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Mathematical modeling of the competitive environment

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Contents

Introduction

Competitive relationships occur in different environments. The main ones are biological and economic. The economic environment regularly undergoes major changes: new jobs are created, production in various industries is growing and technology is improving. In a biological environment, competition within a population and between different populations is also an important biological interaction [1]. Competition is one of many interacting biotic and abiotic factors that affect community structure, species diversity and population dynamics. However, despite considerable success in studying the processes of competition in biological and economic systems, the question of analysis and assessment of the level of competition is still relevant.

The continuous process of evolution has led to the formation of many species. There are the following relationships between species: predator-prey, symbiosis and competition according to biologist classification [2]. In a biological environment an important indicator of competition is the size of populations. On the basis of this, models of population dynamics are considered to describe changes in population size [3].

Various concepts of competition and methods of measurement have been developed. One of them is the loss of most populations due to competition. The paradoxes of coexistence of numerous competing species for resources led to the development of the theory of neutralism, explaining the effect of existence of numerous biological populations on the same trophic resource [4]. Conditions of field research do not allow to establish the level of competition, to find intra-species competition and to evaluate the competitive relationships between individuals [5]. The only experiment was performed by Gause with two species of infusoria. Subsequently, on the basis of the conducted experiment, the principle of competitive exclusion of Gause arises [6].

Mathematical models of interacting populations began to develop in the early 20th century. The first models of Volterra's competition are consistent with Gause's principle [7]. In Russia, mathematical models of interacting populations were developed by A.D.

Bazykin. [8]. Abroad, the most comprehensive justification of such models was given by Murray J. [9]. Most models do not consider the presence of niches, trophic resource, time factor, habitat change [10-11]. Some developed mathematical models are considered obsolete, because they do not characterize the current state of the competitive environment [12-15]. Mathematical models for n populations have not been found either. Some authors carry out theoretical analysis of relationship graphs for n populations. However, in the above literature there are no mathematical models for n populations and simulation [16-19]. In the dissertation work, the developed models consider various factors and simulation for the case of n population.

Over time, mathematical models of population changes began to be applied to economic agents as well. In particular, in economic dynamics these models were used by Prasolov A.V. [20]. In economics, there are methods of measuring the level of competition. The Herfindahl-Hirschman index is used in scientific research. In the Russian legislation, the antimonopoly committee uses the concentration index [21]. In the law, competition is defined as the equality of all interacting entities. Terms related to market concentration have also been introduced. Preliminary estimates indicate a large number of small enterprises, compared to the number of large ones, which indicates to some extent the absence of monopolists [22]. It is necessary to check whether this trend is really observed and whether the hypothesis of the theory of neutralism takes place in the world food market.

The relevance of the research topic

In the Russian language, competition was interpreted as a struggle to achieve great benefits and advantages. At the end of the 19th century in Russia, competition in the field of national economy was understood as "the rivalry of several persons in achieving the same goal" [23]. I. Tugan-Baranovsky considered "free" competition without interference in it, as a beneficial phenomenon, but capable of being reborn into a monopoly under certain conditions. Long before M. I. Tugan-Baranovsky in 1859, B. Kalinowski on the basis of the analysis of the results of free trade in the European countries concluded that it is necessary to permanently restrict it [24]. By now, there are more and more debates about the importance of competition. In some works, the term competition is considered as a concept detached from its essence [25]. In the work the author holds the position that competition is the struggle of old technologies with new ones [26]. With man-made changes, the essence of the concept is also changing. Therefore, the term competitive relationship needs to be redefined.

Aim and tasks

The main purpose of the research is the development of mathematical models of competitive environment, considering hierarchical relationships, trophic resource, change of area, time factor, and development of survival models in competitive environment, as well as analysis of the world market with existing competition indicators and development of new criteria for assessing market competition, threshold values of market concentration and forecasting its development.

The main tasks include:

- Analysis of concentration index, export growth rate and correlation dependencies of market positions in order to identify the most and least competitive industry, as well as entities occupying leading positions;
- Forecasting the concentration index and export volume using time series analysis methods and developed cumulative methods to identify the best forecasting method;
- Development and verification of new criteria for analysis of competitive interaction of economic entities of the market – group, correlation and share criteria;
- Development and verification of new models of competition – operational model of competition, hierarchical model, fragment model, two-chamber model, model of passive avoidance of competition, seasonal model of competition, etc.;
- Simulations to identify population survival and establish concentration index thresholds.

Methodology and research methods

In the process of conducting the research, the author relied on the scientific methodology of conducting researches, generally recognized principles and approaches to research activities in the field of applied mathematics, mathematical modeling, statistics, Theory of systems of ordinary differential equations and systems of partial differential equations.

Numerical results for solving systems of ordinary differential equations are obtained using the **odeint** library of Python. Numerical results for solving systems of partial differential equations are obtained using the **pde** library of Python. The Python Matplotlib library was used to visualize the results.

Scientific novelty

- New methods of analysis of the world food market were developed;
- Three new criteria were developed for assessing the state of the market – group, correlation and share;
- Simulation modeling of competitive systems has been substantiated and carried out;
- Mathematical models of competitive relationships have been developed, considering niches, trophic resource, time factor, hierarchical relationships.

Reliability of the results

The reliability of the results is ensured by the correctness of the problem statement, considering modern concepts and concepts of competitive relationships, as well as by the publication of original results in scientific journals and the discussion of the results at international conferences and scientific seminars. The qualitative theoretical results are consistent with the results of the statistical analysis of the world food market.

Approbation

The results of the dissertation work were presented at scientific seminars of the Department of Computational Methods of Solid Mechanics of St. Petersburg State University and at the following international and all-Russian conferences:

- VII International Scientific Conference of Postgraduate Students and Students "Modern Methods of Applied Mathematics, Management Theory and Computer Technologies (PMTUCT-2014)", 2014, Voronezh, Russia;
- XLVI International Scientific Conference of Postgraduate Students and Students "Management Processes and Sustainability" (CPS'15), 2015, St. Petersburg, Russia;
- IX International Scientific Conference of Postgraduate Students and Students "Modern Methods of Applied Mathematics, Management Theory and Computer Technologies (PMTUCT-2016)", 2016, Voronezh, Russia;
- X International Scientific Conference of Postgraduate Students and Students "Modern Methods of Applied Mathematics, Management Theory and Computer Technologies (PMTUCT-2017)", 2017, Voronezh, Russia;
- All-Russian Scientific Conference of Postgraduate Students and Students "Wealth of Russia", 2018, Moscow, Russia;
- All-Russian Scientific Conference of Postgraduate Students and Students "Wealth of Russia", 2019, Moscow, Russia;
- International Scientific Conference of Postgraduate Students and Students "Management Processes and Sustainability" (CPS'19), 2019, St. Petersburg, Russia;
- XVI International Scientific Conference of Postgraduate Students and Students "Modern Methods of Applied Mathematics, Management Theory and Computer Technologies (PMTUCT-2023)", 2023, Voronezh, Russia.

Publications

The main results on the topic of the dissertation are presented in twenty-seven scientific publications [27–53], five in the publications indexed by SCOPUS and Web of Science [27-31], three in the journals included in the list of the HAC [51-53].

Personal contribution

The dissertation is an independent scientific research, testifying to the professional qualification of its author. Consultations during the development phase of the new models were conducted with E. P. Kolpak. The applicant's personal contribution consists in direct participation in all stages of dissertation research. The author of the dissertation carried out the implementation of the developed methods of solving the problems and writing computer programs. Processing and analysis of the results were also performed by the applicant.

Support

Part of the research was carried out with the financial support of RFBR as part of the research project No 18-31-00323.

Scope and structure of work

The structure of the thesis consists of an introduction, three chapters and a conclusion. The total volume of the dissertation is 135 pages with 55 figures and 26 tables. Bibliography contains 125 items.

Content of work

The introduction presents an analysis of the current state of research on the subject studied, reveals the relevance of the work, its scientific and practical significance. The purpose and objectives of the research are also formulated, the main provisions for the protection and methods of research are given. A summary of each chapter is provided at the end of the introduction.

Chapter 1 presents the basic concepts and definitions of the theory of biological and ecological systems – variants of competitive relationships in the economic environment and ecosystem, description of the concept of ecological niche, its properties and characteristics, the theory of neutralism and the principle of competitive exclusion of Gause.

Chapter 2 is devoted to the statistical analysis of the world food market and the development of new criteria for the analysis and assessment of the market level. Statistical analysis using classical approaches and criteria was carried out. Part of the statistical analysis is the assessment of the level of competition. The world food market is analyzed by 120 items, 12 product categories and 190 economic entities. The choice of the food market is due to its stability and sustainability. The competition criteria used to measure the volume of exports are the concentration index, the Herfindahl-Hirschman index and the Linda index. 3 new criteria for assessing the level of competition have been developed:

- group criterion allows to assess the product category and to identify monopolistic relationships;
- correlation criterion allows to assess the state of the market;
- share criterion allows to establish the core of the market.

Cumulative methods for forecasting food volumes have also been developed and a comparative analysis of existing forecasting methods has been carried out.

Chapter 3 consists of two sections. The first section is devoted to the development of local mathematical models of competitive relationships, the second section is devoted to space-time models. Within the framework of each model, an analysis is carried out for the search and stability of stationary points. The developed mathematical models consider the environmental factor, time factor, trophic resource, hierarchical relationships and other factors. Simulation modeling was carried out for each model, during which the density of the distribution of the level of survival and the concentration index was analyzed, the dependence of the environmental factor was investigated and the threshold values of the number of subjects of the low-concentrated environment were established.

In **conclusion**, the main results of the work are formulated.

Main scientific results

In the dissertation research the statistical analysis of the world market of food export using classical approaches and criteria was carried out. Part of the statistical analysis of the world market is the assessment of the level of competition. The paper analyzes the world market of various product categories over the last 60 years. The criteria used to assess the level of competition are the concentration index, the Herfindahl-Hirschman index and the Linda index. The results of the analysis may serve as a basis for amending the law on protection of competition. The theoretical analysis of the market showed that the existing criteria are not sufficient to draw an unambiguous conclusion about the market situation. The developed criteria for statistical analysis allow to give an accurate assessment of the market state, to establish threshold values of market concentration and to give a forecast of its development [51].

Most existing models do not consider the presence of niches, trophic resource, time factor, habitat change, etc. Mathematical models for n populations are not found either. In the dissertation research, the developed models take these factors into account. Simulations are also carried out for the case of n populations for different types of relationships – interference [52], exploitation [28], hierarchy [53]. The results of the modeling allow to identify factors that influence population death, to assess the dynamics of population survival, to establish the influence of environment on competitive relationships [31].

The results can be used by biological authorities for rational use of protected areas or to control of populations. Also developed criteria and metrics can be used by antimonopoly authorities or other organizations [29, 30] regulating competitive relations in the market to expand, refine or redefine existing methods of assessing competition and identifying monopoly or oligopoly in the market.

Basic provisions for defense

- Share criterion of market analysis;
- Group criterion of market analysis;

- Correlation criterion of market analysis;
- Cumulative forecasting methods;
- Hierarchical competition model;
- Fragmented competition model;
- Resource competition model;
- Operational competition model;
- Two-chamber model of the dynamics of the number of a single population;
- Mathematical model of "passive" avoidance of competition;
- Mathematical model of seasonal competition;
- Spatial operational model of competition;
- Competition on a segment with a meeting on a resource.

Chapter 1. Theory of biological and economic systems

§ 1. Theory of biological systems

1.1.1. Ecosystem competition

Competition in ecosystems is the symbiotic relationship between living beings that compete for resources, namely trophic resource, territory, ecological status, etc. [54-56]. Competition is one of the symbiotic relationships occurring in nature. This interaction is possible both between different populations and within the same population. Individuals of the same population have similar needs in resources, ensuring their survival, growth and reproduction [57]. In the case of biological competition, the following classification is considered [58-61].

Competition by the type of interaction between species:

- *Interspecies competition* occurs when individuals of two species share a limited resource in the same habitat.
- *Intraspecific competition* occurs when members of the same species compete for similar resources in an ecosystem.

Competition by the type of interaction with the resource:

- *Operational competition* occurs indirectly when both populations share a common resource without interacting with individuals of the other species.
- *Interference competition* occurs when organisms interact directly to compete for limited resources.

Competition by proportional characteristics:

Resources can be spatially dispersed. Differential use of resources may be accompanied by divergence in local habitats or geographical distribution. The availability of resources may vary in time (days, seasons).

- *Symmetric competition.* All individuals receive the same amount of resources, regardless of their size.
- *Assymmetric competition.* The largest individuals use all available resources.

Ecological system factors. Territorial distribution of individuals depends on the presence of trophic resource, intraspecies competition, pressure of other populations, climatic conditions, nature of the area and external factors of the ecological system [62]. These factors include:

- *Internal dynamic balance.* Any change in energy, information and quality indicators of systems causes functional-structural qualitative and quantitative changes in the system.
- *Reducing environmental factors* can lead to the death of organisms or the destruction of the ecosystem. In social systems, it is caused by economic conditions and wealth reduction.
- *Decreasing biota* in ecosystems or the population of social systems below the threshold renders the environment or social system unsustainable.
- *In case of external influences* that take the system out of the state of stable equilibrium, the equilibrium shifts in the direction of the lowest effect of external influence.

1.1.2. Ecological niche

Numerous examples of peaceful interaction between plant and animal populations cast doubt on the existence of constant interspecific competition [63]. Many species coexist peacefully on the same trophic resource, without entering into competition with each other. Two main concepts have been proposed to explain this interaction:

- Two species can coexist in the same habitat without rivalry, without diverging on their niches – one of the concepts of the theory of neutralism.
- Species in the process of evolution create or find their disjoint niches – evolutionary concept (competition took place in the past).

The level of competition in different territories and time intervals may vary. Therefore, in the case of competitive relationships, the death of the weak species does not always occur. Even a weaker competitor can displace a stronger rival by gaining a sufficient advantage.

The concept of an ecological niche emerged in the 1970s [64]. Early definitions ranged from a niche as the role played by a species in a community to the type of environment occupied by a species. An ecological niche describes the position of a species in an ecosystem, its ecological role in it, and a number of conditions necessary for the sustainability of the species. The ecological niche is a comprehensive and fundamental concept, as it considers the interactions of species with both biotic and abiotic environments.

The concept of ecological niche also considers the environmental conditions that affect the species, as well as the impact of the species itself on the environment. There are three concepts of ecological niche – niche as a description of the need of the species in the habitat, niche as a functional role of the species in the ecosystem and niche as a dynamic position of the species in the community [65].

Niche as a description of the species' habitat needs. The concept considers the environmental conditions that are necessary for the species to live comfortably and to maintain its population. For the first time this concept was introduced by Grinnell in 1928, where he mentioned a niche as a unit of spatial distribution within which a species can exist [66]. That is, within the framework of this concept an ecological niche is the area occupied by a species, and it is determined by its abiotic needs, food preferences, micro-environment characteristics, avoidance of predators, etc. Knowledge of the ecological niche of the species in question is necessary for understanding and predicting its geographical distribution. Thus, this concept of a niche is more appropriate in biogeography and macroecology than in the ecology of a community or ecosystem [67].

Niche as a functional role of the species in the ecosystem. In this niche concept, each species plays a role in the ecosystem and its dynamics. This role can be played by different species in different territories. The observation of distant species adapted to equivalent ecological roles influenced Charles Elton, who emphasized the functional roles of species. He highlighted the similarities between jerboa and rat-kangaroos, between many placental and marsupial species. Thus, the functional niche refers to the position of the species in the trophic and food chains. This concept is particularly relevant for the ecology of ecosystems.

