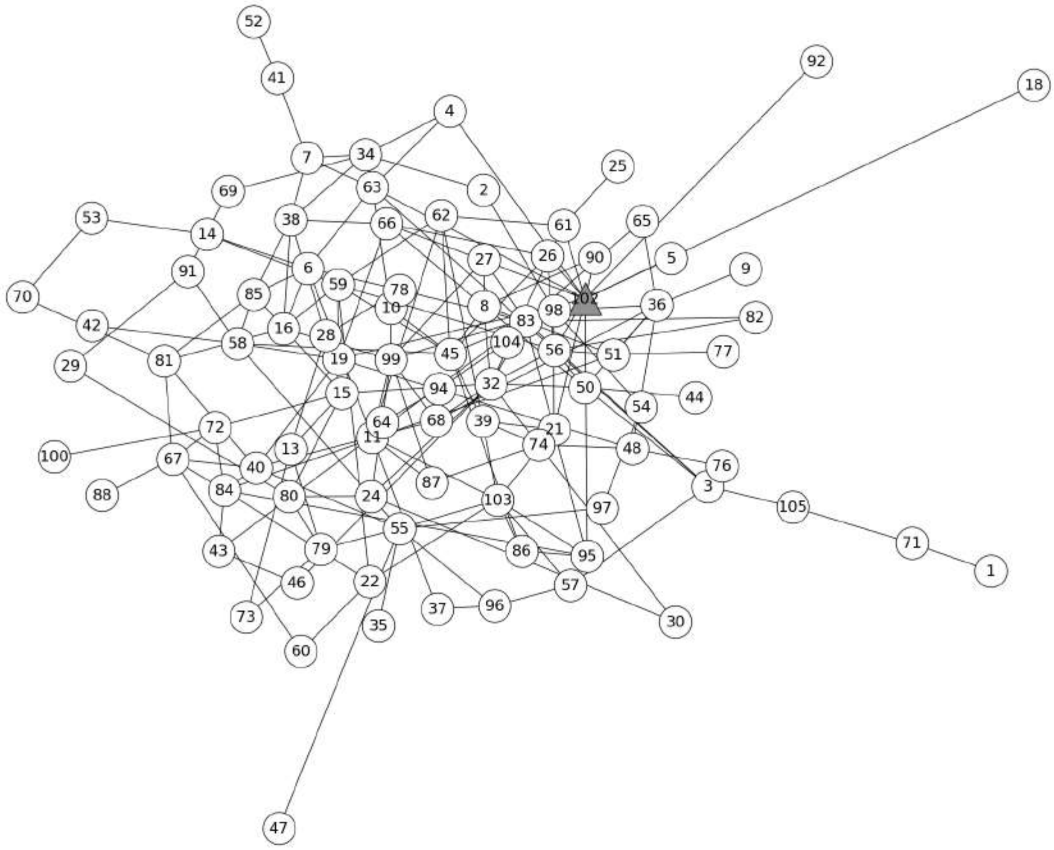


Figure 3.19 — Pre-infestation ant colony interaction graph

Table 15 — Ranks of vertices before the infection

Id	9	37	47	49	50	64	73
Ranks based on potentials	75	54	79	-	47	89	77
Ranks based on the Myerson vector	78	64	76	-	34	90	74

infected ants form their own network of interactions in which some nodes have increasing ranks, which may be due to becoming leaders in their community. The infected individuals were selected from the "foragers" group. All of them, with the exception of vertex 25, were isolated.

Table 16 presents the ranks of the vertices corresponding to the ants that were exposed to the infestation. Close ranks indicate equal roles of vertices in the system. The highest ranked node after infestation is the node corresponding to the queen of the colony.

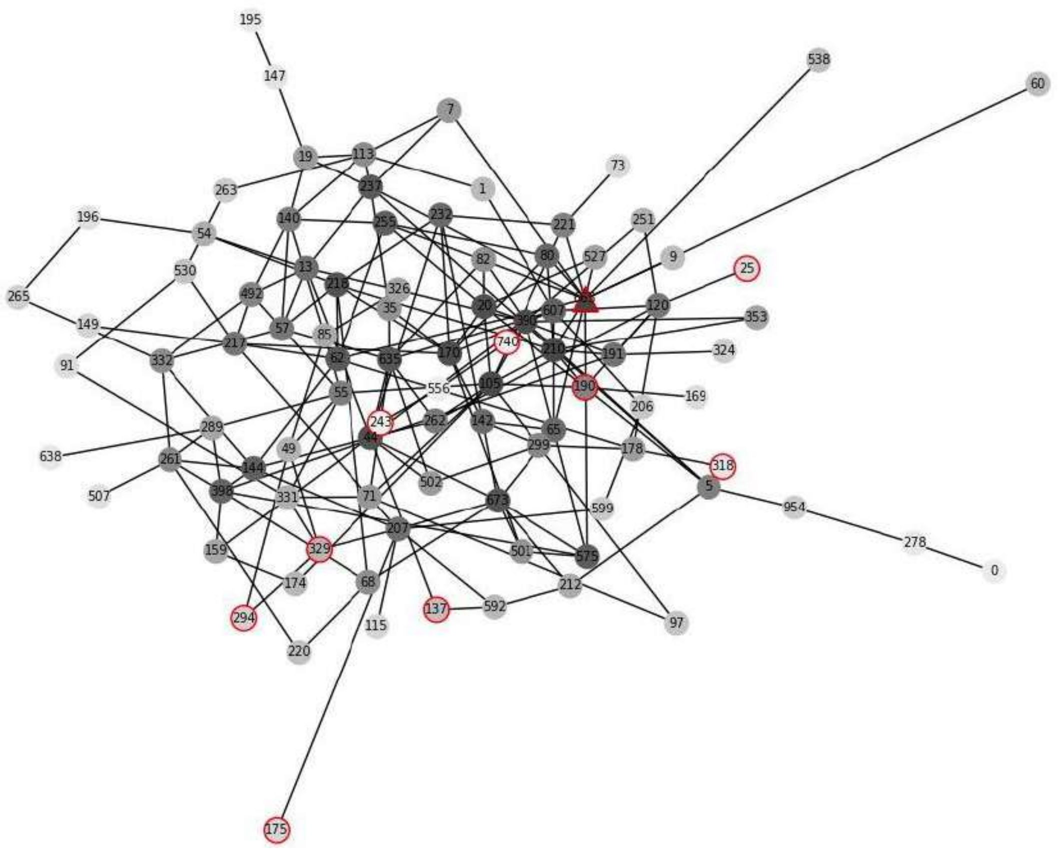


Figure 3.20 — Heat map of graph vertex ranks prior to infection. Modified Myerson value

Table 16 — Ranks of infected vertices

Id	9	37	47	49	50	64	73
Ranks based on potentials	54	74	72	69	78	70	73
Ranks based on the Myerson vector	55	75	72	64	70	49	69

3.3 St. Petersburg Subway

As an object of computational experiment we used the graph constructed on the basis of the St. Petersburg Subway scheme (Fig. 3.25). The corresponding graph (Fig. 3.26) contains 72 vertices.

Let us estimate the centrality of vertices in this graph using the modified Myerson vector for paths of length 5 and 10 (Table 17). The lowest centrality values are observed at Devyatkino, Pr.Veteranov, Parnas, Kupchino, Rybatskoe, Dybenko St., Komendantsky Prospekt, Mezhdunarodnaya, Begovaya stations. All these sta-

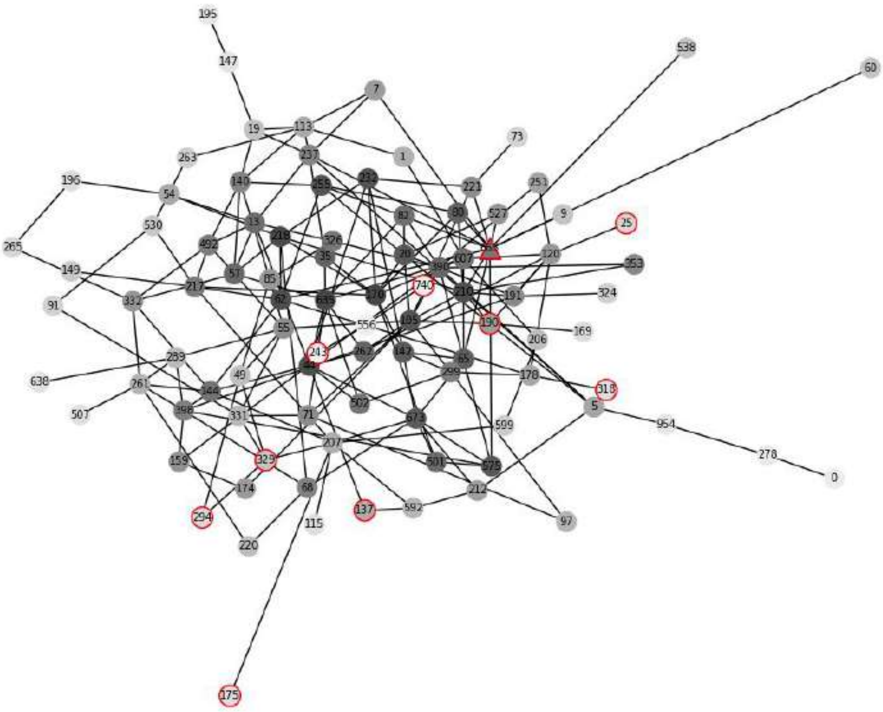


Figure 3.21 — Heat map of graph vertex ranks prior to infection. Absolute potentials

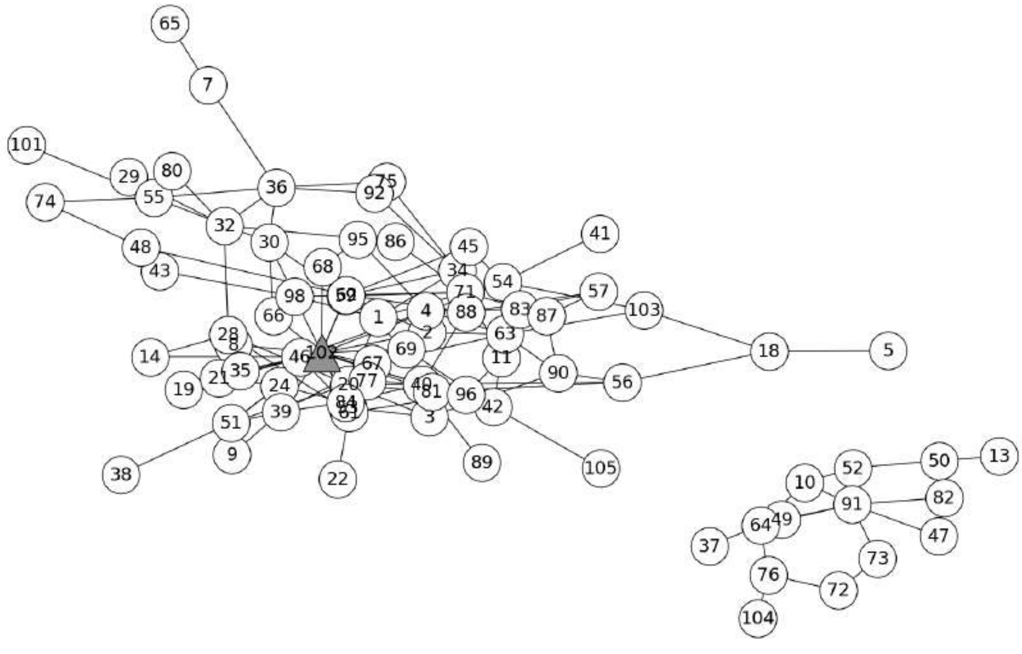


Figure 3.22 — Post-infestation ant colony interaction graph

tions are terminal stations. The highest centrality values are at transfer stations, the highest at Sadovaya, Spasskaya, Sennaya Ploshchad stations. According to St. Petersburg Subway statistics (see appendix 3.4) these stations have the highest daily

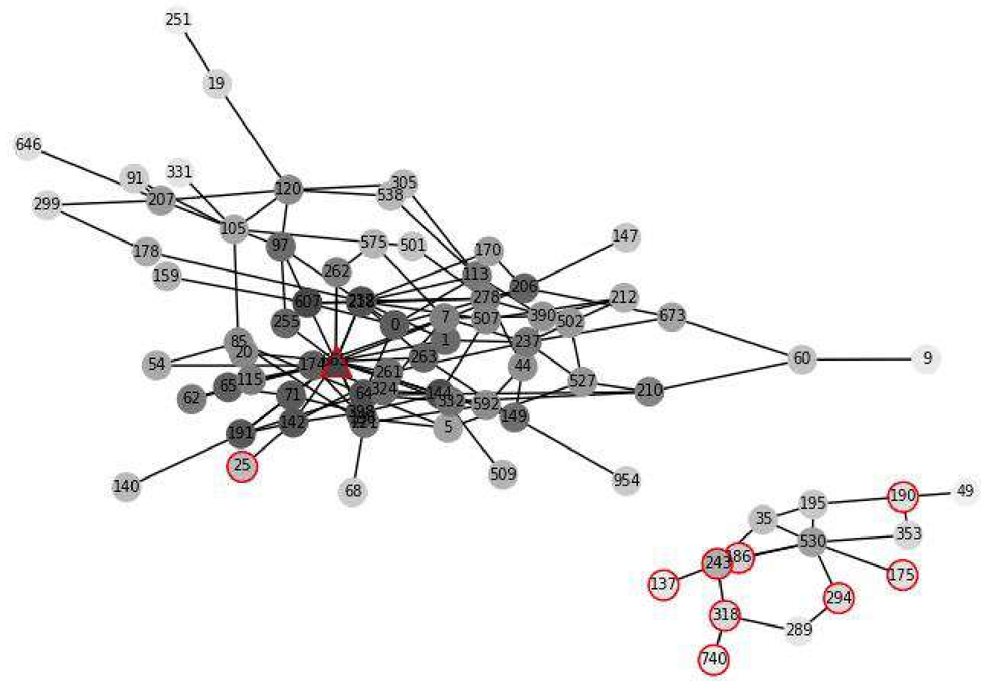


Figure 3.23 — Heat map after infestation. Modified Myerson value

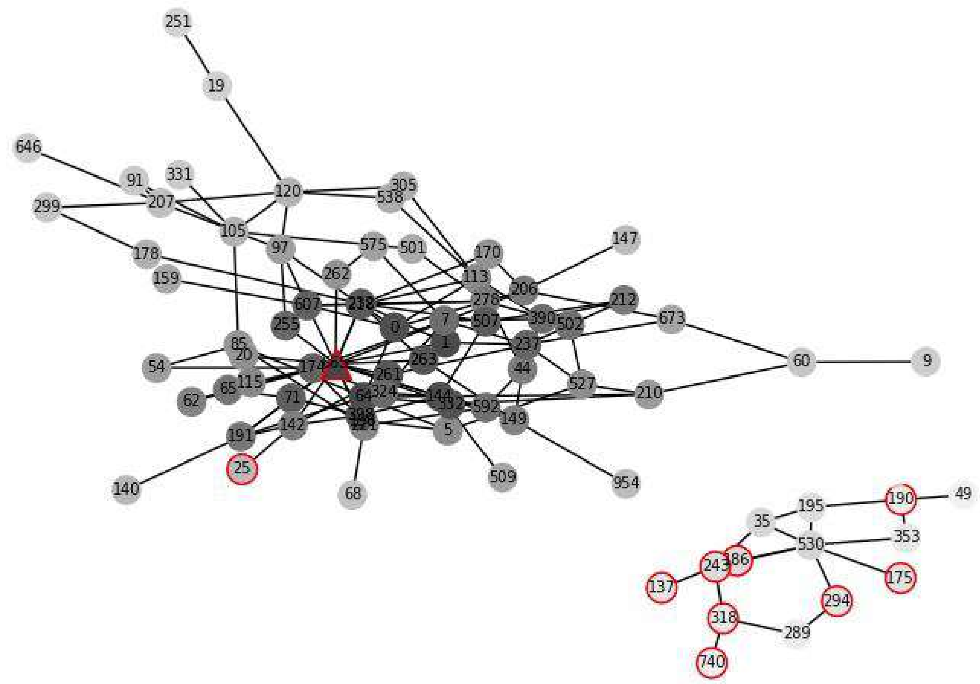


Figure 3.24 — Heat map after infestation. Absolute potentials

and monthly passenger traffic. Although, according to the same statistics, the passenger flow is also quite high at the end stations of dormitory districts.