Niche as a dynamic position of the species in the community. Attention to the diversity of ecological communities has led to the formalization of the concept of niches and their properties that allow different species to coexist in the habitat. In 1957 ecologist G.E. Hutchinson formulated the definition of a niche as a "hypervolume" in a multidimensional ecological space determined by the species' needs for reproduction and survival [68]. Niches are dynamic because the presence of one species limits the presence of another through interspecies competition, which changes the position of niches in multidimensional space. Niche formation is influenced by biologically significant environmental factors [69].

Inextricably linked to the term niche is the Gause principle [6], which states that two species cannot coexist if they occupy the same ecological niche. Gause himself gives this definition of a niche – the place occupied by a species in the community, that is, its habits, food and way of life.

It is believed that there are several factors that allow different species to simultaneously coexist peacefully in the same habitat or to quickly find or create their own ecological niche. Several factors influence the creation of an ecological niche:

- Spatial heterogeneity.
- Climatic fluctuations.
- Evolutionary time.

Restriction of ecological niches in the ecosystem. There is a concept that there are limited niches available for the environment in question, as the ecological space is heterogeneous and the distribution of resources available to the community is limited. There is always a limited potential for the size of the community as determined by the total resources. For an organism in the environment, the number of possible niches is determined by the number of possible ways of using the resource [70-72].

1.1.3. Neutralism theory

The historical development of niche theory is closely linked to the problem of competition and the coexistence of species. From the beginning of the development of the concept of an ecological niche, it was assumed that two species would not be allowed to use the same niche. Gause's "competitive exclusion principle" is that the use of a finite resource leads to its depletion, and population growth leads to a point when the resource level is insufficient for further growth. In the case of two species that share a resource, there is likely to be a level of resource at which the first species can still grow even if the second cannot. This will lead to a further decrease in the growth rate of the second population and to its extinction. Even if two species sharing multiple resources have exactly the same needs and ability to use them, the coexistence of these species is not stable due to ecological or demographic stochasticity: over time, one of the species will eventually accidentally die out.

The "competitive exclusion principle" is a basic principle of community ecology, and much of this area is devoted to exploring how species with similar ecological requirements can coexist. The coexistence of species is often ensured by the division of niches. Niche shifts may result from the competitive exclusion of one species from the part of ecological space where niches overlap, or from the co-evolution of competing species. Dividing niches may be related to processes that have taken place in the past.

If the types are similar in resource usage, the competitive exception can take a very long time. If the rotation of individuals is a random process, then the dominance of a particular species is also a random process. The founder of the theory of neutralism in Russia was Gilyarov A.M., abroad – Hubbell [73]. Now there are two main versions of avoiding competition. The first is finding your spatial or trophic niche. It does not involve coexistence of several species on the same territory. The second concept assumes peaceful coexistence of species without diverging their niches. The main part of this theory is that species can live together precisely because of similar ecological characteristics of each other [74].

§ 2. Theory of economic systems

1.2.1. Competition in economic communities

Humans, relying on natural resources and the richness of ecosystems, thousands of years ago began to create a habitat for themselves in which they created the necessary trophic resources for their existence. These include food, housing, buildings, education systems, necessities and luxury items, etc. The necessary resources have gradually become unevenly distributed both among the members of the communities and among the producers of goods. In the system of production and sale of goods, at some stage of its evolution, numerous producers of the same commodity began to emerge. The abundance of products in markets led to competition between producers (sellers), which came to be called competition. Individual producers have also emerged seeking to monopolize the production and sale of goods or to exclude (including destruction) similar producers.

It is believed that competition determines the nature of interaction in the sphere of economy between producers of products and buyers [75]. It is the principle of the structure of the market economy, the mechanism of interaction and the way of realization of economic goals. Competition is proclaimed an enduring value that ensures the progressive movement of humanity, and is also, by some perceptions, a natural law in social society. In some markets, the competitive process may be subject to regulation [76]. Competitive relationships have a continuous impact on the state of the market. Stability of the market is due to the following conditions [77-79]:

- The rank hierarchy of subjects is defined and may change over time;
- Control of subjects is carried out uniformly;
- There are no barriers to the movement of goods, services, capital (freedom of entry to the market);
- Ensured homogeneity of products.

In the work of Vasilyev A.N. competition was treated as a struggle for achieving great benefits and advantages [80]. At the end of the XIX century in Russia, competition in the field of national economy was understood as "the rivalry of several persons to achieve the same goal".

The contradiction of the definition of competition is also reflected in the Russian competition law [21] – competition is understood as "the competition of economic entities, in which the independent actions of each of them exclude or limit the possibility of each of them unilaterally to influence the general conditions of circulation of goods on the relevant product market. At the same time, the concept of unfair competition is introduced as "any actions of economic entities (group of persons) that are aimed at obtaining advantages in carrying out entrepreneurial activities…". Competition should therefore be understood as an environment of production and sale of goods and services in which the acting organizations and persons do not interfere with each other in any way.

The law also describes the signs of restriction of competition, which can be considered as a kind of "blurred" border between "competition" and "unfair competition". Quantitative criteria for identifying the presence of competition is not provided in the law. Only the criterion for determining the dominant position of the three economic facilities is given. There is also a concept of unfair competition in foreign sources [81].

In the framework of unfair competition, the concept of a niche for an economic entity in the economy may be related to the ability of the entity to prevent competition with other entities and, consequently, to obtain the maximum level of protection.

In the case of competition in the economic environment, the following classification is considered [82].

Types of competition by the scale of development:

- Individual;
- Territorial;
- Intraindustry;
- Interindustry;
- Global.

Types of competition by nature of development:

- Free/Adjustable;
- Price/Non-price.

In the case of price competition, there is an artificial change in the prices of the goods. In the case of non-price competition, the product itself is directly changed by improving production technologies.

Types of competition by direction:

- Competition for resources;
- Competition for cargo/services.

In the dissertation work, mathematical models of competitive relationships of populations in economic and ecological systems are developed. The population can be interpreted as a community of people living in the territory and connected with each other through economic relations, the association of collectives producing specific types of goods and services, regions and states economically interacting with each other. Intra-species competition is the cost of production, and inter-species competition is the form of competition arising from the sale of goods in the market.

1.2.2. Competition indicators and market models

The hypothesis that competition depends on the number of participants in the process of production and sale of goods is taken as a basis [26]. Depending on the number of participants, several types of market are identified. The main ones are monopoly, oligopoly, monopolistic competition, pure competition. There are no thresholds to define "quantitative" boundaries between these markets. This classification also does not reflect the intensity of the processes taking place in the markets.

One of the concepts of analysis of the intensity of competition in the market is based on the use of financial indicators of firms. A low rate of return indicates little market power, and a high rate of return indicates a power that restricts competition [83].

The following approach assumes that the level of competition is inversely dependent on the concentration of market shares of firms in the market – the higher the concentration of economic agents, the less competitive the market environment is considered. In economics, there are methods of measuring competition. Scientific research uses concentration index [84], Herfindahl–Hirschman index [85], relative concentration coefficient [86], entropy coefficient [87], modified Ginny coefficient [88], rank index of concentration [89] and other indices. Indices in which the quantitative assessment of the concentration of production is estimated by the share of the economic entity in the general balance of goods sold on the market are considered below.

The market concentration ratio CR_n is calculated using the formula:

$$CR_n = \frac{\sum_{k=1}^{n} V_k}{\sum_{k=1}^{N} V_k},$$

where V_k $(k = 1,2,\ldots,N)$ – share of the k-th entity on the market, n – the number of the largest market entities, N – the total number of market entities. Concentration index is measured in relative fractions (or percentages). With equal proportions of all operators $CR_n = \frac{n}{N}$. If the choice of n is fixed, then $CR_n \to 0$, if $N \to \infty$.

Concentration index thresholds:

- $CR_3 \in [0,\ 0.45)$ – unconcentrated market;
- $CR_3 \in [0.45,\ 0.7)$ – moderately concentrated market;
- $CR_3 \in [0.7,\ 1]$ – highly concentrated market.

The Herfindahl-Hirschman index considers the shares of all firms and is calculated using the formula:

$$H_N = \sum_{k=1}^{N} V_k^2.$$

In the case of equal fractions $(V_k = \frac{1}{N}, k = 1,2, \dots N)$ $H_N = \frac{1}{N}$. That is, as the number of peers increases, the Herfindahl–Hirschman index decreases. If the proportion of at least one operator increases and the others decreases, then H increases, approaching 1.

Herfindahl–Hirschman index thresholds:

- $H \in [0,0.1)$ – unconcentrated market;
- $H \in [0.1,0.2)$ – moderately concentrated market;
- $H \in [0.2,1)$ – highly concentrated market.

The Linda index is used in the European Union to analyse differences in a group of major economic agents. It defines the "core" of the market. For this purpose, the market shares of economic agents are arranged in descending order $V_1 \geq V_2 \geq \cdots V_N$.

The Linda index for the two subjects is calculated using the formula:

$$IL = \frac{V_1}{V_2};$$

for three subjects –

$$IL = \frac{1}{2}\left(2\frac{V_1}{V_2 + V_3} + \frac{V_1 + V_2}{2V_3}\right);$$

for four subjects –

$$IL = \frac{1}{3}\left(3\frac{V_1}{V_2 + V_3 + V_4} + \frac{V_1 + V_2}{V_3 + V_4} + \frac{1}{3}\frac{V_1 + V_2 + V_3}{V_4}\right);$$

for five subjects –

$$IL = \frac{1}{4}\left(4\frac{V_1}{V_2 + V_3 + V_4 + V_5} + \frac{3}{2}\frac{V_1 + V_2}{V_3 + V_4 + V_5} + \frac{2}{3}\frac{V_1 + V_2 + V_3}{V_4 + V_5} + \frac{1}{4}\frac{V_1 + V_2 + V_3 + V_4}{V_5}\right).$$

If the shares of all economic agents are the same, the Linda index is equal to one. If the share of one subject grows, the Linda index will increase. Therefore, the addition of each new entity should be accompanied by a reduction in the index. If, starting with $k + 1$-th subjects, the index increases, then the first k subjects form the "core" of the market.

Chapter 2. Statistical analysis of global food market competition

This chapter presents the main results of the statistical analysis of the world export market. The work analyzed data on 120 market positions of the world food market from the open official UN source FAOSTAT for the time period 1961-2019 [90]. The food market was chosen because it is stable and sustainable [91]. The global food market is heterogeneous. Heterogeneity refers to the wide variation in export volumes of economic entities and the lack of statistical data for a certain time period of the time interval under consideration. The economic entity is understood as the exporting country.

All market positions were divided into product categories. Combining products into homogeneous groups allows us to make an unambiguous conclusion about the state of the market for each product category. A total of 12 product categories were considered. A similar approach has not been found in the literature. A full statistical analysis and prediction of values was carried out on the presented data, in particular:

- Analysis of the world food market using independent methods;
- Forecasting the level of competition using cumulative methods;
- Development of new criteria for measuring competition - group, equity and correlation;
- Algorithm for quantitative analysis of competition in the market;
- Development of an algorithm for determining threshold values of market intensity;
- Testing the Gause principle on the world food market.

2.1. Group criterion of market analysis

The concentration index was calculated for all market positions from 1961 to 2019. The nature of the change in the concentration index depends on the type of market. For the special case of the market position "apples", Figure 1 (A) shows the result of the change in the concentration index. For the time interval 1961-1981 – $CR_3 \in [0.45, 0.7)$, for 1981-2019 $CR_3 \in [0, 0.45)$. That is, for the market position "apples" the market is considered unconcentrated. An example of a moderately concentrated market is the global banana export volume. Figure 1 (B) shows the change in the concentration index for the market position "bananas". For the time interval 1961-2019 гг – $CR_3 \in [0.45, 0.7)$.

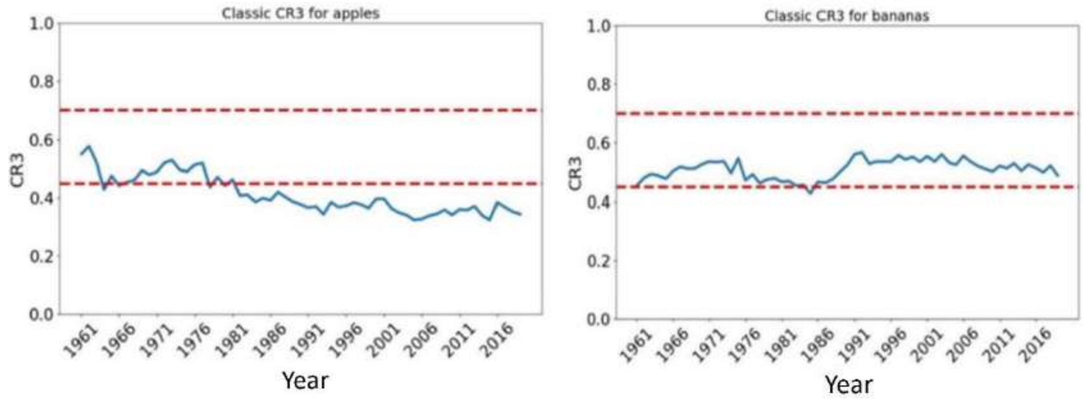


Fig.1. A – Concentration index for the volume of apples export,
B - Concentration index for the volume of bananas exports.

Figure 2 (A) shows the change in the index of concentration for buckwheat exports. Concentration index for the time interval 1961-2012 $CR_3 \in [0.7, 1)$, in 2012-2019 $CR_3 \in [0.45, 0.7)$. The market was highly concentrated, but since 2012 the market has become moderately concentrated. The up to 2012 changes can be explained by the presence of monopolistic competition. Later, the market stabilized as new entrants appeared. Figure 2 (B) shows the change in the concentration index for garlic exports. Concentration index for time interval 1961-1997 $CR_3 \in [0.45, 0.8)$, in 1997-2019 $CR_3 \in [0.7, 1)$. The market was moderately concentrated, but since 1997 the market has become highly concentrated. According to the analysis, the garlic export market is moving from medium to high concentration.

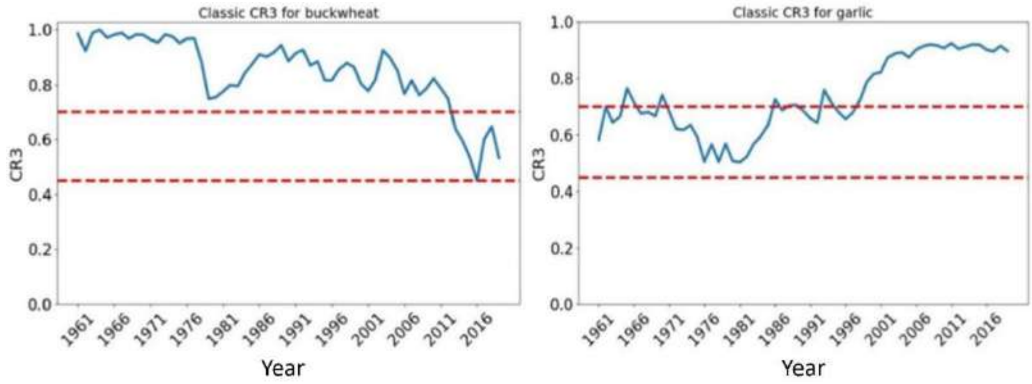


Fig.2. A - Concentration index for buckwheat exports,
B – Concentration index for the volume of garlic exports.

Table 1 shows the results of an analysis of the global export market for 120 market positions using the concentration index. Statistical data for 2019 are used in the calculation of the concentration index. For each concentration index interval, the share of market positions that meet this interval is calculated.