Figure 3.25 — Scheme of St. Petersburg Subway

Next, we rank the vertices of the metro graph based on the values of absolute potentials of the corresponding nodes of the electric circuit. Since the information on the length of distances between stations is not freely available, the inverse of the time required to travel between pairs of neighbouring stations was chosen as the weights of the edges. Fig. 3.27 shows a heat map of the graph vertex ranks. A darker colouring corresponds to a higher rank of a vertex; the numbers indicated on the vertices correspond to the ranks. The highest ranks were obtained for stations Vladimirskaya, Dostoevskaya, Spasskaya, Sadovaya, Sennaya, Ligovsky Prospekt. The worst ranks are observed on the red line: Akademicheskaya, Grazhdansky Prospekt, Devyatkino stations.

In the case of metro graph analysis, for example, in order to evaluate more favourable places for advertising placement, information about passenger flow by stations per day can be used as the weights of vertices (appendix 3.4). The ranking

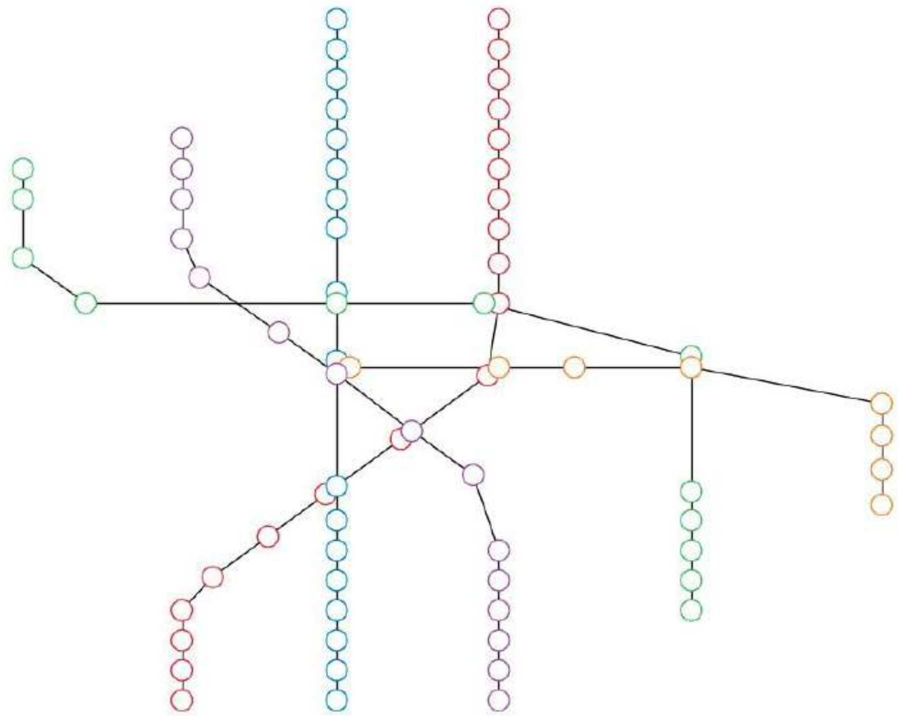


Figure 3.26 — Graph of St. Petersburg Subway

result considering passenger flow is shown in Fig. 3.28. Here, the vertex with the highest rank is Ploshchad Vosstaniya (in the ranking considering only the edge weights, this station had a rank of 9). It is worth noting that some peripheral stations, which had the worst ranks, became significant in this approach, because the flow at these terminal stations of branches is quite high.

3.4 The social network of the Star Wars saga

Dr Evelina Gabasova – a member of the Alan Turing Institute – conducted a study based on the scripts[3] of the Star Wars film series [78]. As a result, the social network graph of the whole saga was constructed (Fig. 3.29), as well as the social network graphs of each episode (Fig. 3.30). The nodes of the graphs correspond to the characters in the films; the presence of a connection between a pair of vertices

[3]`https://imsdb.com/`

Table 17 — Centrality values of some vertices, modified Myerson value

Station	$s_i(5)$	$\sigma_i(5)$	$s_i(10)$	$\sigma_i(10)$
Devyatkino	44	7.33	1 588	144.36
Prospect Veteranov	44	7.33	1 784	162.18
Parnas	44	7.33	1 598	145.27
Kupchino	44	7.33	1 648	149.82
Rybatskoe	44	7.33	2 154	195.82
Komendantsky Prospekt	44	7.33	1 992	181.09
Dybenko Street	46	7.67	2 927	266.09
Mezdunarodnaya	46	7.67	4 281	389.18
Begovaya	46	7.67	3 563	323.91
Spasskaya	1 988	331.33	964 437	87 676.09
Sadovaya	2 578	429.67	1 224 429	111 311.73
Sennaya Ploshchad	2 810	468.33	1 342 204	122 018.55

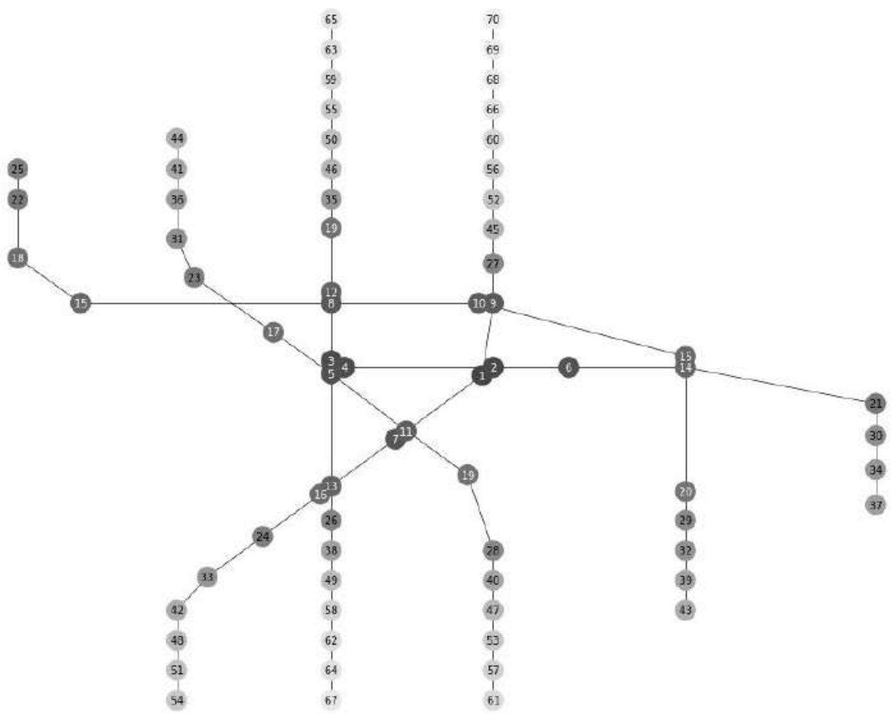


Figure 3.27 — Heat map of vertex ranks of the St. Petersburg subway graph (weighted edges)

means that the corresponding characters both speak in the same scene. The size of each vertex corresponds to the total number of scenes in which the character

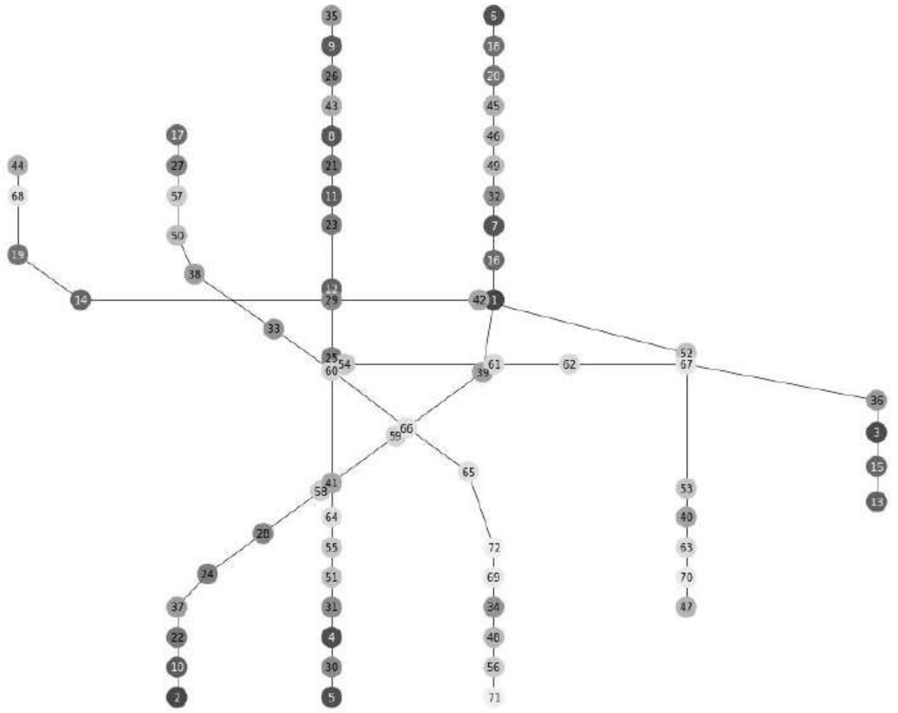


Figure 3.28 — Heat map of vertex ranks of the St. Petersburg subway graph (weighted vertices and edges)

appears. It can be seen that the saga graph is visually divided into two parts (corresponding to the classic trilogy and the prequel trilogy), which are connected by the nodes Obi-Wan Kenobi, R2-D2 and C-3PO. These characters appear in all episodes. There are a total of 144 characters in the film series.