Table 1. Proportion of groups and intervals of concentration index.

The Group	$CR_3 \in [0, 0.45)$	$CR_3 \in [0.45, 0.7)$	$CR_3 \in [0.7, 1)$
Fruits	0.23	0.64	0.13
Vegetables	0.1	0.57	0.33
Cereals	0.21	0.64	0.15
Dairy products	0.67	0.165	0.165
Sweets	0.6	0.2	0.2
Oils	0.33	0.67	0
Meat	0	0.83	0.17
Nuts and dried fruits	0	0.375	0.625
Spices	0.2	0.2	0.6
Alcoholic beverages	0.6	0.2	0.2
Non-alcoholic beverages	0.2	0.8	0
Others	0.4	0.6	0

From the studies carried out it follows that the market shares in the considered intervals of the threshold value of the concentration coefficient are unbalanced. Therefore, it is difficult to draw an unequivocal conclusion about the state of the market. The following objective function is entered:

$$f(x1, x2, x3) = w_1x_1 + w_2x_2 + w_3x_3 \rightarrow max$$

$$w_1 = \frac{3}{10}, w_2 = \frac{2}{10}, w_3 = -\frac{1}{2}$$

$$x_1 = share_{cr0.45}, x_2 = share_{cr0.7}, x_3 = share_{cr1}$$

where w_i – the weights of shares, x_i – the share of market positions in the product category. Weight coefficients are normalized as follows:

$$\sum_i |w_i| = 1$$

This criterion allows you to evaluate each product category as a whole. The weighting factors of the function prioritize the impact of market positions on the group's overall evaluation. The following group criterion for the analysis of product categories is introduced.

Criterion: The market is considered stable and competitive if the objective function is non-negative. Otherwise, monopolistic competition is considered to be present in the market

$$f(x1, x2, x3) \geq 0.$$

According to the results, stable and balanced global export markets include the following product categories: dairy products, butter and soft drinks. Monopolized markets include the group of spices, nuts and dried fruits.

Table 2. Group Criterion Values $f(x)$.

The Group	$f(x)$
Fruits	0.1
Vegetables	0.0
Cereals	0.1
Dairy products	0.2
Sweets	0.1
Oils	0.2
Meat	0.1
Nuts and dried fruits	-0.2
Spices	-0.2
Alcoholic beverages	0.1
Non-alcoholic beverages	0.2
Others	0.2

Conclusion:

- A group criterion has been developed, allowing to assess the market condition of the product category under consideration. If the value of the objective function is non-negative, then the market is considered to be competitive and stable. Otherwise, there is monopolistic competition in the market.
- The analysis of the competition between the market positions of various groups showed that the main trend is an improvement in the market situation, or the continuation of the observed trend.
- The analysis of the concentration coefficient within the groups showed that 1/3 of the product categories under consideration are the most competitive. 1/2 of the categories are moderately concentrated. In the global food market, for some product categories (1/6) there are certain restrictions on entering the market and internal or external leverage on it.

2.2. Correlation criterion of market analysis

In each product category, a Pearson and Spearman correlation matrix was constructed for each market position. A high level of correlation is considered:

$$corr_{ij} > 0.95.$$

It follows from the analysis that the correlation coefficient of Pearson and Spearman yield a similar result. In this case, the analysis identifies a number of economic agents that may be interdependent. Further, the hypothesis of the relationship between high correlation and the level of competition is advanced.

It is difficult to establish an unambiguous relationship between the correlation coefficient and the concentration index. Therefore, the number of dependent economic entities was averaged for each concentration index interval:

$$\overline{object_{\Delta CR}} = \frac{\sum_i object_i}{amount_{object}}$$

where ΔCR – concentration index interval, $\sum_i object_i$ – total number of dependent economic entities that are within the concentration index interval ΔCR, $amount_{object}$ – the number of measurements that are in ΔCR. These values were approximated by an exponential function:

$$f(x) = \mathrm{e}^{a+\mu x}, a = 4.4, \mu = -0.63$$

Figure 3 shows the average values of the number of dependent economic subjects for each concentration index interval, which are approximated by an exponential function. The system of black points corresponds to the average values intervally – $\overline{object_{\Delta CR}}$. The blue line corresponds to an exponential approximation.

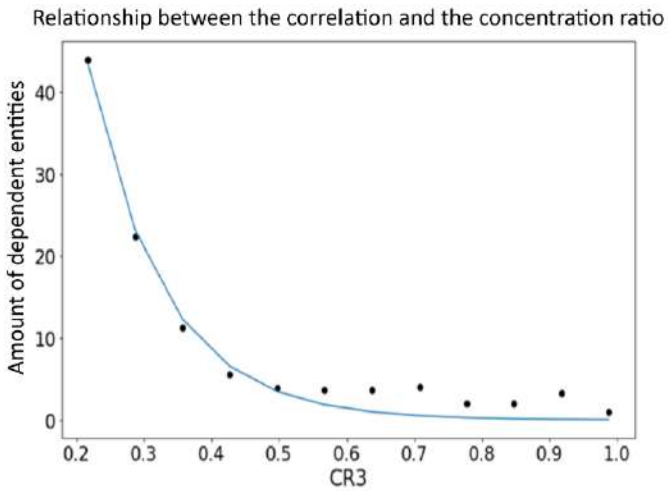


Fig. 3. Approximation of averaged data by an exponential function

The analysis shows that there is a relationship between the number of correlations between economic agents and the concentration index. The developed criterion allows to assess the level of market competition through the correlation coefficient. The more relationships between economic agents are present in the market, the lower the concentration index and the higher the correlation coefficient. Table 3 shows the thresholds of the number of interdependent entities for the corresponding level of market concentration.

Table 3. Threshold values of the number of dependent entities.

Concentration of the market	Threshold value of the number of dependent entities
Low	$k > 5$
Moderate	$k \in [0, 5]$
High	$k = 0$

Criterion: the market is considered stable and competitive if $k > 5$, that is, the number of interdependent entities is not less than 6. Otherwise, the market is considered moderately or highly concentrated.

2.3. Share distribution of economic entities

For each market position of all product categories, a histrogram of frequencies is constructed. The number of bins in the histogram was assumed as follows:

$$bins = 0.4 * len(data)$$

where $len(data)$ – number of economic entities on the market of the considered market position. Two trends are observed - an even distribution of entities in the market and the formation of various groups of economic entities. The number of groups lies in the interval [2, 3]. Figure 4 shows the histogram of the volume of apple exports. In this example, there are two main groups of subjects.

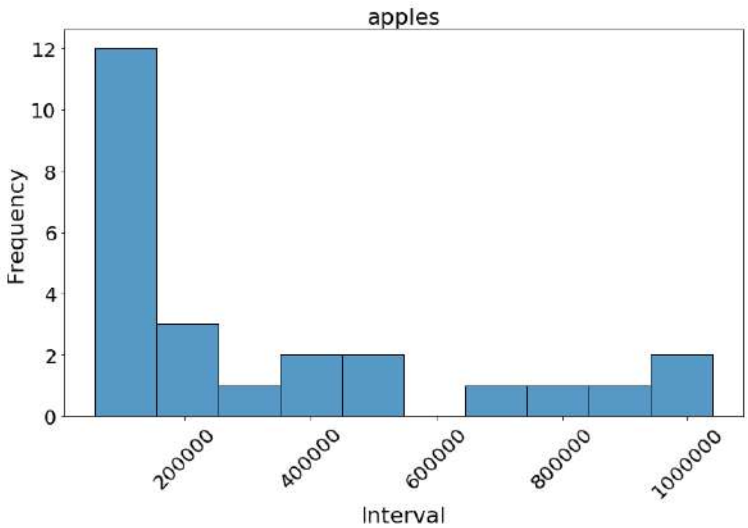


Fig. 4. Histogram of the volume of export of apples.

Table 4 presents the results of the formation of several groups for individual positions and their number. All food industries, except the sweets group, are predisposed to form 2 groups. There is a tendency to be evenly distributed with respect to the formation of 3 or more groups.

Table 4. Formation of product category groups

The Group	No group	2 groups	3 groups and more
Fruits	8/22 = 36%	13/22 = 59%	1/22 = 5%
Vegetables	8/22 = 36%	11/22 = 50%	3/22 = 14%
Cereals	2/14 = 14%	10/14 = 71%	2/14 = 15%
Dairy products	1/6 = 17%	4/6 = 67%	1/6 = 16%
Sweets	1/5 = 20%	1/5 = 20%	3/5 = 60%
Oils	1/3 = 33%	2/3 = 67%	0%
Meat	2/6 = 33%	3/6 = 50%	1/6 = 17%

Nuts and dried fruits	2/14 = 14%	10/14 = 71%	2/14 = 15%
Spices	2/5 = 40%	3/5 = 60%	0%
Alcoholic beverages	1/5 = 20%	3/5 = 60%	1/5 = 20%
Non-alcoholic beverages	1/5 = 20%	3/5 = 60%	1/5 = 20%
Others	0%	4/5 = 80%	1/5 = 20%

Conclusion: the results of the analysis show that, depending on the product category, the market may lead to several groups. The most common is the split into two groups of economic agents.

2.4. Cumulative forecasting methods

The local cumulative method consists of summing up export volumes for each year of the time interval under consideration:

$$v_{year} = \sum_{countries} \left(v_{year}^{country_1} + \cdots + v_{year}^{country_N}\right)$$

The local cumulative method was applied to all market positions. For most market positions there is an exponential dependence. For cumulatively generated export volumes, the concentration index is as follows:

$$CR_n = \frac{\sum_{k=1}^{n} \bar{V}_k}{\sum_{k=1}^{N} \bar{V}_k},$$

where $\bar{V}_k$ – local cumulative volume of exports of the economic entity.

The local cumulative method was applied to all market positions. Figure 5 shows the results of the concentration index calculation for the market position of apples. The cumulative method smooths the concentration index curve. For the rest of the market positions, similar results are observed.

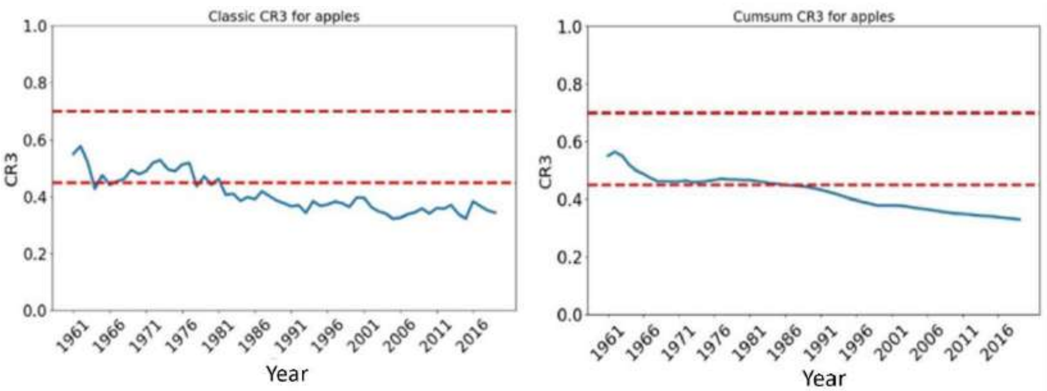


Fig. 5. Results of application of the concentration index for the local cumulative method for apples.

Full cumulative method. In the full cumulative method, complete summation occurs according to the formula:

$$v_{year} = \sum_{years} \left(\sum_{countries} \left(v_{year_1}^{country_1} + \cdots + v_{year_t}^{country_N}\right)\right)$$

Exports are measured in market units. The full cumulative method was applied for all market positions. Figure 6 shows the results of local (A) and full (B) cumulative methods. The yellow system of points is the result of application of the method for the market position "almonds". Export volumes of economic entities using annual and local cumulative methods were approximated by exponential function. For a given market position, it can be argued that the cumulative volume of exports changes exponentially.

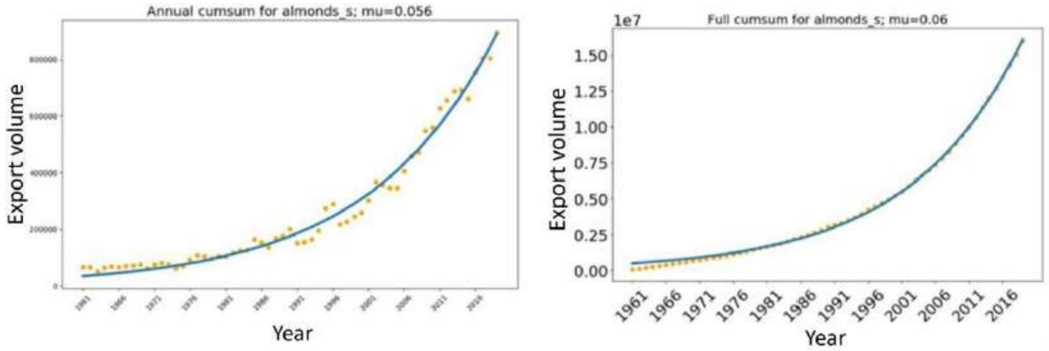


Fig. 6. A - Result of local cumulative method for almonds
B - Results of the full cumulative method for almonds.

2.5. Forecasting the level of competition by cumulative methods

For all market positions of all product categories, the forecast volume of exports was calculated by developed cumulative methods. The prediction limit is 10%, i.e. the allowable deviation from the forecast is ∓10%. Figure 7 shows the result of the comparative analysis for the product category nuts and dried fruits. The red horizontal line corresponds to a valid prediction boundary.

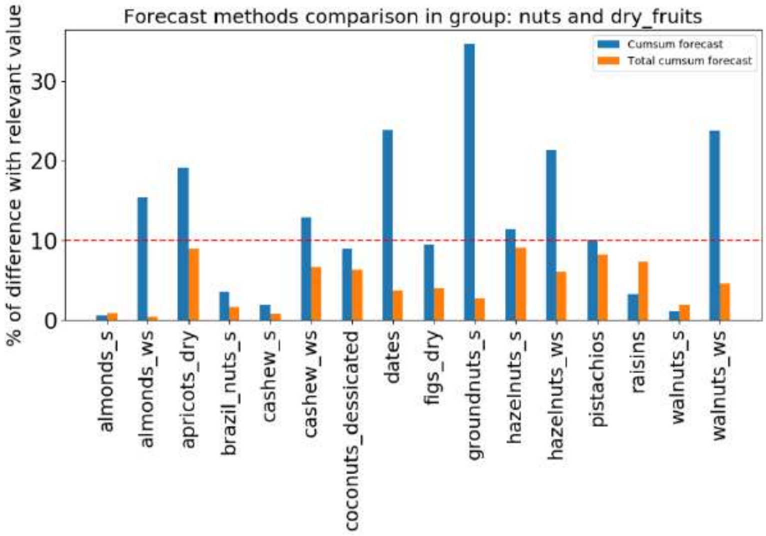


Fig. 7. The result of a comparative analysis for the food industry – nuts and dried fruits

For the nuts and dried fruits industry, there is a permissible deviation of export volumes calculated using the full cumulative method for all market positions. In most cases, the local cumulative method is not within the acceptable range. For the rest of the product industries, a similar trend is observed.

Table 5 presents the results of a comparative analysis of the two cumulative methods for each product category, namely the market share for which the method is effective.

Table 5. Comparative analysis of cumulative methods.