Let us analyse the overall graph of the saga and the graphs of the individual episodes using the methods proposed in the thesis. These graphs are undirected. The weight of an edge is equal to the number of scenes in which the characters corresponding to the vertices incident to the edge participate together. The weight of a vertex is equal to the number of scenes in which the corresponding character participates.

The 18 table lists the most central characters in the saga, determined using the Myerson vector, absolute circuit potentials, total charge transfer work, and integral centrality. These ranking results can be compared with the results obtained earlier in [78] using degree centrality and betweenness centrality.

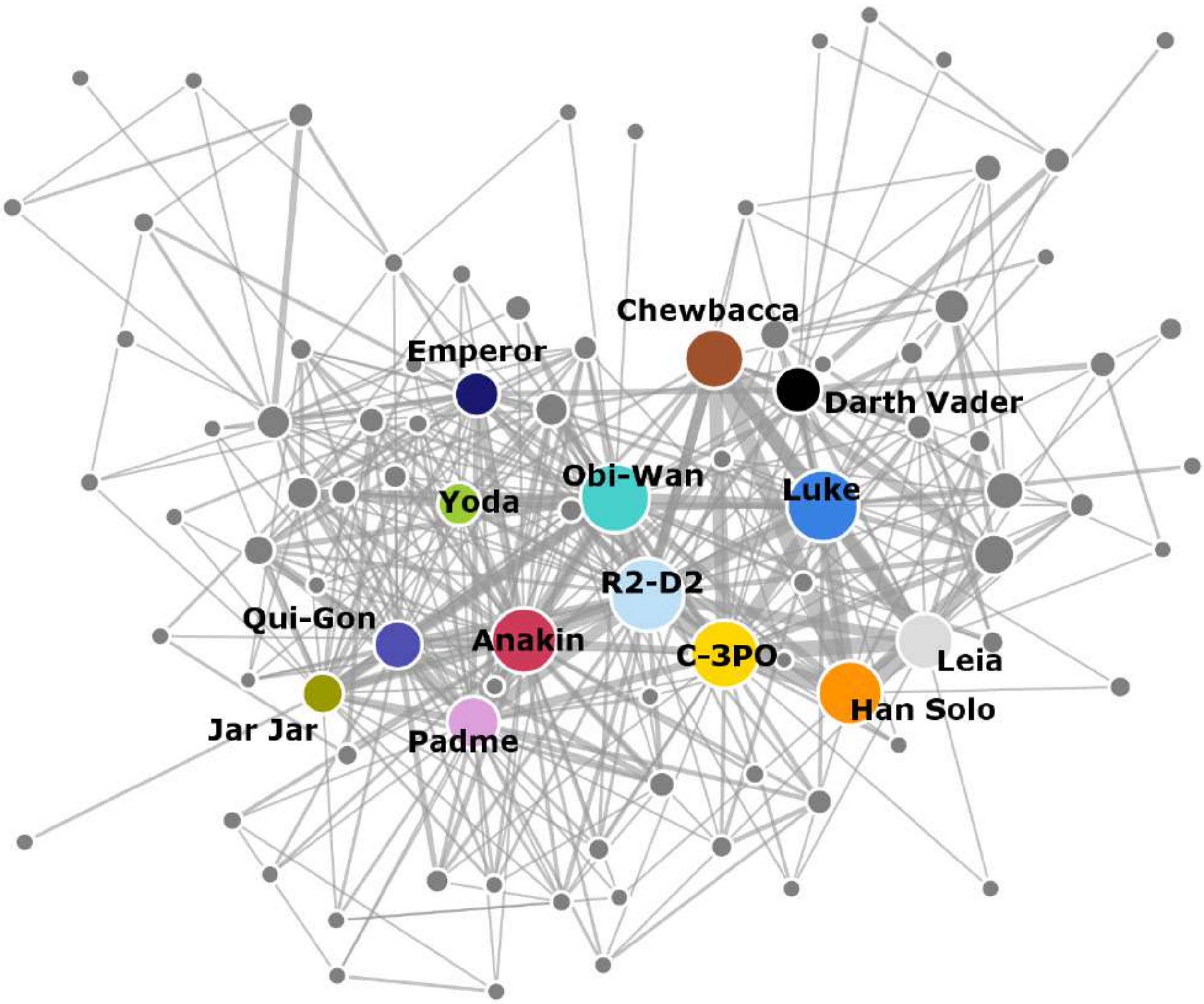


Figure 3.29 — Star Wars social network graph

Tables 19 – 24 list character ranks for each episode of the saga separately, obtained using the Myerson vector, absolute potentials, total charge transfer work, and integral centrality.

Star Wars creator George Lucas said in an interview that the saga is the story of Anakin Skywalker, but analysing the centrality of the vertices in the overall graph, we can also conclude that Obi-Wan Kenobi, Han Solo and Chewbacca are the links of the story, despite the fact that the second and third characters take part only in the classic trilogy.

In the first three episodes of the saga, the vertex corresponding to Anakin has the highest centrality, but calculating centrality using the total charge transfer work indicates the centrality of the R2-D2 vertex. This approach to calculating centrality takes into account the weight of the vertex, in this case the number of episodes in

Table 18 — Star Wars character ranks

Rank	Myerson vector	Absolute potentials	Total work
1	ANAKIN	OBI-WAN	CHEWBACCA
2	C-3PO	R2-D2	HAN
3	OBI-WAN	ANAKIN	LEIA
4	PADME	PADME	REY
5	R2-D2	QUI-GON	LUKE
Rank	Integral centrality	Degree centrality	Betweenness centrality
1	HAN	ANAKIN	OBI-WAN
2	CHEWBACCA	R2-D2	PADME
3	C-3PO	OBI-WAN	R2-D2
4	LEIA	PADME	C-3PO
5	LUKE	C-3PO	LUKE

Table 19 — Episode I character ranks

Rank	Myerson vector	Absolute potentials
1	QUI-GON	QUI-GON
2	ANAKIN	PADME
3	PADME	OBI-WAN
4	JAR JAR	ANAKIN
5	OBI-WAN	R2-D2
Rank	Total work	Integral centrality
1	R2-D2	QUI-GON
2	PADME	ANAKIN
3	OBI-WAN	JAR JAR
4	QUI-GON	PADME
5	ANAKIN	OBI-WAN

which the character has participated, but R2-D2 is not an actor in his own right, most often he accompanies Anakin or Obi-Wan Kenobi.

Table 20 — Episode II character ranks

Rank	Myerson vector	Absolute potentials
1	ANAKIN	ANAKIN
2	OBI-WAN	PADME
3	PADME	OBI-WAN
4	MACE WINDU	R2-D2
5	COUNT DOOKU	MACE WINDU
Rank	**Total work**	**Integral centrality**
1	R2-D2	ANAKIN
2	PADME	PADME
3	OBI-WAN	OBI-WAN
4	LAMA SU	R2-D2
5	ANAKIN	C-3PO

Table 21 — Episode III character ranks

Rank	Myerson vector	Absolute potentials
1	ANAKIN	ANAKIN
2	OBI-WAN	R2-D2
3	BAIL ORGANA	OBI-WAN
4	EMPEROR	C-3PO
5	PADME	BAIL ORGANA
Rank	**Total work**	**Integral centrality**
1	R2-D2	ANAKIN
2	OBI-WAN	OBI-WAN
3	ANAKIN	R2-D2
4	C-3PO	EMPEROR
5	YODA	PADME

In the classic trilogy (episodes IV, V and VI) in the ranking that takes into account the frequency of appearances on the screen, the most important vertices are

Table 22 — Episode IV character ranks

Rank	Myerson vector	Absolute potentials
1	LUKE	LUKE
2	LEIA	CHEWBACCA
3	C-3PO	R2-D2
4	R2-D2	C-3PO
5	BIGGS	OBI-WAN
Rank	Total work	Integral centrality
1	R2-D2	LUKE
2	C-3PO	HAN
3	CHEWBACCA	C-3PO
4	LUKE	CHEWBACCA
5	OBI-WAN	LEIA

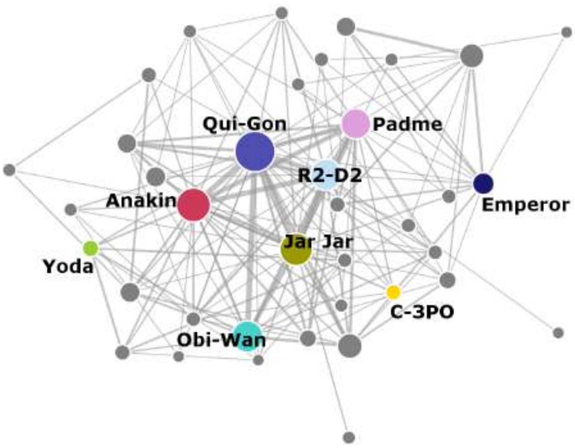
Table 23 — Episode V character ranks

Rank	Myerson vector	Absolute potentials
1	HAN	CHEWBACCA
2	LEIA	HAN
3	LUKE	LEIA
4	CHEWBACCA	C-3PO
5	DARTH VADER	LANDO
Rank	Total work	Integral centrality
1	C-3PO	HAN
2	CHEWBACCA	LEIA
3	LEIA	C-3PO
4	HAN	CHEWBACCA
5	R2-D2	LANDO

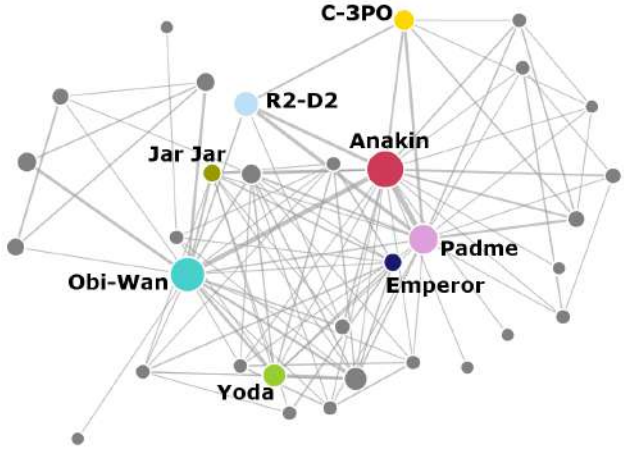
those corresponding to Chewbacca, R2-D2 and C-3PO, the other methods indicate the importance of vertices corresponding to Han Solo, Luke and Leia Skywalker. But the Darth Vader vertex is in the top 5 only in episode V for the Myerson vector.

Table 24 — Episode VI character ranks

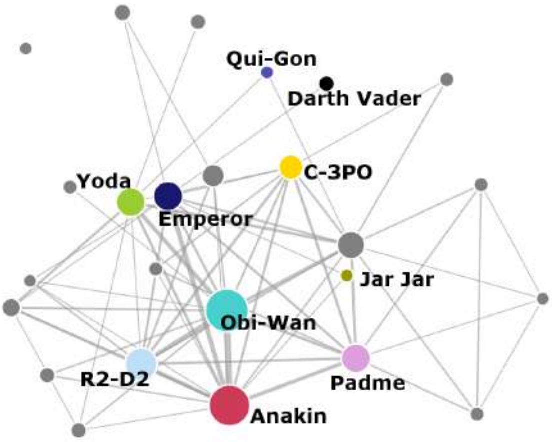
Rank	Myerson vector	Absolute potentials
1	LUKE	HAN
2	C-3PO	CHEWBACCA
3	HAN	LEIA
4	LANDO	R2-D2
5	MON MOTHMA	LUKE
Rank	Total work	Integral centrality
1	CHEWBACCA	HAN
2	HAN	C-3PO
3	R2-D2	LEIA
4	LEIA	CHEWBACCA
5	C-3PO	LUKE



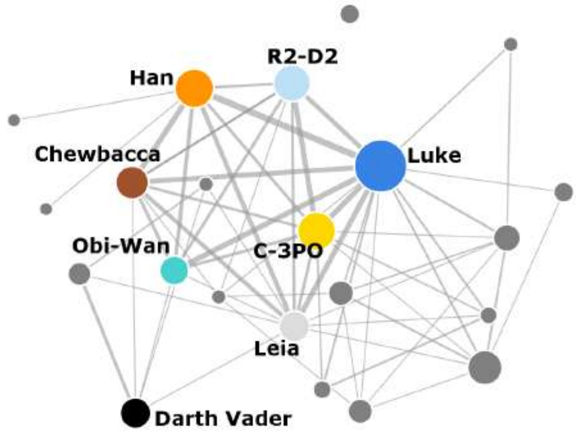
a) Episode I: The Phantom Menace;



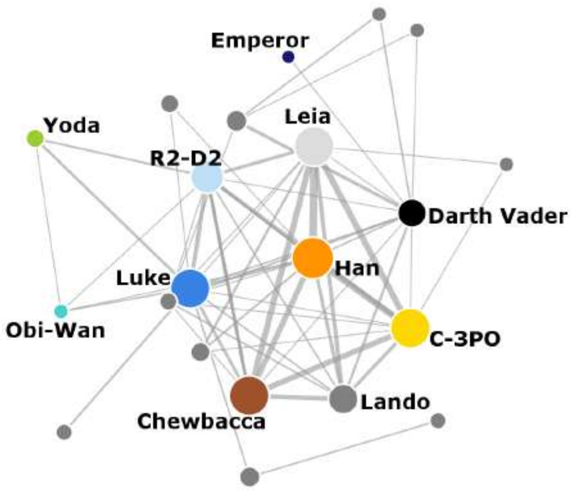
b) Episode II: Attack of the Clones;



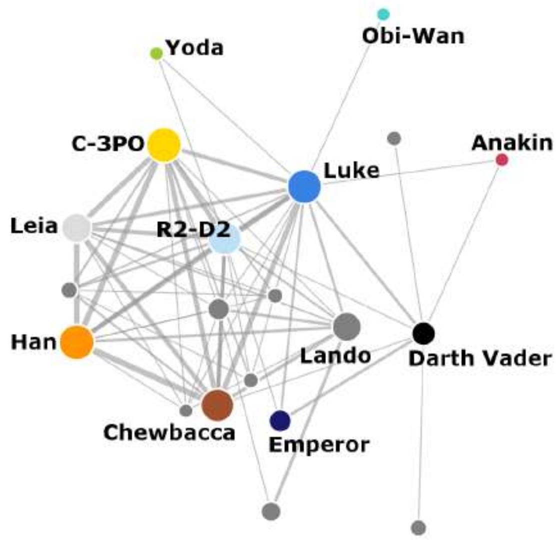
c) Episode III: Revenge of the Sith;



d) Episode IV: A New Hope;



e) Episode V: The Empire Strikes Back;



f) Episode VI: Return of the Jedi;

Figure 3.30 — Social network graphs of the saga's episodes.

Conclusion

The main results of the thesis work are as follows:

1. In the course of this work we proposed a method for computing the centrality of vertices of a directed graph as the Myerson value in a cooperative game, where the characteristic function is defined in terms of the number of simple paths of fixed length in the subgraph generated by the coalition. To calculate centrality, formulas were proposed to find the number of appearances of a vertex in simple paths of lengths 2 and 3 through the adjacency matrix.
2. We introduced the notion of integral centrality through the value of the definite integral of the function defining the payoff in a cooperative game, whose characteristic function is given by the number of total appearances of a graph vertex in simple paths. It is shown that the Boldi – Vigna axioms are valid for such a measure of centrality.
3. For vertices of undirected graphs, we modified the Myerson value in the cooperative game, where the characteristic function corresponds to the number of simple paths of fixed length, allowing cycles to be included in the consideration.
4. A method for estimating the centrality of vertices of a graph with weighted edges was developed, based on the values of absolute potentials of nodes of an electric circuit calculated on the basis of Kirchhoff's rules using the Laplace matrix. After calculating the absolute potentials, a two-stage ranking procedure using the voting theory approach is proposed. Rank distributions are found for a number of special cases (for clique, star graph, double star and complete bipartite graph).
5. A method for finding the centrality values of the vertices of a graph with weighted vertices and edges was proposed, based on the values of the total work required to move charges between the nodes of an electric circuit. The

choice of different characteristics as vertex weights allows us to evaluate the importance of vertices from different points of view. For a number of special cases (for clique, star graph and complete bipartite graph) analytical expressions for finding centrality values are found.