The Group	Local Cumulative Method	Full Cumulative Method
Nuts and dried fruits	0.2	0.8
Fruits	0.2	0.8
Dairy products	0.33	0.67
Alcoholic beverages	0.2	0.8
Non-alcoholic beverages	0.2	0.8
Meat	0.17	0.83
Cereals	0.3	0.7
Vegetables	0.2	0.8

Sweets	0	1
Spices	0	1
Oils	0	1
Others	0.2	0.8

Conclusion:

- A comparative analysis of the two cumulative methods was carried out. Comparing the two methods, it was found that the full cumulative method is generally more efficient than the local one.
- Exponential approximation was used to forecast export volumes. The hypothesis of exponential growth of global exports is correct.
- Full cumulative method is applied for 90% of market positions.

2.6. Forecasting the concentration index by time series analysis methods

In this chapter, the results of forecasting the concentration index by time series analysis methods. For all market positions, the forecast values for 2019 were calculated using 5 different methods:

- AR - Autoregressive model;
- MA - Moving Average Model;
- ARMA - Autoregression model – moving average;
- ARIMA - Boxing - Jenkins model;
- SES - Exponential Smoothing.

Figure 8 shows the results of the comparative analysis of time series. The Box–Jenkins model and exponential smoothing show the best results.

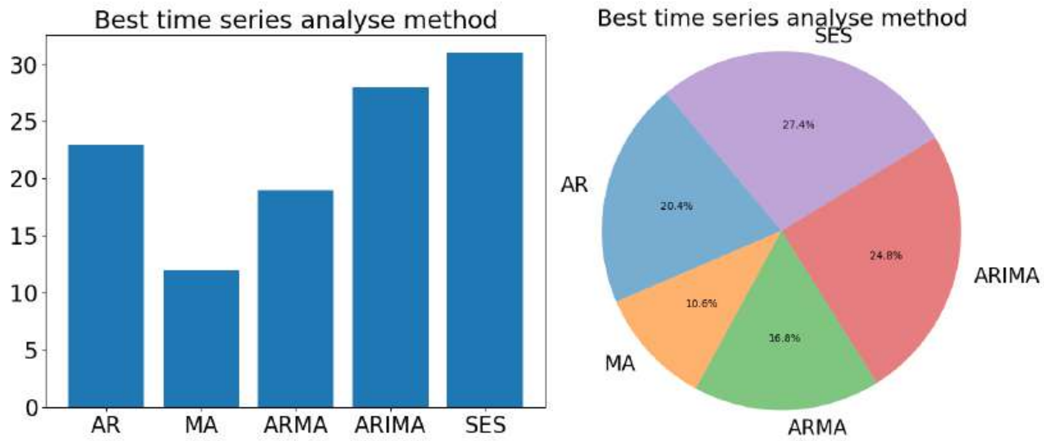


Fig. 8. Result of comparative analysis of time series

Figure 9 shows the results of the comparative analysis for the product category nuts and dried fruits. Red dotted lines represent the threshold values of the concentration index. The forecasting margin is 10%, i.e. the accuracy of the forecast is ∓10%. All items have accuracy within the specified interval. Similar results are observed for the rest of the product categories. Table 6 shows the results of the analysis of the concentration index using time series methods.

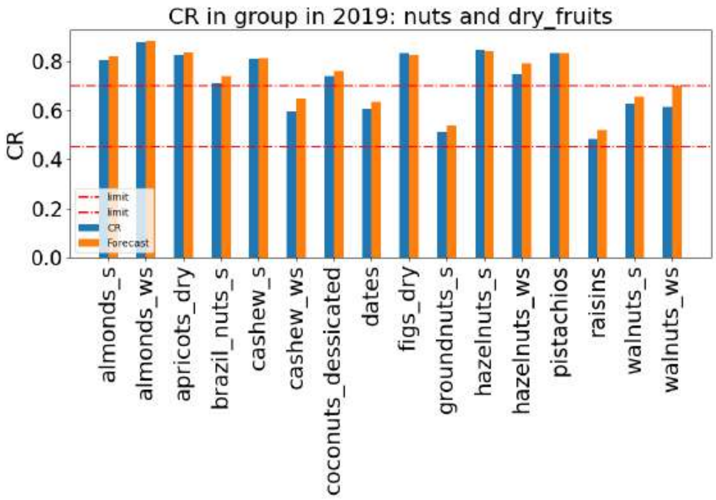


Fig. 9. Results of comparative analysis for the product category – nuts and dried fruits

Table 6. Accuracy of time series analysis methods and standard deviation from the true value for all product categories.

The Group	**Accuracy of methods**	**Mean deviation from initial value**
Nuts and dried fruits	75.0 %	3.9 %
Fruits	81.8 %	2.7 %
Dairy products	66.7 %	3.5 %
Alcoholic beverages	80.0 %	9.3 %
Non-alcoholic beverages	100 %	0.7 %
Meat	80.0 %	3.1 %
Cereals	71.4 %	4.7 %
Vegetables	90.4 %	1.6 %
Sweets	80.0 %	2.2 %
Spices	100 %	1.8 %
Oils	100 %	0.2 %
Others	80.0 %	2.4 %

Conclusion: a comparative analysis of 5 methods of time series forecasting was carried out. The analysis revealed that the best prediction methods are the Box–Jenkins model and exponential smoothing. The methods considered give the required accuracy in 84% of cases. For product categories such as soft drinks, spices and oils, 100% accuracy of time series analysis methods is observed, as well as a low standard deviation from the true value of the concentration index.

2.7. Analysis of export growth rates

One of the characteristics of competition is the rate of production growth - μ. Export growth rates have been calculated for all market items of each product category. Figure 10 shows the results for all product categories. Point systems correspond to all product categories. Points correspond to market positions in the group. The dotted line corresponds to the global rate of population growth [92].

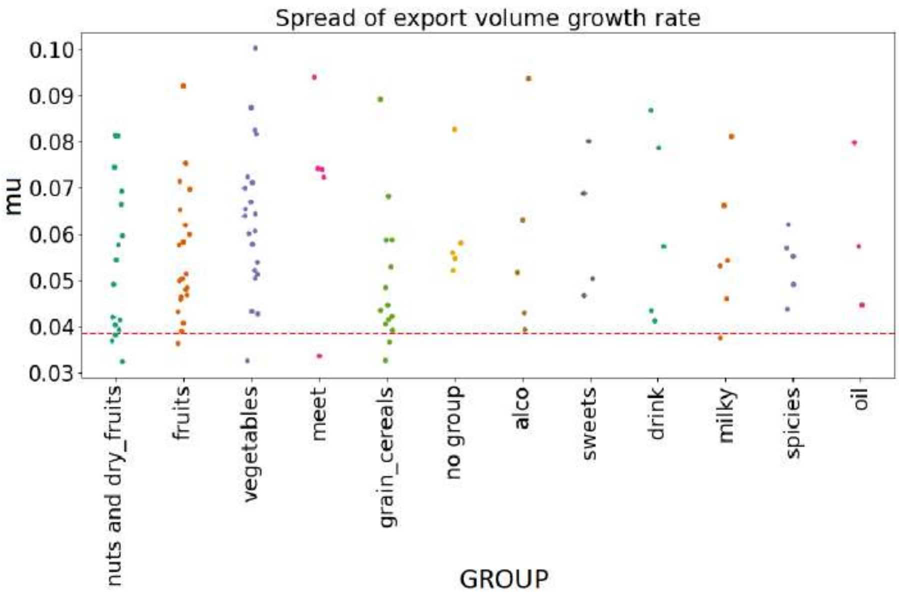


Fig. 10. Export growth rate for all product categories

Table 7 shows the interval of the growth rate within each group. The interval $[0.032, 0.1]$ is common for all market positions. As can be seen from Table 6 and Figure 10, all product categories develop evenly.

Table 7. Interval of change of growth rate within each product category.

The Group	μ_{min}	μ_{max}
Nuts and dried fruits	0.032	0.081
Fruits	0.036	0.092
Dairy products	0.038	0.081
Alcoholic beverages	0.039	0.094
Non-alcoholic beverages	0.041	0.087
Meat products	0.034	0.095
Grain	0.033	0.089
Vegetables	0.033	0.1
Sweets	0.047	0.08
Others	0.052	0.083

Spices	0.044	0.062
Oils	0.045	0.08

Conclusion: All product categories develop evenly. From the analysis it follows that the market for global food exports develops uniformly and is stable.

2.8. Frequency criterion of market analysis

To analyze the core of the market, it is customary to use the Linda index, which allows to determine the degree of inequality between the leading players in the market. According to the theoretical analysis, the Linda index yields contradictory results. With this in mind, we propose an alternative way of analyzing market leaders - frequency criterion. The frequency of the emergence of leading countries is calculated as follows:

$$frequency_{country} = \frac{\sum_{t_1}^{t_2} top_{value}}{\Delta t},$$

where Δt – investigated interval, top_{value} indicates the frequency of the appearance of the country in the top three positions during the interval considered. This criterion was applied to research market leaders for all market positions within each group. Figure 11 shows the distribution of leaders for the apple export market.

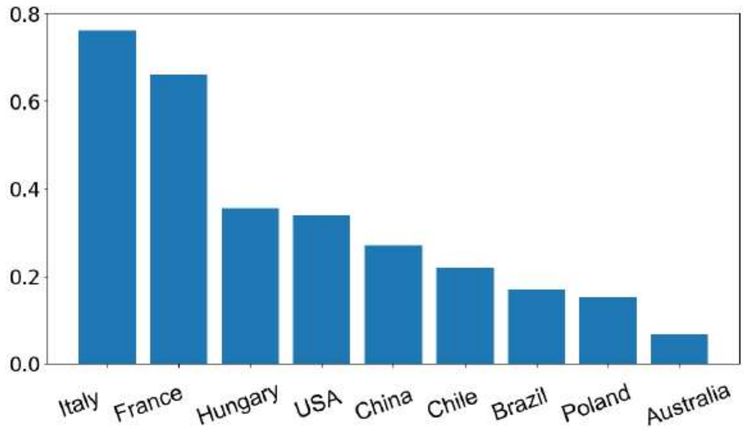


Fig. 11. Results of application of frequency criterion for apples

Italy led by 3/4 of the time span, followed by France, which was 2/3 of the time span. The rest of the countries were in the lead for less than half of the time. Similar results were obtained for the remaining market positions. Table 8 presents the result of applying the frequency criterion for all product categories, namely the share of market positions within the product category, when the entity held the leading positions.

Table 8. Results of application of frequency criterion

The Group	Group Leaders
Nuts and dried fruits	0.56 - USA, 0.25 - Turkey
Fruits	0.41 - Spain, 0.32 - Italy, USA, 0.18% - Mexico
Dairy products	0.5 - Spain

Alcoholic beverages	0.6 - France, 0.4 - United Kingdom, Spain, Italy
Non-alcoholic beverages	0.6 - Germany, 0.4 - Netherlands
Meat products	0.4 - Netherlands, USA, France
Grain	0.57 - USA, 0.5 - Canada, 0.28 - Germany
Vegetables	0.48 - Netherlands, 0.38 - Mexico, 0.33 - Spain
Sweets	0.4 - China, Netherlands, Germany
Others	0.6 - USA, 0.4 - Greece
Spices	0.4 - India, China, Indonesia, Singapore
Oils	0.33 - Italy, Tunisia, Spain, Indonesia, USA

Conclusion: a frequency criterion was developed to analyze leading positions for product categories. All product categories were analyzed using this criterion. For each group, the leading countries were identified. The U.S. is a frequent leader. According to the results of the group criterion (Chapter 2, 2.1, p. 28), the monopolized market is the product category "nuts and dried fruits", in which the USA also leads. Similar studies are carried out in the work [93].

2.9. Share criterion of market analysis

For the analysis of export volumes, a share criterion of market core analysis was developed. The proportion criterion for the sorted cumulative export volumes of the market item under consideration is calculated as follows:

$$countries = \sum country_{share} < 0.8$$

where $country_{share}$ – the share of exports of the economic entity.

This criterion allows us to estimate the number of economic entities that form 80% of the export market. This set of economic agents can also be called the core of the market.

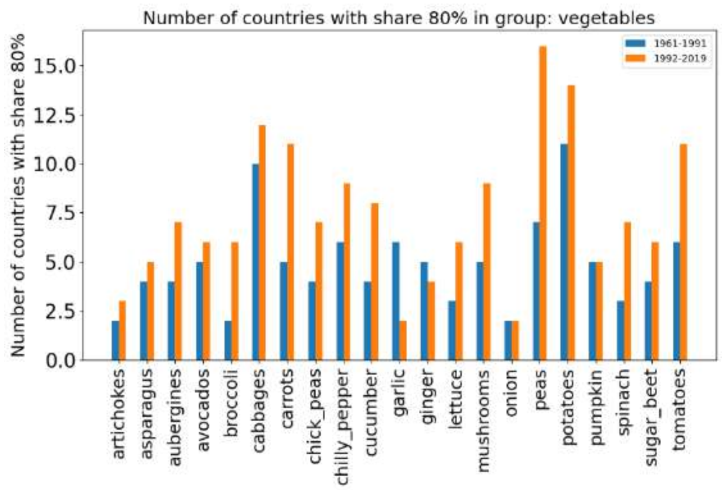


Fig. 12. Results of share criterion for 1991 and 2019 for vegetables

In order to analyse the change in the core of the market over time, the time span was divided into two parts, [1961, 1991] and [1991, 2019]. Figure 12 shows the results of the share criterion for 1991 and 2019 for the food category – vegetables. The blue column corresponds to the number of countries forming the core of the market by 1991, the orange column by 2019. For almost all market positions, there is an expansion of the core of the market. It follows that the market is stabilizing and becoming less concentrated. Table 9 shows the average number of countries within the product industry that form the core of the market at the end of 1991 and 2019.

Table 9. Average number of countries in the core market by 1991 and 2019.

The Group	Average number of countries in the core of the market in 1991	Average number of countries in the core of the market in 2019
Nuts and dried fruits	2.5	3.9
Fruits	6.3	8.3
Dairy products	5.5	10.2
Alcoholic beverages	7	11.8
Non-alcoholic beverages	6.6	11.6
Meat products	3.8	6
Grain	5.7	8.5
Vegetables	4.9	7.4
Sweets	11.2	17
Others	6	10.4
Spices	5.8	6.8
Oils	6.3	8.3

Figure 13 shows the number of countries in the core market of each market position within the group by the end of 1991 and 2019. Here, there is a similar tendency to increase the core of the market. It follows that the food market as a whole becomes less concentrated over time.

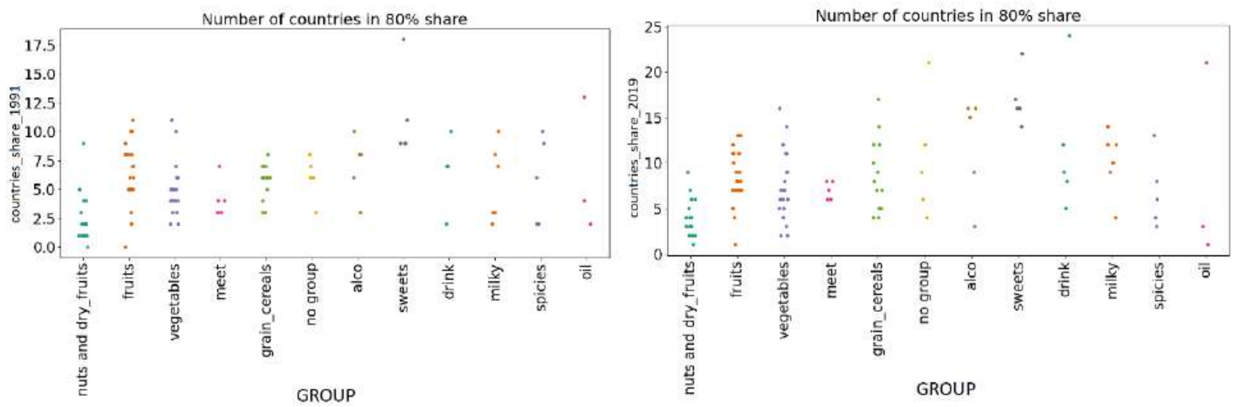


Fig. 13. Share criterion results for 1991 and 2019

Consider the dependence of the concentration index in 2019 and the number of economic entities in the core of the market. Figure 14 shows this constraint. There is a correlation between the concentration ratio and the number of economic entities in the core of the market – the larger the number of countries in the core of the market, the more stable and lowly concentrated the market.