6. The graph model of the transport system of the city of Petrozavodsk is constructed in this paper. The intersections of motorways are considered as the vertices of the graph. The distribution of city residents and organisations registered on the territory of Petrozavodsk urban district on the graph vertices is performed.
7. The proposed methods of graph vertex ranking have been applied to the graph of the transport network of Petrozavodsk, to the graph of districts of Petrozavodsk, to the graphs of interactions between individuals of an ant colony, to the graph of the St. Petersburg subway, as well as to the social graph of the Star Wars saga.

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Bibliography

1. *Gubanov D.A.* Social'nye seti: modeli informacionnogo vliyaniya, upravleniya i protivoborstva [Social media: models of information influence, control and confrontation] [Text] / Gubanov D.A., Novikov D.A., Chkhartishvili A.G. ; ed. by RAS correspondent member Novikov D.A. — Moscow : Izdatel'stvo fiziko-matematicheskoj literatury, 2010. — 228 p.
2. *Moreno, J. L.* Sociometry, experimental method and the science of society: An approach to a new political orientation [Text] / J. L. Moreno. — Beacon, N.Y. : Beacon House, 1951.
3. *Gradosel'skaya G.V.* Setevye izmereniya v sociologii [Network dimensions in sociology] [Text] / Gradosel'skaya G.V. ; ed. by Batygin G.S. — Moscow : Izdatel'skij dom "Novyj uchebnik", 2004. — 248 p.
4. *Toropov B. A.* Modeli terroristicheskih setej i teoretiko-igrovoj podhod k ocenke central'nosti ih uchastnikov [Models of terrorist networks and a game-theoretic approach to assessing the centrality of their members] [Text] / Toropov B. A., Tagirov Z. I. // Voprosy bezopasnosti. — 2016. — Vol. 6, no. 6. — P. 77—89. — URL: https://doi.org/10.7256/2409-7543.2016.6.21436.
5. A systematic survey of centrality measures for protein-protein interaction networks [Text] / M. Ashtiani [et al.] // BMC Systems Biology. — 2018. — No. 1. — URL: https://doi.org/10.1186/s12918-018-0598-2.
6. Analiz vzaimodejstviya genov nejronal'nogo apoptoza i genov bol'shogo depressivnogo rasstrojstva v ego associativnoj gennoj seti [Analysing the interaction between neuronal apoptosis genes and major depressive disorder genes in its association gene network] [Text] / Yankina M. A. [et al.] // Dnevnik nauki. — 2017. — No. 9. — P. 1—14.

7. Analiz vzaimodejstviya genov nejronal'nogo apoptoza v associativnoj gennoj seti bolezni Parkinsona [Analysis of neuronal apoptosis gene interactions in the associative gene network of Parkinson's disease] [Text] / Yankina M. A. [et al.] // Vavilovskij zhurnal genetiki i selekcii. — 2018. — 22(1). — P. 153—160.

8. *Pechenkin*, *V.* Applied Aspects of Ranking Algorithms for Oriented Weighted Graphs (on The Example of Social Network Graphs) [Text] / V. Pechenkin, M. Korolev, L. Dimitrov // SPIIRAS Proceedings. — 2018. — Dec. — Vol. 6, no. 61. — P. 94—118. — URL: https://doi.org/10.15622/sp.61.4.

9. *Korzun*, *D.* Structured Peer-to-Peer Systems [Text] / D. Korzun, A. Gurtov. — Springer New York, 2013. — URL: https://doi.org/10.1007/978-1-4614-5483-0.

10. Peer-to-Peer Systems and Applications [Text] / ed. by R. Steinmetz, K. Wehrle. — Springer Berlin Heidelberg, 2005. — URL: https://doi.org/10.1007/11530657.

11. Energy-aware load monitoring for improving battery life of mobile peer-to-peer nodes [Text] / E. Harjula [et al.] // Sustainable Computing: Informatics and Systems. — 2016. — Dec. — Vol. 12. — P. 43—54. — URL: https://doi.org/10.1016/j.suscom.2016.09.002.

12. *Gurtov*, *A.* Cyclic ranking in single-resource peer-to-peer exchange [Text] / A. Gurtov, J. Koskela, D. Korzun // Peer-to-Peer Networking and Applications. — 2017. — July. — Vol. 11, no. 3. — P. 632—643. — URL: https://doi.org/10.1007/s12083-017-0578-0.

13. *Brass*, *D.* Centrality and power in organizations. [Text] / D. Brass, M. Burkhardt // Networks and organizations. — Boston, 1992. — Vol. 46, no. 1. — P. 191—215.

14. *Bavelas*, *A.* Communication Patterns in Task-Oriented Groups [Text] / A. Bavelas // The Journal of the Acoustical Society of America. — 1950. — Nov. — Vol. 22, no. 6. — P. 725—730. — URL: https://doi.org/10.1121/1.1906679.

15. *Leavitt, H. J.* Some effects of certain communication patterns on group performance. [Text] / H. J. Leavitt // The Journal of Abnormal and Social Psychology. — 1951. — Jan. — Vol. 46, no. 1. — P. 38—50. — URL: https://doi.org/10.1037/h0057189.

16. *Mizruchi, M. S.* Centrality and power revisited: actor success in group decision making [Text] / M. S. Mizruchi, B. B. Potts // Social Networks. — 1998. — Oct. — Vol. 20, no. 4. — P. 353—387. — URL: https://doi.org/10.1016/s0378-8733(98)00009-4.

17. *Freeman, L. C.* A Set of Measures of Centrality Based on Betweenness [Text] / L. C. Freeman // Sociometry. — 1977. — Vol. 40, no. 1. — P. 35—41. — URL: https://doi.org/10.2307/3033543.

18. *Newman, M. J.* A measure of betweenness centrality based on random walks [Text] / M. J. Newman // Social Networks. — 2005. — Vol. 27, no. 1. — P. 39—54. — URL: https://doi.org/10.1016/j.socnet.2004.11.009.

19. The PageRank Citation Ranking : Bringing Order to the Web [Text] / L. Page [et al.] // Proceedings of the 7_{th} International World Wide Web Conference. — Brisbane, Australia, 1998. — P. 161—172.

20. Alpha Current Flow Betweenness Centrality [Text] / Avrachenkov K.E. [et al.] // Algorithms and Models for the Web Graph. WAW 2013. Lecture Notes in Computer Science. — 2013. — No. 8305. — P. 106—117. — URL: https://doi.org/10.1007/978-3-319-03536-9_9.

21. *Avrachenkov K.E.* Beta Current Flow Centrality for Weighted Networks [Text] / Avrachenkov K.E., Mazalov V.V., Tsynguev B.T. // Computational Social Networks. CSoNet 2015. Lecture Notes in Computer Science. — 2015. — No. 9197. — P. 216—227. — URL: https://doi.org/10.1007/978-3-319-21786-4_19.

22. *Brandes*, *U.* Centrality Measures Based on Current Flow [Text] / U. Brandes, D. Fleischer // STACS 2005. Lecture Notes in Computer Science. — 2005. — Vol. 3404. — P. 533—544. — URL: https://doi.org/10.1007/978-3-540-31856-9_44.

23. Centrality and power in social networks: a game theoretic approach [Text] / D. Gómez [et al.] // Mathematical Social Sciences. — 2003. — Vol. 46, no. 1. — P. 27—54. — URL: https://doi.org/10.1016/s0165-4896(03)00028-3.

24. *Mazalov*, *V. V.* Networking games: network forming games and games on networks [Text] / V. V. Mazalov, J. V. Chirkova. — Academic Press, 2019.

25. Efficient Computation of the Shapley Value for Game-Theoretic Network Centrality [Text] / T. P. Michalak [et al.] // Journal of Artificial Intelligence Research. — 2013. — Vol. 46, no. 1. — P. 607—650. — URL: https://arxiv.org/abs/1402.0567.

26. *Li*, *J.* Importance of Agents in Networks: Clique Based Game-Theoretic Approach [Text] / J. Li, A. Tur, M. Zavrajnov // Contributions to Game Theory and Management. — 2022. — Vol. 15. — P. 189—199. — URL: https://doi.org/10.21638/11701/spbu31.2022.14.

27. *Mazalov*, *V. V.* Generating functions and the Myerson vector in communication networks [Text] / V. V. Mazalov, L. I. Trukhina // Discrete Mathematics and Applications. — 2014. — Vol. 24, no. 5. — P. 295—303. — URL: https://doi.org/10.1515/dma-2014-0026.