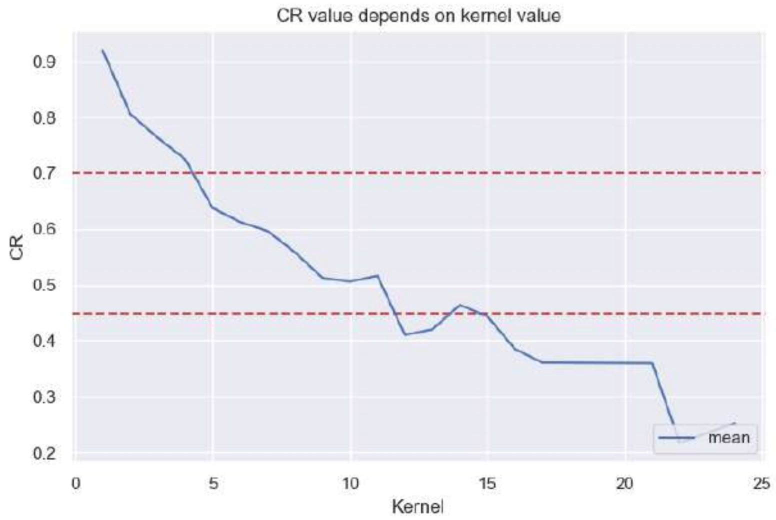


Figure 14: Dependence of the concentration index on the number of economic entities in the core of the market

Criterion: in order for the market to be stable, competitive and have low concentration, it is necessary that the number of economic entities in the core of the market should be at least 12 participants.

Chapter 3. Mathematical models of competition

This chapter is devoted to mathematical modeling of biological and economic processes. Mathematical modeling allows to evaluate the behavior of a biological or economic process. The most common are population models for a single species, as well as models of interacting populations. The description of population changes over time is the subject of population dynamics. Population models or growth models assume that the population growth rate is proportional. The first model of continuous growth was proposed by Malthus in 1798 in the book "On Population Growth" and has the form [94]:

$$\frac{dN}{dt} = bN - dN$$

where b, d – positive constants, $N(0) = N_0$ – initial population number. In this model, the population grows exponentially. This effect later became known as Malthus Law. For most species, this law does not apply because there are external constraints and factors that stop exponential growth. However, there is an initial exponential growth for some species, but then the pattern of the curve changes. The exception is the growth of the human population, which until recently grew faster than the exponent. In mathematical models, there may be an autocatalytic term that describes the increase in the rate of change of the value with the increase of the value itself. The basic model of limited growth is the Verhulst model [95]:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

At low values of x, this function increases exponentially, as in Malthus' law, but at high values, it approaches K, where K is the population capacity or the ecological niche capacity. The capacity of the population is systemic and can depend on many factors, such as limited food base, presence of a niche, etc. The Mono model (Michaelis-Menten) is used to limit the food resource, which describes the dependence of the growth rate μ on the resource [96]:

$$\mu(S) = \frac{\mu_0 S}{K_s + S},$$

where K_s – concentrating resource, μ_0 – maximum growth rate. In conditions of limited resource, the rate of population growth is proportional to the concentration of the resource. If the resource is surplus, the population stabilizes over time.

Also, in modeling one can find the Allee effect [97]. It is a phenomenon in biology characterized by a correlation between the size or density of a population and the average individual fit of a population or species (often measured as the rate of population growth per capita).

The population model has 2 levels of description:

- Individual. At this level, the state of the species, age, death, the influence of the external environment on the individual is considered.
- Population-related. At this level, the initial distribution of individuals, the intensity of production, the current distribution of the territory is defined.

There are many biological processes, such as management of renewable resources, inter-species communities, inter-species relationships, interactions between populations, etc. There are three main types of interactions between populations:

- Predator-prey relationship system. A case of decreasing the growth rate of one population and increasing the other [98-99].
- Competition. When the growth rate of both populations decreases.
- Mutualism or symbiosis. Increase in the growth rate of both populations [100].

If we consider a system of differential equations (1), the coefficients of the model will determine the nature of population interactions.

$$\begin{cases} \dfrac{dx_1}{dt} = a_1 x_1 + b_{12} x_1 x_2 - c_1 x_1^2 \\ \dfrac{dx_2}{dt} = a_2 x_2 + b_{21} x_1 x_2 - c_2 x_2^2 \end{cases} \quad (1)$$

The dependence of the types of interactions between populations on the value of the coefficients is shown in Table 10.

Table 10. Types of interaction of species.

	a_i	c_i	
Symbiosis	+	+	$b_{12}, b_{21} > 0$
Commensalism	+	0	$b_{12} > 0, b_{21} = 0$
Predator prey	+	-	$b_{12} > 0, b_{21} < 0$

Amensalism	0	-	$b_{12} = 0, b_{21} < 0$
Competitiveness	-	-	$b_{12}, b_{21} < 0$
Neutralism	-	0	$b_{12} = b_{21} = 0$

This chapter consists of two parts. The first part presents the analysis and development of local models, the second - spatial models. In the presented chapter:

- Interference and operational mathematical models of competition have been developed, considering the change of habitat, time factor, presence of niches, trophic resource, seasonality of competition and other factors. The environmental factor is introduced within each model.
- For the case of two subjects, the stability state of the system is analyzed and the stability is proved. Stability of trivial states is analyzed for distributed systems.

Simulations were carried out within each model developed. Three scenarios are considered:

- The first scenario validates Gause's "competitive exclusion principle" by constructing a population survival distribution density and a concentration ratio distribution density.
- In the second scenario, the influence of the environmental factor on the population system is analyzed and the threshold values of the environmental factor for the appropriate level of concentration of the environment are established.
- In the third scenario, thresholds are set for the number of subjects within the appropriate level of concentration of the environment.

The numerical solution is obtained in Python using the **odeint** module from the scipy.integrate package to solve systems of ordinary differential equations. The **pde** library is used to solve systems of partial differential equations.

§ 1. Local mathematical models of competition

3.1.1. Mathematical models of competition

The first mathematical models of interacting populations in the 1930s were developed by Volterra [7]. His work focuses on the predator-prey system and species that "devour" each other. By the 1980s, numerous mathematical models appeared, which were systematized in Bazykin's work [8]. One of Volterra's first models was that of species struggling for the same food resource. For the case of two types, the model is as follows:

$$\begin{cases} \dfrac{dN_1}{dt} = N_1(\varepsilon_1 - \gamma_1 F(N_1, N_2)) \\ \dfrac{dN_2}{dt} = N_2(\varepsilon_2 - \gamma_2 F(N_1, N_2)) \end{cases}$$

where ε_1 , ε_2 – increase ratio, N_1, N_2 – number of populations, γ_1, γ_2 – constants corresponding to the food needs of species. This model describes the case of extinction of one species while the number of a second species increases in the struggle for food. $F(N_1, N_2)$ – function that describes the level of interaction.

In this model, one of the species dies. For the case of n species, the term mutual competitive exclusion or Gause principle is introduced, since all species die except those that reproduce fastest with minimal resource consumption. Over time, this model has come to be seen as a model of competition.

Later, the model of two species devouring each other came into form [101] (2):

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1}{K_1} - b_{12}\dfrac{N_2}{K_1}\right) \\ \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2}{K_2} - b_{21}\dfrac{N_1}{K_2}\right) \end{cases} \quad (2)$$

where r_1, r_2 – population growth rate, N_1, N_2 – number of populations, K_1, K_2 – medium capacity, b_{12}, b_{21} – measures of the influence of species on each other. This model is considered in the current literature as a model of two types of competition.

3.1.2. Interference competition model

Interference competition is a competition where members of biological or economic communities interact directly, competing for limited resources, obstructing or displacing each other. In animals, competition with intervention is a strategy that is mainly used by larger and stronger organisms in the habitat. Thus, populations with a high level of competitive intervention have cycles of generations managed by adults. The mathematical model (2) by substituting variables $t = \frac{\tau}{r_1}, N_1 = K_1u_1, N_2 = K_2u_2$ allows to bring the system of equations (2) to the form (3):

$$\begin{cases} \frac{du_1}{d\tau} = u_1(1 - u_1 - \gamma_1 u_2) \\ \frac{du_2}{d\tau} = \gamma u_2(1 - u_2 - \gamma_2 u_1) \end{cases} \quad (3)$$

where u_1, u_2 – number of populations, γ_1, γ_2 – parameters that characterize the inhibitory effect on each other. The model considers both external and internal competition.
The system of equations (3) has four stationary points:

1. $u_1 = 0, u_2 = 0.$
2. $u_1 = 1, u_2 = 0.$
3. $u_1 = 0, u_2 = 1.$
4. $u_1 = \frac{1-\gamma_1}{1-\gamma_1\gamma_2}, u_2 = \frac{1-\gamma_2}{1-\gamma_1\gamma_2}$, if $\gamma_1 > 1$ and $\gamma_2 > 1$ or $\gamma_1 < 1$ and $\gamma_2 < 1$.

The first stationary point is unstable, the second will be stable if $\gamma_1 < 1$ and $\gamma_2 > 1$, and third, if $\gamma_1 > 1$ and $\gamma_2 < 1$. The fourth stationary point is implemented and will be stable if $\gamma_1 < 1$ and $\gamma_2 < 1$. At the fourth stationary point, the eigenvalues of the Jacobi matrix on the right side of the equations (3), are the roots of the quadratic equation

$$\lambda^2 + (u_1 + \gamma u_2)\lambda + \gamma(1 - \gamma_1\gamma_2)u_1u_2 = 0$$

and are negative. So, this stationary point will be a steady focus.

Thus, the competition model (3) allows for the simultaneous stable existence of two biological or economic agents at $\gamma_i < 1$. Figure 15 shows the behavior of population numbers over time with different parameters.

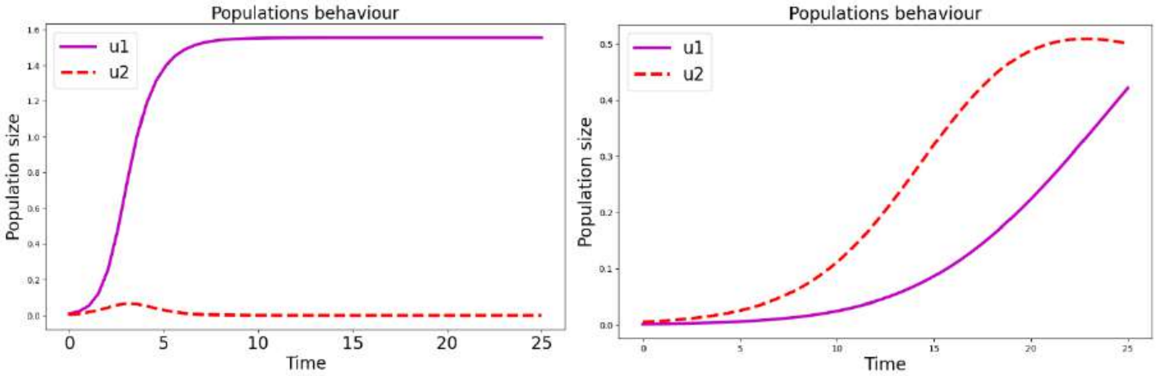


Fig. 15. A - Change in the number of two populations for the case $\gamma = 0.2, \gamma_1 = 1.03, \gamma_2 = 0.8, K_1 = 1.4, K_2 = 1.6,\ u_{10} = 0.002, u_{20} = 0.003$,
B - Change in the number of two populations for the case $\gamma = 0.3, \gamma_1 = 0.6, \gamma_2 = 0.5, K_1 = 1, K_2 = 1.6, u_{10} = 0.001, u_{20} = 0.002$.

Simulation also shows coexistence of populations when the condition for γ is met. In other cases, the principle of competitive exclusion of Gause is observed, namely the death of one of the two species [102].

For the case of n types, model (3) takes the form of (4):

$$\frac{du_i}{dt} = \mu_i u_i \left(1 - u_i - \sum_{j=1, j\neq i}^{n} \gamma_{ij} u_j\right) \quad i = 1, \dots, n \tag{4}$$

where γ_{ij} – parameters characterizing the inhibition of i – th species j-th species. In general, it is considered that$\gamma_{ij} \neq \gamma_{ji}$. In the absence of interaction, at $\gamma_{ij} = 0$, species "exist" independently of each other. The stationary equilibrium position $u_i = 0$ will be unstable and $u_i = 1$ will be stable. The stationary state of the system of equations (4) is found as the solution of the system of equations:

$$1 - u_i - \sum_{j=1, j\neq i}^{n} \gamma_{ij} u_j = 0 \quad (i = 1,2, \dots, n).$$

At low values of the parameters γ_{ij}, due to continuous dependence of the solution on the parameters, the solution of the system of equations (4) will be close to the solution $u_i = 1$ $(i = 1,2, \dots, n)$. The eigenvalues of the Jacobi matrix on the right side of equations (4) will have negative real parts if, according to Gershgorin's theorem [103], for all $i = 1, \dots, n$ inequalities are held $\sum_{j=1, j\neq i}^{n} \gamma_{ij} < 1$.

In this case, the solution of the system of equations (4) with small values of γ_{ij} in the vicinity of point $u_i = 1$ can have a stable solution. If the values of any γ_{ij}, parameters are sufficiently large, as follows from the analysis of the system of equations (3), some populations may die.

In the model (4), competition is determined by the direct contact of individuals in populations. Competition is also influenced by the overall state of the ecosystem, with interspecies interactions unchanged. This can be determined by climatic conditions, anthropogenic pressure and other factors.

The competition of economic entities may depend on the level of technological differences of the entities, market segmentation, resource constraints and growth of needs, etc. In the modern technological structure, new technologies are being introduced quite quickly by most entities under favorable market conditions and the absence of barriers to entry into the market.

Given that the inter-species interaction in the model (4) occurs when at least one of the values of the parameters γ_{ij} becomes positive, the ecosystem factor can be accounted for by introducing the parameter ε:

$$\frac{du_i}{dt} = \mu_i u_i \left(1 - u_i - \varepsilon \sum_{j=1, j \neq i}^{n} \gamma_{ij} u_j \right) \quad i = 1, \dots, n \qquad (5)$$

At $\varepsilon = 0$, there is no interspecies competition, while at $\varepsilon \sum_{j=1,j\neq i}^{n} \gamma_{ij} > 1$, one of the species may disappear, as in the case of the system of equations (4). The parameter ε determines the intensity of competition or the level of environmental impact on individuals of the biological community. In an economic community, this parameter may describe the level of technological difference.

The model (5) allows to explain the disappearance of individual competing species not by their interaction, but by a general change in the state of the ecosystem, which strengthens or weakens the competitive relationship between all species without introducing time dependencies into the right side of the equations (4).

As follows from the analysis of the obtained results at $\varepsilon = 0$, the number of all populations is equal to 1. The emergence of competitive relationships leads to a decrease in the number of all competing populations.

Figure 16 shows the distribution of surviving populations at $\varepsilon = 0.003$ and $\varepsilon = 0.5$. The survival rate of populations decreases with increasing intensity of competition ε, which proves the assumption that the ecosystem is undergoing change in general rather than at the individual population level.

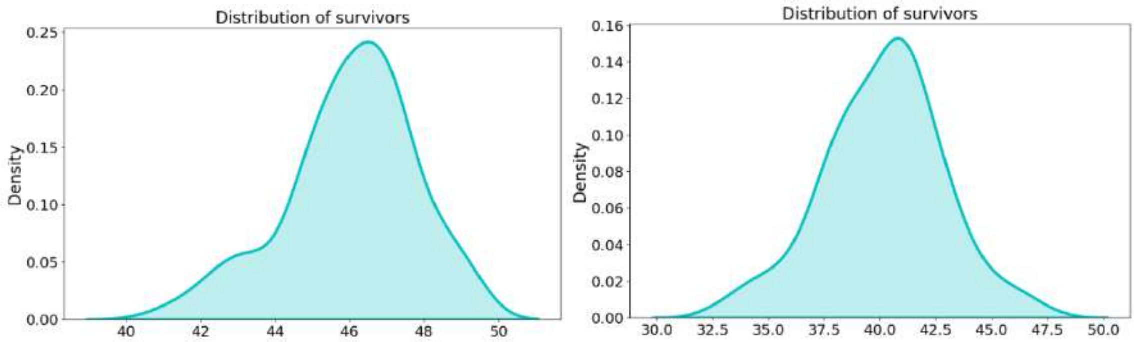


Fig. 16. A – Density of distribution of surviving populations at $\varepsilon = 0.003$,
B – Density of distribution of surviving populations at $\varepsilon = 0.5$.

Scenario No1. Competition of n populations in conditions of low environmental factor. The system of equations includes the parameters μ_i and γ_{ij} $(i,j = 1, 2, \dots, n)$. A specific set of these parameters can be compared with the ecological characteristics of specific populations [104] and the characteristics of the economic environment [105]. By selecting these parameters randomly for n populations from a set interval of variation of these parameters, probabilistic distribution of "survivors" in population competition can be constructed.

Competing species are assumed to live together and have similar ecological characteristics. In terms of one individual, different species will retain approximately the same probability of multiplying, dying, populating free space. Since competition occurs at the individual level rather than at the species level, the parameters μ_i and γ_{ij} are not significantly different in simulations. For competition among economic agents, it is also assumed that these parameters are close.

This scenario presents simulation results for model (5) under low environmental factor intensity conditions. Number of populations $n = 50$, number of experiments $N =$

500. In this scenario, each experiment contains a system solution (5) for 50 populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1},\ \gamma_{ij} = rand_1,\ u_{i0} = rand_1,\ \varepsilon = 0.003$$

where $rand_{0.03,0.1}$ corresponds to a random rational number in the interval $[0.03, 0.1]$, $rand_1$ corresponds to a random rational number in the interval $[0, 1]$.

The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 17 (A) shows the distribution density of the CR_3 concentration coefficient for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.22$, which indicates a low concentration of the competitive environment. Figure 17 (B) shows the population survival density distribution for 500 experiments. The median survival rate is $alive = 47$. Thus, out of 50 populations, 94% of the populations in this scenario survive.

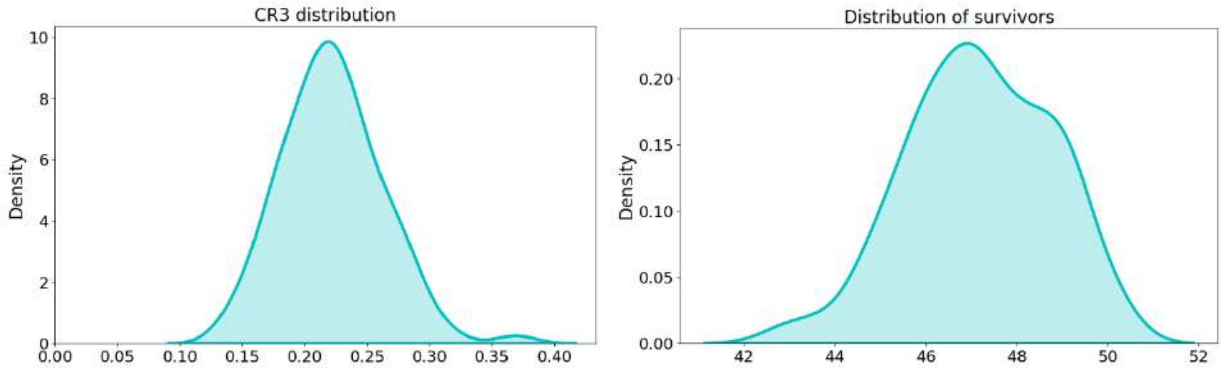


Fig. 17. A – Density of concentration index distribution
B – Density of distribution of surviving populations

Scenario No2. Competition of n populations under conditions of variation of environmental factor intensity. Number of populations $n = 50$, number of experiments $= 500$. The coefficients are generated in the same way, except ε:

$$\varepsilon \in [0.01, 150]$$

This scenario considers the dependence of concentration index and population survival on environmental factors. Figure 18 (A) shows the dependence of the concentration factor on the environmental factor. Figure 18 (B) shows the dependence of the number of surviving populations on the environmental factor. The higher the environmental factor, the lower the survival rate of populations – 46 populations survive at low competition intensity, or 92%. At a high intensity of $\varepsilon = 150$, the survival rate is

10 populations or 20%. With the environmental factor $\varepsilon > 10$, the survival curve (B) begins to decrease.

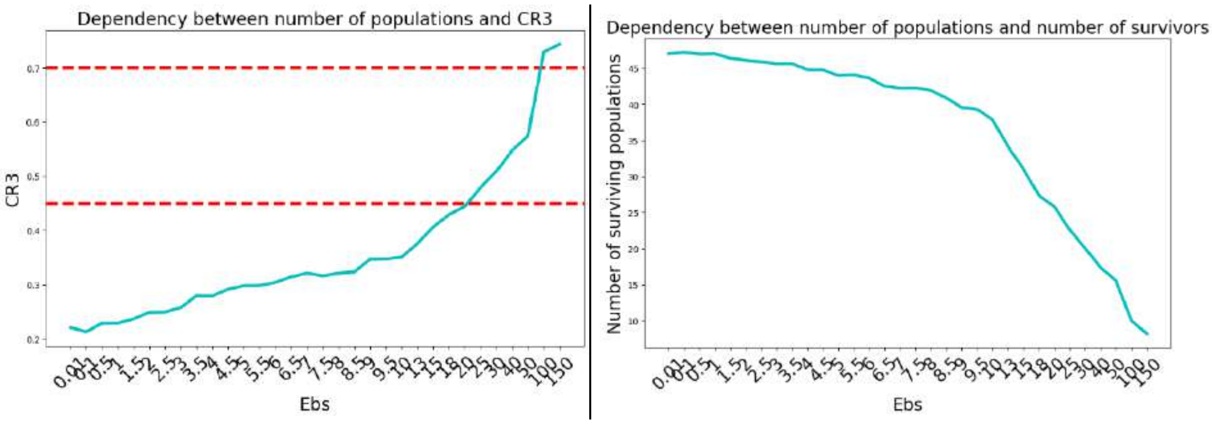


Fig. 18. A – Dependence of the concentration index on the environmental factor
B – Dependence of the number of surviving populations on the environmental factor

The concentration index and survival thresholds for concentration are shown in Table 11.

Table 11. Threshold values of environmental factor and survival rate

Concentration of the environment	Threshold values of the environment factor	Survival thresholds
Low	$\varepsilon < 18$	$alive > 25$
Moderate	$\varepsilon \in [18, 50)$	$alive \in [13, 25)$
High	$\varepsilon > 50$	$alive < 13$

Figure 19 shows the dependence of population size on the environmental factor. The increase of the environmental factor has no effect on the number of populations – the populations remain numerous.

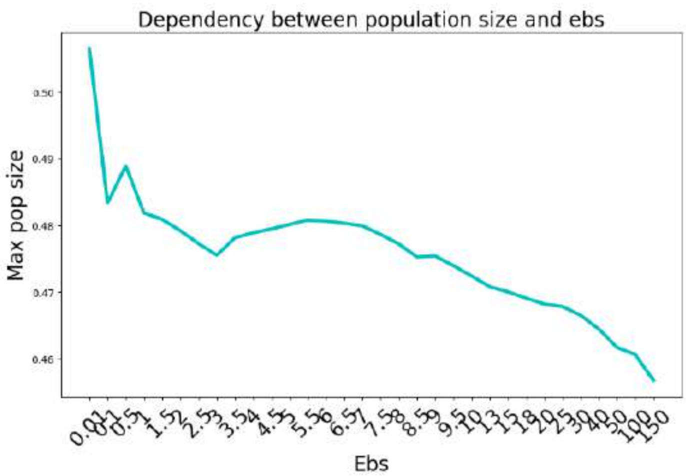


Fig. 19. Dependence of population size on environment factor

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. Each step involves 500 experiments. In this interaction scenario, in each experiment there is a solution of the system (5) for k populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1},\ \gamma_{ij} = rand_1,\ u_{i0} = rand_1,\ \varepsilon = 0.003,\ i = 1, \dots, k$$

This scenario considers the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations in the same way as the previous scenario. Figure 20 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a competitive environment, the required number of participants should be at least 19 subjects. Figure 20 (B) shows the dependence of the number of surviving populations on the number of populations. Survival rates are increasing proportionately.

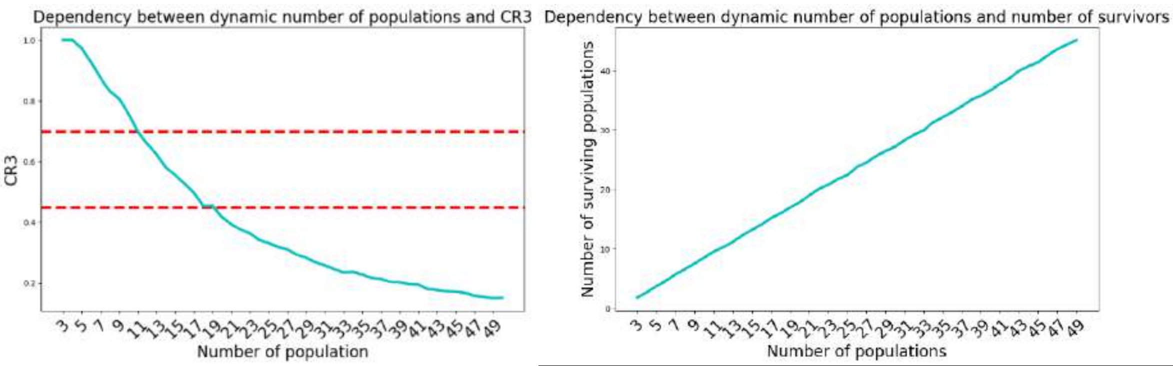


Fig. 20. A – Dependence of the concentration coefficient on the total number of populations, B – Dependence of the number of surviving populations on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 12.

Table 12. Threshold values of the number of subjects

Concentration of the environment	Threshold values of the number of subjects
Low	$k > 19$
Moderate	$k \in [11, 19)$
High	$k < 11$

3.1.3. Hierarchical competition model

The existence of competing species is due to limited resources, different influences of predators on individuals, environmental conditions and many other factors. Individuals in the same population will always have the same needs, so their competition for limited resources becomes more intense. Organisms compete for partners, breeding territory, food, space.

Spatial distribution of populations in an ecosystem can result in overlapping habitats with the emergence of a hierarchical structure of competing populations. Populations living in groups or populations in the same ecosystem can share resources. Individuals receive resources according to their rank and status. Their ranking system is called social hierarchy. Most often, there is a dominant population in the rank system, which occupies the highest rank. High-ranked populations have obvious advantages: they eat first. Dominant males leave more offspring than subordinate males.

The structure of the group provides more opportunities for finding food and increases the chances of survival of individual members in difficult conditions or in the presence of predators. The group provides care for the young individuals of the population through social connections and increases the chances of survival of the populations.

Different types of social hierarchy:

- *Linear hierarchy of dominance.* Each population is recognized as either dominant or subordinate to any other population, creating a linear rank distribution.
- *Complex hierarchy of dominance.* The group has a social structure controlled by the dominant member.

Hierarchies are established by populations or individuals competing for rank. This implies aggression and struggle at an early stage, but as soon as the competitive relationship is stabilized, the struggle stops [107]. The struggle will only begin if a new population or individual emerges that resumes the competitive relationship.

The mathematical model of hierarchical competitive relations is represented by a system of equations:

$$\frac{du_i}{dt} = \mu_i u_i \left(1 - u_i - \varepsilon \sum_{j=1, j\neq i}^{i-1} \gamma_{ij} u_j \right) \quad i = 1, \dots, n \qquad (6)$$

where γ_{ij} – matrix of competitive relationships ($\gamma_{ij} = 0$, if $j \geq i$). Then the matrix has the form:

$$\Upsilon = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ \cdots & \ddots & 0 \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}_{n\times n}$$

In this model, the population ranking first is not influenced by competitors, i.e. the population is dominant. The latter is influenced only by the first population, and so on. The last population is affected by all the populations above the rank. The stationary points of the system of equations (6) are from the recurrent relations:

$$u_1 = 0, u_i = 1 - \varepsilon \sum_{j=1, j\neq i}^{i-1} \gamma_{ij} u_j \quad (i = 2, \dots, n)$$

This assumes that $u_i = 0$, if $1 - \varepsilon \sum_{j=1}^{i-1} \gamma_{ij} u_j \leq 0$. Depending on the values of the parameters $\varepsilon\gamma_{ij}$, the populations that are not the weakest may die.

In Russian and international sources, no mathematical interpretation of the hierarchical model of competition was found.

Scenario No1. Competition of n populations in low environmental factor conditions. This scenario presents simulation results for the model (6) in a low-intensity environment. Number of populations $n = 50$, number of experiments $N = 500$. In this scenario, each experiment contains a system solution (5) for 50 populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1},\ \gamma_{ij} = rand_1,\ u_{i0} = rand_1,\ \varepsilon = 0.003$$

where $rand_{0.03,0.1}$ corresponds to a random rational number in the interval $[0.03, 0.1]$, $rand_1$ corresponds to a random rational number in the interval $[0, 1]$.

The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 21 (A) shows the distribution density of the CR_3

concentration coefficient for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.12$, which indicates a low concentration of the competitive environment. Figure 21 (B) shows the population survival density distribution for 500 experiments. The median survival rate is $alive = 34$. Thus, out of 50 populations, 68% of the populations in this scenario survive.