28. *Suna*, *P.* Two-stage network games modeling the Belt and Road Initiative [Text] / P. Suna, E. M. Parilina, H. Gaob // Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes. — 2022. — Vol. 18, no. 1. — P. 87—98. — URL: https://doi.org/10.21638/11701/spbu10.2022.107.

29. *Tur, A.* Communication Restriction-Based Characteristic Function in Differential Games on Networks [Text] / A. Tur, L. Petrosyan // Mathematical Optimization Theory and Operations Research. — Springer Nature Switzerland, 2023. — P. 314—324. — URL: https://doi.org/10.1007/978-3-031-35305-5_22.

30. Network partitioning algorithms as cooperative games [Text] / Avrachenkov K.E. [et al.] // Computational Social Networks. — 2018. — Vol. 5, no. 1. — P. 1—28. — URL: https://doi.org/10.1186/s40649-018-0059-5.

31. *Parilina, E.* Stable cooperation in graph-restricted games [Text] / E. Parilina, A. Sedakov // Contributions to Game Theory and Management. — 2014. — Vol. 7. — P. 271—281. — URL: http://mi.mathnet.ru/cgtm238.

32. *Parilina, E.* Stable Bank Cooperation for Cost Reduction Problem [Text] / E. Parilina, A. Sedakov // Czech Economic Review. — 2014. — Vol. 8, no. 1. — P. 7—25. — URL: https://ideas.repec.org/a/fau/aucocz/au2014_007.html.

33. *Parilina, E.* Stable Cooperation in a Game with a Major Player [Text] / E. Parilina, A. Sedakov // International Game Theory Review. — 2016. — June. — Vol. 18, no. 02. — P. 1640005. — URL: https://doi.org/10.1142/s0219198916400053.

34. *Myerson, R. B.* Graphs and Cooperation in Games [Text] / R. B. Myerson // Mathematics of Operations Research. — 1977. — Vol. 2, no. 3. — P. 225—229. — URL: https://doi.org/10.1287/moor.2.3.225.

35. *Shapley, L. S.* 17. A Value for n-Person Games [Text] / L. S. Shapley // Contributions to the Theory of Games (AM-28), Volume II. — Princeton University Press, 12/1953. — P. 307—318. — URL: https://doi.org/10.1515/9781400881970-018.

36. *Aumann, R. J.* Endogenous formation of links between players and of coalitions: An application of the Shapley value [Text] / R. J. Aumann, R. B. Myerson // The Shapley Value: Essays in Honor of Lloyd S. Shapley. — 1988. —

No. 12. — P. 175—192. — URL: https://doi.org/10.1017/cbo9780511528446.013.

37. *Talman*, *D.* Average tree solution and subcore for acyclic graph games [Text] / D. Talman, Y. Yamamoto // Journal of the Operations Research Society of Japan. — 2008. — Vol. 51, no. 3. — P. 203—212. — URL: https://doi.org/10.15807/jorsj.51.203.

38. *Borm*, *P.* On the Position Value for Communication Situations [Text] / P. Borm, G. Owen, S. Tijs // SIAM Journal on Discrete Mathematics. — 1992. — Aug. — Vol. 5, no. 3. — P. 305—320. — URL: https://doi.org/10.1137/0405023.

39. *Borm*, *P.* Cooperation and Communication Restrictions: A Survey [Text] / P. Borm, A. van den Nouweland, S. Tijs // Theory and Decision Library. — Springer Netherlands, 1994. — P. 195—227. — URL: https://doi.org/10.1007/978-94-011-1370-0_9.

40. *Slikker*, *M.* Link monotonic allocation schemes [Text] / M. Slikker // International Game Theory Review. — 2005. — Dec. — Vol. 07, no. 04. — P. 473—489. — URL: https://doi.org/10.1142/s021919890500065x.

41. Directed networks, allocation properties and hierarchy formation [Text] / M. Slikker [et al.] // Mathematical Social Sciences. — 2005. — Jan. — Vol. 49, no. 1. — P. 55—80. — URL: https://doi.org/10.1016/j.mathsocsci.2004.04.005.

42. *Calvo*, *E.* Values of games with probabilistic graphs [Text] / E. Calvo, J. Lasaga, A. van den Nouweland // Mathematical Social Sciences. — 1999. — Jan. — Vol. 37, no. 1. — P. 79—95. — URL: https://doi.org/10.1016/s0165-4896(98)00013-4.

43. *Jackson*, *M. O.* Allocation rules for network games [Text] / M. O. Jackson // Games and Economic Behavior. — 2005. — Apr. — Vol. 51, no. 1. — P. 128—154. — URL: https://doi.org/10.1016/j.geb.2004.04.009.

44. *Boldi*, *P.* Axioms for Centrality [Text] / P. Boldi, S. Vigna. — 2013. — URL: https://arxiv.org/abs/1308.2140.

45. *Jackson M.O.* Social and economic networks [Text] / Jackson M.O. — Princeton University Press, 2010. — 520 p.

46. *Shaw*, *M. E.* Communication Networks [Text] / M. E. Shaw // Advances in Experimental Social Psychology. — Elsevier, 1964. — P. 111—147. — URL: https://doi.org/10.1016/s0065-2601(08)60050-7.

47. *Nieminen*, *U.* On the centrality in a directed graph [Text] / U. Nieminen // Social Science Research. — 1973. — Dec. — Vol. 2, no. 4. — P. 371—378. — URL: https://doi.org/10.1016/0049-089x(73)90010-0.

48. *Bavelas*, *A.* A Mathematical Model for Group Structures [Text] / A. Bavelas // Human Organization. — 1948. — Vol. 7. — P. 16—30. — URL: https://api.semanticscholar.org/CorpusID:143816855.

49. *Freeman*, *L. C.* A Set of Measures of Centrality Based on Betweenness [Text] / L. C. Freeman // Sociometry. — 1977. — Mar. — Vol. 40, no. 1. — P. 35—41. — URL: https://doi.org/10.2307/3033543.

50. *Beauchamp*, *M. A.* An improved index of centrality [Text] / M. A. Beauchamp // Behavioral Science. — 1965. — Vol. 10, no. 2. — P. 161—163. — URL: https://doi.org/10.1002/bs.3830100205.

51. *Sabidussi*, *G.* The centrality index of a graph [Text] / G. Sabidussi // Psychometrika. — 1966. — Dec. — Vol. 31, no. 4. — P. 581—603. — URL: https://doi.org/10.1007/bf02289527.

52. *Rochat*, *Y.* Closeness Centrality Extended to Unconnected Graphs: the Harmonic Centrality Index [Text] / Y. Rochat //. — 2009. — URL: https://api.semanticscholar.org/CorpusID:10755931.

53. *Bianchini*, *M.* Inside PageRank [Text] / M. Bianchini, M. Gori, F. Scarselli // ACM Transactions on Internet Technology. — 2005. — Feb. — Vol. 5, no. 1. — P. 92—128. — URL: https://doi.org/10.1145/1052934.1052938.

54. *Mazalov, V. V.* A modified Myerson value for determining the centrality of graph vertices [Text] / V. V. Mazalov, V. A. Khitraya // Automation and Remote Control. — 2021. — Vol. 82, no. 1. — P. 145—159. — URL: https://doi.org/10.1134/s0005117921010100.

55. *Mazalov V.V.* Matematicheskaya teoriya igr i prilozheniya [Mathematical game theory and applications] [Text] : uchebnoe posobie dlya vuzov [textbook for universities] / Mazalov V.V. — St.Petersburg : Lan', 2023. — 500 p.

56. *Shcherbakova N.G.* Aksiomatika central'nosti v kompleksnyh setyah [Axiomatics of centrality in complex networks] [Text] / Shcherbakova N.G. // Problemy informatiki. — 2013. — Vol. 3, no. 28. — P. 3—14.

57. *Wolfram MathWorld.* Euler-Mascheroni Constant [Text] / Wolfram MathWorld. — 2023. — URL: https://mathworld.wolfram.com/Euler-MascheroniConstant.html (visited on 08/15/2023).

58. *Mazalov, V. V.* Ranzhirovanie vershin grafa s ispol'zovaniem absolyutnyh potencialov uzlov elektricheskoj cepi [Graph vertex ranking using absolute potentials of the nodes of an electric circuit] [Text] / V. V. Mazalov, V. A. Khitraya // Vestnik Sankt-Peterburgskogo universiteta. Prikladnaya matematika. Informatika. Processy upravleniya. — 2023. — Vol. 19, no. 2. — P. 233—251. — URL: https://doi.org/10.21638/11701/spbu10.2023.209.