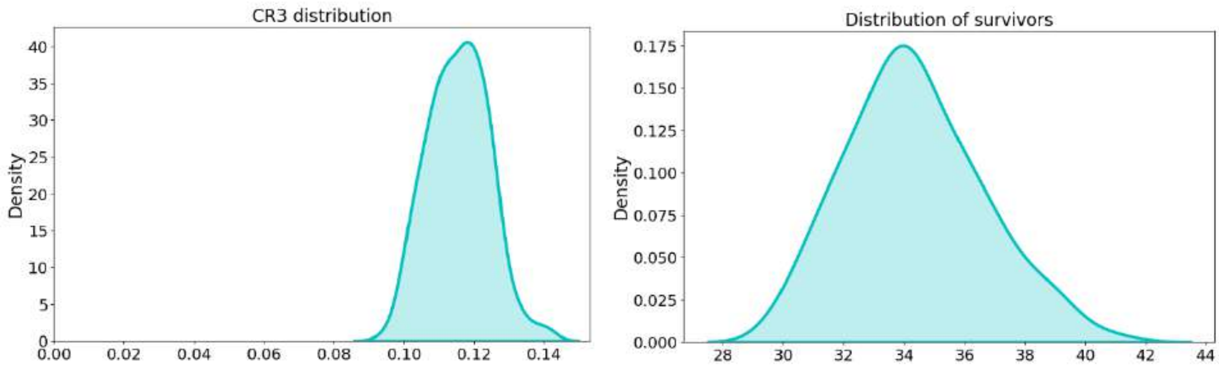
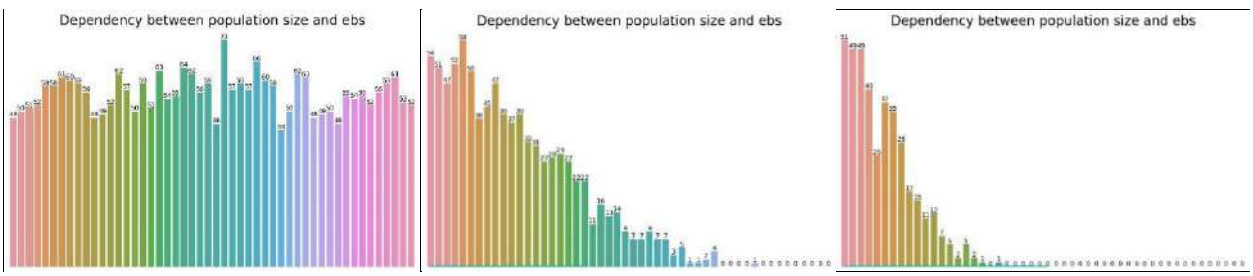


Fig. 21. A - Density of distribution of concentration index,
B - Density of distribution of surviving populations

Figure 22 shows three diagrams of the distribution of surviving populations within positions with low $\varepsilon = 0.003$, medium $\varepsilon = 10$ and high environmental factor $\varepsilon = 50$. The first column corresponds to the rank of the first population in the model, which is not affected by any of the other populations. The second column corresponds to the rank of the second population, which is influenced only by the first population, and so on. The survival probability of a population depends on the environmental factor. At medium and high intensity of the environmental factor, the probability of survival of populations with a rank greater than 20 is low. The fewer populations interact, the more likely they are to survive. When the intensity level is $\varepsilon > 100$, only the dominant, i.e., the population with the highest rank, survives.



(a) (б) (в)

Fig. 22. Density of distribution of surviving populations at $\varepsilon = 0.003, \varepsilon = 10$ and at $\varepsilon = 50$

Scenario No2. Competition of n populations under conditions of variation in the intensity of the environmental factor. Number of populations $n = 50$, number of experiments $N = 500$. The coefficients are generated in the same way, except for ε:

$$\varepsilon \in [0.01, 150]$$

This scenario considers the dependence of concentration index and population survival on environmental factors. Figure 23 (A) shows the dependence of the concentration factor on the environment factor. The hiarchical model is suitable for describing ecological or economic communities in which competitive relationships are pronounced, as opposed to the interference model. Figure 23 (B) shows the dependence of the number of surviving populations on the environmental factor. At a low competition intensity, 18 populations survive, which accounts to 36%. This supports the assumption of a highly concentrated environment in the case of a hierarchical model. At a high intensity of $\varepsilon = 150$, the survival rate is 0 populations, i.e., all populations die out.

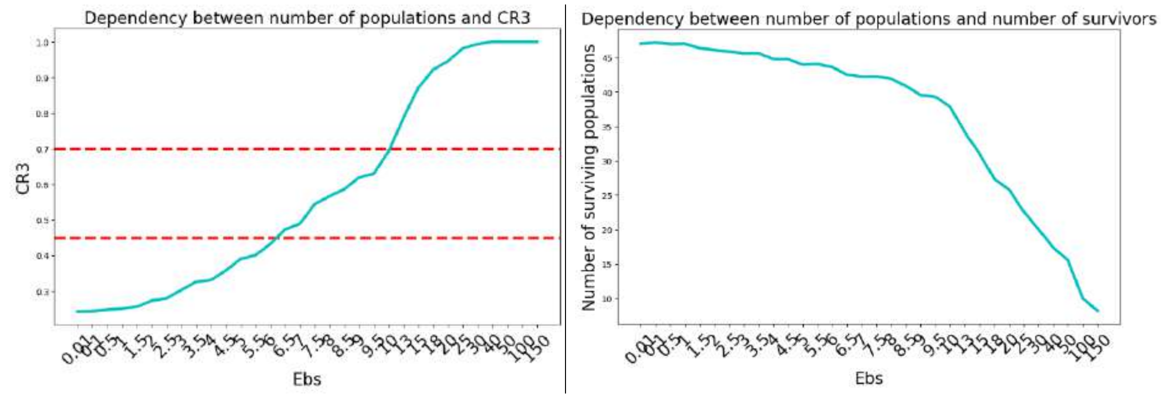


Fig. 23. A – Dependence of the concentration coefficient on the environment factor
B – Dependence of the number of surviving populations on the environment factor

The concentration index and survival thresholds for concentration of environment are shown in Table 13.

Table 13. Threshold values of environmental factor and survival rate

Concentration of the environment	Threshold values of the environment factor	Survival thresholds
Low	$\varepsilon < 6$	$alive > 7$
Moderate	$\varepsilon \in [6, 10]$	$alive \in [3, 7]$
High	$\varepsilon > 10$	$alive < 3$

Figure 24 shows the dependence of population size on the environmental factor. The curve decreases monotonically. Despite the decreasing curve, all populations remain small within the model.

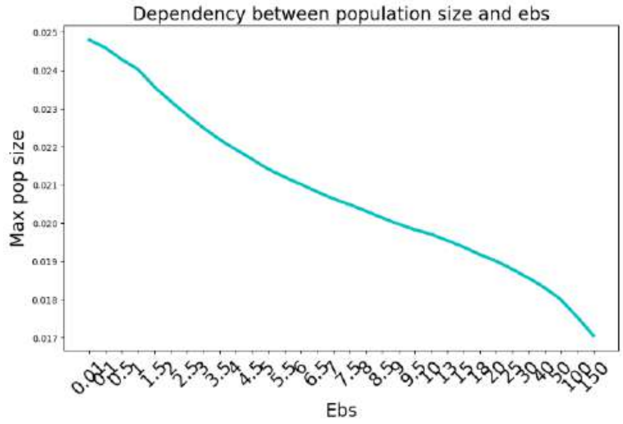


Fig. 24. Dependence of population size on environment factor

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. Each step involves 500 experiments. In this interaction scenario, in each experiment there is a solution of the system (6) for k populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1},\ \gamma_{ij} = rand_1,\ u_{i0} = rand_1,\ \varepsilon = 0.003,\ i = 1, \dots, k$$

This scenario considers the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations in the same way as the previous scenario. Figure 25 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a competitive environment, the required number of participants should be at least 25 subjects. Figure 25 (B) shows the dependence of the number of surviving populations on the number of populations. Survival rates are increasing proportionately.

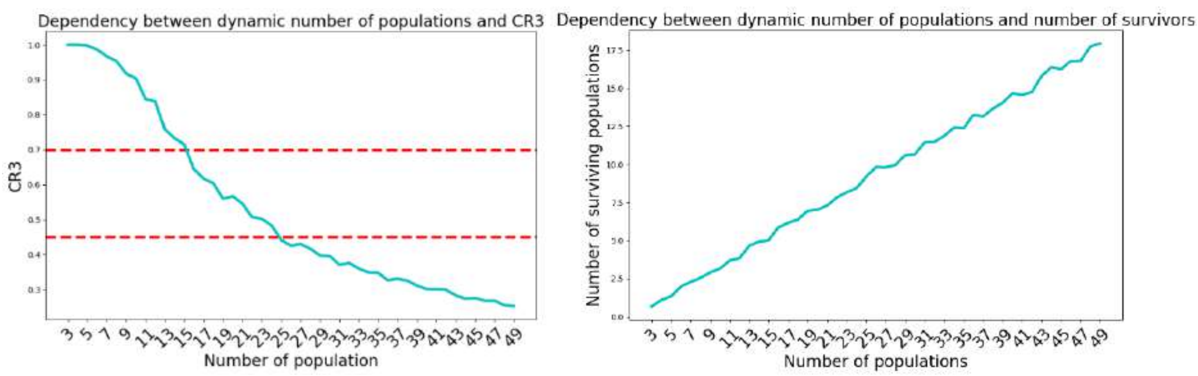


Fig. 25. A – Dependence of the concentration index on the total number of populations,
B – Dependence of the number of surviving populations on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 14.

Table 14. Threshold values of the number of subjects

Concentration of the environment	Threshold values of the number of subjects
Low	$k > 25$
Moderate	$k \in [15, 25]$
High	$k < 15$

3.1.4. Fragmented competition model

Not all participants of the ecological or economic community may be participants of competitive interaction. Let $\frac{u_1}{b_1+u_1}$ part of the first community and $\frac{u_2}{b_2+u_2}$ part of the second community participate in the competition. This is the main difference between a fragmentary model and an interference model of competition. In this case, the model (5) takes the form (7):

$$\begin{cases} \dfrac{du_1}{d\tau} = \mu_1 u_1 \left(1 - u_1 - a_1 \dfrac{u_1}{b_1 + u_1} \dfrac{u_2}{b_2 + u_2}\right) \\ \dfrac{du_2}{d\tau} = \mu_2 u_2 \left(1 - u_2 - a_2 \dfrac{u_1}{b_1 + u_1} \dfrac{u_2}{b_2 + u_2}\right) \end{cases} \tag{7}$$

where a_1, a_2, b_1 and b_2 are non-negative parameters. At stationary points:

- $u_1 = 0, u_2 = 0$
- $u_1 = 1, u_2 = 0$
- $u_1 = 0, u_2 = 1$

The Jacobi matrix on the right side of equations (7) has a positive value. Therefore, these points are unstable. Nontrivial stationary points are found from a system of equations (8):

$$\begin{cases} u_2 = 1 + \dfrac{a_2}{a_1}(u_1 - 1) \\ 1 - u_1 - a_1 \dfrac{u_1}{b_1 + u_1} \dfrac{1 + \frac{a_2}{a_1}(u_1 - 1)}{b_2 + 1 + \frac{a_2}{a_1}(u_1 - 1)} = 0 \end{cases} \tag{8}$$

The left part of the second equation when $u_1 = 0$ takes a positive value, and when $u_1 = 1$ takes a negative value. Therefore, the second equation has roots at the interval $0 < u_1 < 1$. And from the first equation in (8) it follows, that the physical meaning will have such a solution $u\_1 = u_1^*$, which satisfies the condition $u_1^* > 1 - \frac{a_1}{a_2}$. When $u_1 = 1 - \frac{a_1}{a_2}$, the right part of the second equation in (8) takes positive value equals $\frac{a_1}{a_2}$. This means that the system of equations has a solution such as $0 < u_1 < 1$ and $0 < u_2 < 1$.

The components of the Jacobi matrix at this stationary point are counted according to the formulas:

$$J_{11} = -\mu_1 u_1 \left(1 + a_1 \frac{b_1}{(b_1 + u_1)^2} \frac{u_2}{b_2 + u_2}\right)$$

$$J_{12} = -\mu_1 u_1 a_1 \frac{b_2}{(b_2 + u_2)^2} \frac{u_1}{b_1 + u_1}$$

$$J_{21} = -\mu_2 u_2 a_2 \frac{b_1}{(b_1 + u_1)^2} \frac{u_2}{b_2 + u_2}$$

$$J_{22} = -\mu_2 u_2 \left(1 + a_2 \frac{b_2}{(b_2 + u_2)^2} \frac{u_1}{b_1 + u_1}\right)$$

The sum of the eigenvalues of this matrix $\lambda_1 + \lambda_2 = -(J_{11} + J_{22})$ takes a negative value, and the product is positive:

$$\lambda_1 \lambda_2 = \mu_1 u_1 \mu_2 u_2 \left(1 + a_1 \frac{b_1}{(b_1 + u_1)^2} \frac{u_2}{b_2 + u_2}\right)\left(1 + a_2 \frac{b_2}{(b_2 + u_2)^2} \frac{u_1}{b_1 + u_1}\right)$$

Accordingly, this stationary point will be stable.

For n populations, the model takes the form:

$$\frac{du_i}{dt} = \mu_i u_i \left(1 - u_i - \varepsilon \frac{u_i}{b_i + u_i} \sum_{j=1,}^{n} \gamma_{ij} \frac{u_j}{b_j + u_j}\right) \quad i = 1, \dots, n \qquad (9)$$

$$a_{ii} = 0, A = \{\gamma_{ij}\}$$

where A – population interaction matrix, ε – environment factor.

Scenario No1. Competition of n populations in low environmental factor conditions. This scenario presents simulation results for the model (9) at low intensity. Number of populations $n = 50$, number of experiments $N = 500$. In this interaction scenario, in each experiment there is a system solution (9) for 50 populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1}, b_i = rand_1, \gamma_{ij} = matrix(rand_1), u_{i0} = rand_{0.01}, \varepsilon = 0.003$$

where $rand_{0.03,0.1}$ corresponds to a random rational number in the interval $[0.03, 0.1]$, $rand_1$ corresponds to a random rational number in the interval $[0, 1]$, $rand_{0.01}$ corresponds to a random rational number in the interval $[0, 0.01]$, $matrix(rand_1)$ is a matrix $n \times n$ with random rational numbers in the interval $[0,1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 26 (A) shows the distribution density of the CR_3 concentration coefficient for 500 experiments.

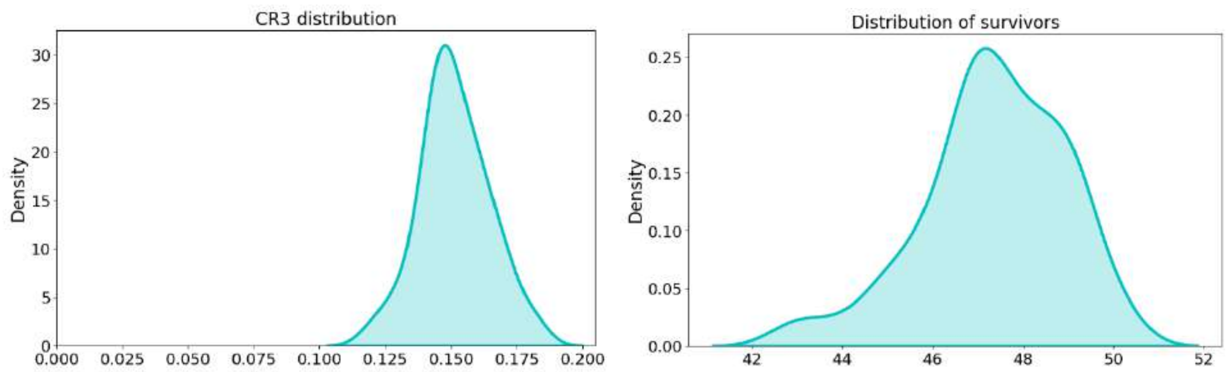


Fig. 26. A - Density of distribution of concentration index,
B - Density of distribution of surviving populations

The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.15$, which indicates a low concentration of the competitive environment. Figure 26 (B) shows the population survival density distribution for 500 experiments. The median survival rate is $alive = 47$. Thus, out of 50 populations, 94% of the populations in this scenario survive.