59. *Aleskerov F. T.* Binarnye otnosheniya, grafy i kollektivnye resheniya. Primery i zadachi: ucheb. posobie dlya vuzov [Binary relations, graphs and collective solutions. Examples and problems: textbook for universities] [Text] / Aleskerov F. T., Habina E. L., Shvartz D. A. — Moscow : YUrajt, 2023. — 458 p.

60. *Khitraya, V. A.* Teoretiko-igrovaya central'nost' vershin orientirovannogo grafa [Game-theoretic centrality of vertices of an directed graph] [Text] / V. A. Khitraya, V. V. Mazalov // Matematicheskaya teoriya igr i ee prilozheniya. — 2023. — Vol. 15, no. 3. — P. 64—87.

61. *Mazalov, V. V.* Metody teorii kooperativnyh igr v zadache ranzhirovaniya tekstov [Cooperative game theory methods in the text ranking] [Text] / V. V. Mazalov, V. A. Khitraya, A. V. Khitryj // Vestnik Sankt-Peterburgskogo universiteta. Prikladnaya matematika. Informatika. Processy upravleniya. — 2022. — Vol. 18, no. 1. — P. 63—78. — URL: https://doi.org/10.21638/11701/spbu10.2022.105.

62. *Kondratev, A. Y.* A Ranking Procedure with the Shapley Value [Text] / A. Y. Kondratev, V. V. Mazalov // Intelligent Information and Database Systems. ACIIDS 2017. Lecture Notes in Computer Science. — 2017. — Vol. 10192. — P. 691—700. — URL: https://doi.org/10.1007/978-3-319-54430-4_66.

63. *Khitraya, V. A.* Ranzhirovanie vershin grafa kak kooperativnaya igra na elektricheskoj cepi [Graph vertex ranking as a cooperative game on an electrical circuit] [Text] / V. A. Khitraya // Processy upravleniya i ustojchivost'. — 2023. — Vol. 10. — P. 482—489.

64. *Sheffi, Y.* Urban Transportation Networks: Equilibrium Analysis With Mathematical Programming Methods [Text] / Y. Sheffi. — Englewood Cliffs, N.J. : Prentice-Hall, 1985. — 399 p.

65. *Nagurney, A.* Variational Inequality Theory [Text] / A. Nagurney // Network Economics. Advances in Computational Economics. — Boston, MA : Springer US, 1999. — P. 3—48. — URL: https://doi.org/10.1007/978-1-4757-3005-0_1.

66. *Kerner, B. S.* Introduction to Modern Traffic Flow Theory and Control [Text] : The Long Road to Three-Phase Traffic Theory / B. S. Kerner. — Springer Berlin, Heidelberg, 2009. — URL: https://doi.org/10.1007/978-3-642-02605-8.

67. Vvedenie v matematicheskoe modelirovanie transportnyh potokov [Introduction to mathematical modelling of traffic flows] [Text] / Gasnikov A. V. [et al.]. — Moscow : MCNMO, 2013. — 196 p.

68. *Nesterov, Y.* Stationary Dynamic Solutions in Congested Transportation Networks: Summary and Perspectives [Text] / Y. Nesterov, A. de Palma // Networks and Spatial Economics. — 2003. — Vol. 3, no. 3. — P. 371—395. — URL: https://doi.org/10.1023/a:1025350419398.

69. *Nurminskij E. A.* Prognoznoe modelirovanie avtomobil'nogo trafika Vladivostoka [Predictive modelling of Vladivostok car traffic] [Text] / Nurminskij E. A., SHamraj N. B. // Trudy MFTI. — 2010. — Vol. 2, 4(8). — P. 119—129.

70. *Wardrop, J. G.* Some Theoretical Aspects of Road Traffic Research [Text] / J. G. Wardrop, J. I. Whitehead // Proceedings of the Institution of Civil Engineers. — 1952. — Vol. 1, no. 5. — P. 767—768. — URL: https://doi.org/10.1680/ipeds.1952.11362.

71. Riding to Wall Street: determinants of commute time using Citi Bike [Text] / W. Ford [et al.] // International Journal of Logistics Research and Applications. — 2019. — Vol. 22, no. 5. — P. 473—490. — URL: https://doi.org/10.1080/13675567.2019.1584164.

72. *Shvetsov, V. I.* Mathematical Modeling of Traffic Flows [Text] / V. I. Shvetsov // Automation and Remote Control. — 2003. — Vol. 64, no. 11. — P. 1651—1689. — URL: https://doi.org/10.1023/a:1027348026919.

73. *Shvetsov, V. I.* Algorithms for distributing traffic flows [Text] / V. I. Shvetsov // Automation and Remote Control. — 2009. — Vol. 70, no. 10. — P. 1728—1736. — URL: https://doi.org/10.1134/s0005117909100105.

74. *OpenStreetMap contributors.* Planet dump retrieved from https://planet.osm.org [Text] / OpenStreetMap contributors. — 2023. — URL: https://www.openstreetmap.org (visited on 09/15/2023).

75. *Khitraya, V. A.* Veb-servis dlya vizualizacii dorozhnoj seti goroda Petrozavodsk [Web service for visualisation of the Petrozavodsk road network] [Text] / V. A. Khitraya, A. V. Khitryj // Trudy Karel'skogo nauchnogo centra RAN. — 2023. — Vol. 4. — P. 54—63. — URL: https://doi.org/10.17076/mat1780.

76. Modeling of the City's Transport Network Using Game-Theoretic Methods on the Example of Petrozavodsk [Text] / N. A. Ermolin [et al.] // Contributions to Game Theory and Management. — 2022. — Vol. 15. — P. 18—31. — URL: https://doi.org/10.21638/11701/spbu31.2022.02.

77. Social network plasticity decreases disease transmission in a eusocial insect [Text] / N. Stroeymeyt [et al.] // Science. — 2018. — Vol. 362, no. 6417. — P. 941—945. — URL: https://doi.org/10.1126/science.aat4793.

78. *Gabasova*, *E.* Star Wars social network [Text] / E. Gabasova. — 2016. — URL: https://zenodo.org/record/1411479.

Figure list

Table list

Appendix A. Daily passenger traffic at St. Petersburg Subway stations

Station	Passenger traffic (persons/day)
Devyatkino	71400
Grazhdansky Prospekt	45421
Akademicheskaya	42991
Polytechnicheskaya	26456
Ploshchad Muzhestva	26736
Lesnaya	25774
Vyborgskaya	30957
Ploshchad Lenina	65354
Chernyshevskaya	45976
Ploshchad Vosstaniya	91531
Vladimirskaya	26879
Pushkinskaya	18527
Tekhnologichesky Institut-1	17188
Baltiyskaya	38076
Narvskaya	40650
Kirovsky Zavod	29174
Avtovo	40703
Leninsky Prospekt	56625
Prospect Veteranov	82598
Parnas	29648
Prospekt Prosveshvaniya	60942
Ozerki	38083
Udelnaya	26386
Pionerskaya	61119
Chyornaya Rechka	39185

Station	**Passenger traffic (persons/day)**
Petrogradskaya	57749
Gorkovskaya	38473
Nevsky Prospekt	58222
Sennaya Ploshchad	40241
Tekhnologichesky Institut-2	30187
Frunzenskaya	16484
Moskovskie Vorota	22981
Elektrosila	24378
Park Pobedy	31589
Moskovskaya	76041
Zvyozdnaya	31866
Kupchino	73487
Begovaya	29387
Zenit	10056
Primorskaya	44434
Vasileostrovskaya	52817
Gostiny Dvor	35434
Mayakovskaya	25889
Ploshchad Aleksandr Nevsky-1	23577
Elizarovskaya	22595
Lomonosovskaya	30471
Proletarskaya	17646
Obukhovo	9064
Rybatskoe	27158
Spasskaya	22603
Dostoevskaya	17178
Ligovsky Prospekt	18143
Ploshchad Aleksandr Nevsky-2	11213
Novocherkasskaya	29485

Station	**Passenger traffic (persons/day)**
Ladozhskaya	79378
Prospekt Bolshevikov	48245
Ulitsa Dybenko	52545
Komendantsky Prospekt	47872
Staraya Derevnya	38496
Krestovsky Ostrov	18534
Chkalovskaya	25204
Sportivnaya	30374
Admiralteyskaya	33079
Sadovaya	17068
Zvenigorodskaya	15852
Obvodny Kanal	17707
Volkovskaya	5722
Bukharestskaya	11064
Mezdunarodnaya	32718
Prospekt Slavy	26409
Dunayskaya	21420
Shushary	8039