Scenario No2. Competition of n populations under conditions of variation in the intensity of the environmental factor. Number of populations $n = 50$, number of experiments $N = 500$. The coefficients are generated in the same way, except for ε:

$$\varepsilon \in [0.01, 150]$$

This scenario considers the dependence of the concentration index on the intensity of environmental factor, as well as the dependence of population survival on the intensity of environmental factor.

Figure 27 (A) shows the dependence of the concentration coefficient on the intensity of competition. It can be assumed that in this model the threshold of competition intensity is high. The fragmentary model is suitable for describing weakly or moderately competitive communities.

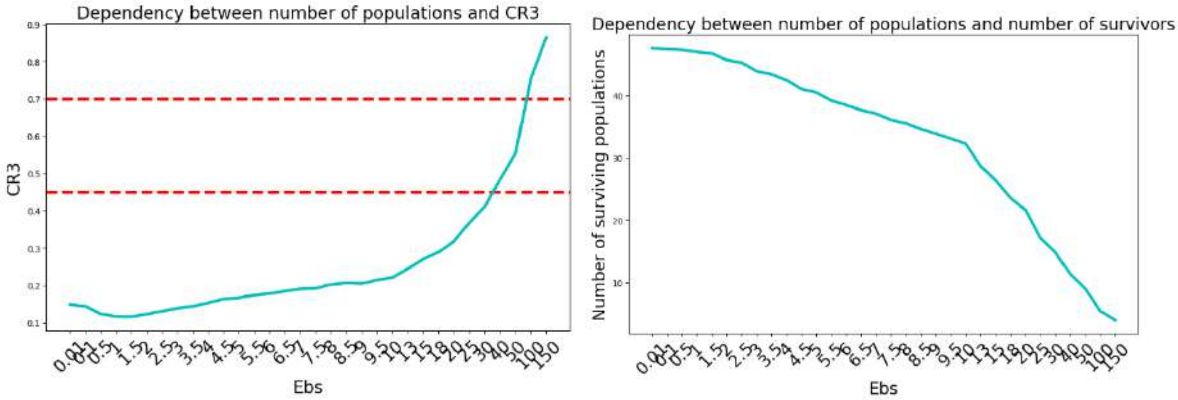


Fig. 27. A – Dependence of the concentration coefficient on the environmental factor
B – Dependence of the number of surviving populations on the environmental factor

Figure 27 (B) shows the dependence of the number of surviving populations on the environmental factor. The higher the intensity of competition, the lower the survival rate of populations - with low intensity of competition, 46 populations survive, which is 92%. At high intensity $\varepsilon = 150$, the survival rate is 2 populations, or 4%, that is, almost all populations die. At $\varepsilon = 10$, a change in the slope of the curve is observed, followed by an increase in the rate of decrease of the curve. For the fragmented competition model, the dependence of threshold values on the level of environmental concentration is presented in Table 15.

Table 15. Threshold values of environmental factors and survival rates

Concentrated environment	Threshold values of environmental factors	Survival thresholds
Low	$\varepsilon < 30$	$alive > 13$
Moderate	$\varepsilon \in [30, 100)$	$alive \in [3, 13)$
High	$\varepsilon > 100$	$alive < 3$

Figure 28 shows the dependence of population numbers on environmental factors. The population size decreases monotonically with increasing environmental factors. In the fragmentary model, with increasing intensity, medium-sized populations become small in number.

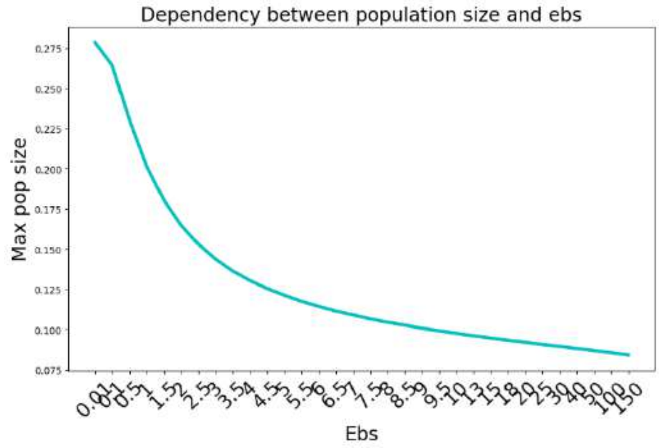


Fig. 28. Dependence of population numbers on environmental factors

Scenario No2. Dynamic simulation modeling. Number of populations $n = 50$, number of experiments $N = 500$. At the first step, 2 populations are considered, at the second step, 3 populations, and so on until 50 populations are considered. At each step, 500 experiments are carried out and the average value is selected. In this interaction scenario, in each experiment, a solution to system (9) is found for k populations with the following generation of coefficients:

$$\mu_i = rand_{0.03,0.1},\ b_i = rand_{0.1}, \gamma_{ij} = matrix(rand_{0.1}),$$

$$u_{i0} = rand_{0.01},\ \varepsilon = 0.003,\ i = 1, \dots, k$$

This scenario examines the dependence of the concentration index on the total number of populations, as well as the dependence of the survival rate of species on the total number of populations in a similar manner relative to the previous scenario. Figure 29 (A) shows the dependence of the concentration coefficient on the total number of populations. That is, in order to ensure competitive environmental conditions, the required number of competition participants must be at least 14 entities within the framework of a fragmented model. The fragmented model is suitable for moderately or weakly competitive communities.

Figure 29 (B) shows the dependence of the number of surviving populations on the number of populations. Linear growth is observed. The survival rate increases proportionally. In the fragmented model, every 10th population dies.

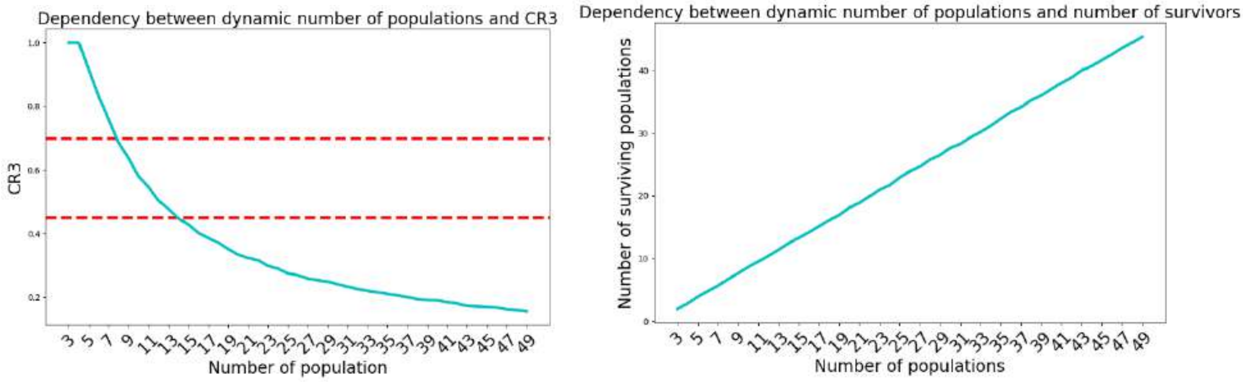


Fig. 29. A – Dependence of the concentration coefficient on the total number of populations, B – Dependence of the number of surviving populations on the total number of populations

Threshold values for the number of subjects to determine the concentration of the environment are given in table 16.

Table 16. Threshold values for the number of subjects

Concentrated environment	Subject Number Thresholds
Low	$k > 14$
Moderate	$k \in [8, 14]$
High	$k < 8$

3.1.5. Resource competition model

The generalized Volterra model does not consider the resource consumed by both populations. Model (4), in fact, is a model of mutual destruction of two populations. Let S be a trophic resource used by populations. Then the model of competition between two populations on a trophic resource can be represented by a system of equations (10):

$$\begin{cases} \dfrac{du_1}{d\tau} = \mu_1 u_1 \left(1 - \dfrac{u_1}{K_1} - \dfrac{\gamma_{12} u_2 S}{S + b_{12}}\right) \\ \dfrac{du_2}{d\tau} = \mu_2 u_2 \left(1 - \dfrac{u_2}{K_2} - \dfrac{\gamma_{21} u_1 S}{S + b_{21}}\right) \\ \dfrac{dS}{dt} = \mu_s S \left(1 - \dfrac{S}{K_s}\right) - \left(\alpha_1 \mu_1 \dfrac{\gamma_{12} S}{S + b_{12}}\right) u_1 u_2 \end{cases} \tag{10}$$

In this model, in the absence of a resource (at $S = 0$), populations exist independently of each other. System of equations (10) has four stationary points:

1. $u_1 = 0, u_2 = 0, S = K_s$. Unstable stationary point.
2. $u_1 = 1, u_2 = 1, S = 0$. At this stationary point, the eigenvalues of the Jacobian matrix on the right side of equations (10) will be $\lambda_1 = -\mu_1,\ \lambda_2 = -\mu_2, \lambda_s = \mu_s - (\frac{\alpha_1}{b_1} + \frac{\alpha_2}{b_2})$. If the inequality $\mu_s > \frac{\alpha_1}{b_1} + \frac{\alpha_2}{b_2}$ is satisfied, the stationary point will be unstable. Since the parameters α_1, α_2 and b_1, b_2 characterize the rate of resource consumption, then at low specific rates of resource replenishment μ_s it will be exhausted.
3. $u_1 = 1, u_2 = 0, S = K_s$. At this stationary point, the eigenvalues of the Jacobian matrix of the right side of the equations will be $\lambda_1 = -\mu_1$, $\lambda_2 = \mu_2 \left(1 - \gamma_2 \frac{K_s}{b_2 + K_s}\right), \lambda_s = -\mu_s$. When the inequality $1 < \gamma_2 \frac{K_s}{b_2 + K_s}$ is satisfied this stationary point will be stable. That is, the second species dies. In model (4) the second type survives if the inequality $\gamma_2 < 1$ is satisfied, in model (10) - if the inequality $\gamma_2 < 1 + \frac{b_2}{K_s}$ is satisfied. It follows that the survival threshold (the upper limit of the parameter γ_2) increases with a decrease in the capacity of the trophic resource environment (at $K_s \to 0$).

4. $u_1 \neq 0, u_2 \neq 0$. The stationary point at which $u_1 \neq 0, u_2 \neq 0$ is found as a solution to the system of equations:

$$\begin{cases} u_1 = \frac{1}{d}\left(1 - \gamma_1 \frac{S}{b_1 + S}\right) \\ u_2 = \frac{1}{d}\left(1 - \gamma_2 \frac{S}{b_2 + S}\right) \\ \mu_s S\left(1 - \frac{S}{K_s}\right) - \frac{1}{d^2}\left(\alpha_1 \frac{S u_1}{b_1 + S} + \alpha_2 \frac{S u_2}{b_2 + S}\right) = 0 \end{cases}$$

where $d = 1 - \gamma_2 \frac{S}{b_2+S} \gamma_1 \frac{S}{b_1+S}$. The function

$f(S) = \mu_s S\left(1 - \frac{S}{K_s}\right) - \frac{1}{d^2}\left(\alpha_1 \frac{S u_1}{b_1+S} + \alpha_2 \frac{S u_2}{b_2+S}\right)$ in $S = 0$, goes to (10). If $S \to \infty, f(S) \to \infty$.

$$f(0) = \mu_s - \left(\frac{\alpha_1}{b_1} + \frac{\alpha_2}{b_2}\right) \tag{11}$$

Therefore, if the inequalities$\gamma_1 \frac{K_s}{b_1+K_s} < 1, \gamma_2 \frac{K_s}{b_2+K_s} < 1, \frac{\alpha_1}{b_1} + \frac{\alpha_2}{b_2} < \mu_s$ are satisfied, the system of equations (10) will have at least one solution lying in the first octant. That is, with the abundance of the resource and its rapid restoration.

As an example of resource exhaustion, presented in Figure 30, we consider the case $\mu_1 = 1, \mu_2 = 0.5, \mu_s = 0.18, b_1 = 0.015, b_2 = 0.2, \alpha_1 = 0.07, \alpha_2 = 0.06, K_s = 2.0$, and coefficients γ_1 and γ_2 depend on time $\gamma_1 = \frac{1-\sin(\pi\omega t)}{2}, \gamma_2 = \frac{1-\cos(\omega t)}{4}, \omega = 0.02$. For a given set of constants, the inequality holds $\mu_s < \frac{\alpha_1}{b_1} + \frac{\alpha_2}{b_2}$. The figure shows the dependence of the functions $u_1(t), u_2(t), S(t)$ on time. Over time, the resource is exhausted and competition ceases.

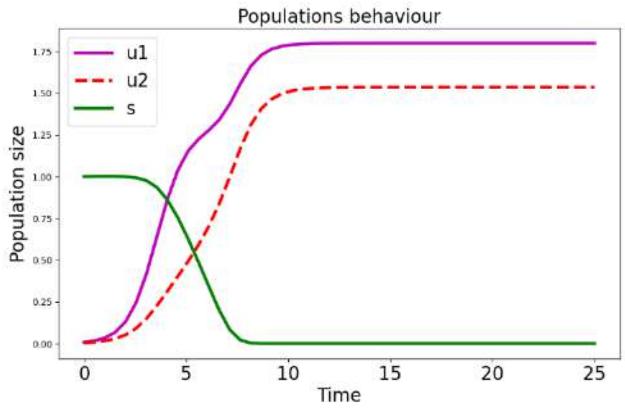


Fig. 30. Changes in the size of two populations over time

Let us assume that n populations participate in competitive relationships on trophic resource S. Interference competition, considering the resource, is described by a system of differential equations (12):

$$\begin{cases} \dfrac{du_i}{d\tau} = \mu_i u_i \left(1 - \dfrac{u_i}{K_i} - \varepsilon \displaystyle\sum_{j=1, j\neq i}^{n} \dfrac{\gamma_{ij} u_j S}{S + b_{ij}}\right) \\ \dfrac{dS}{dt} = \mu_s S \left(1 - \dfrac{S}{K_s}\right) - \varepsilon \displaystyle\sum_{i=1}^{n} \alpha_i \mu_i u_i \sum_{j=1, j\neq i}^{n} \gamma_{ij} \dfrac{S}{b_{ij} + S} u_j \end{cases} \quad i = 1, \dots, n \qquad (12)$$

Scenario No1. Competition of n populations under conditions of low environmental factor. Today's system of economic relations, production technologies, personnel training, etc allow participants in the production and sale of goods to have similar rates of production and sales of products. The same is true for the biological competitive environment. That is, within the framework of model (12), we can assume that the values of the coefficients $\mu_i, b_i, \alpha_i, K_i$ $(i = 1,2, \dots, n)$ are close. Therefore, during the analysis, we select their values randomly:

$$\mu_i = rand_{0.03,0.1},\ K_i = rand_{0.1},\ \alpha_i = rand_{0.1},\ u_{i0} = rand_{0.01},$$
$$b_{ij} = matrix(rand_{0.1}),\ \gamma_{ij} = matrix(rand_{0.1}),\ \varepsilon = 0.003,$$
$$\mu_s = rand_{0.1},\ K_s = rand_{0.1},\ i = 1, \dots, n$$

where $rand_1$ corresponds to a random rational number in the interval $[0, 1]$, $rand_{0.01}$ corresponds to a random rational number in the interval $[0, 0.01]$, $matrix(rand_1)$ is an $n \times n$ matrix with random rational numbers in the interval $[0, 1]$.

The distribution density of the concentration index CR_3 and the survival rate of populations is constructed. Figure 31 (A) shows the density distribution of the concentration coefficient CR_3 for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.15$, which indicates a low concentration of the competitive environment. Figure 31 (B) shows the density distribution of the population survival rate for 500 experiments. The average survival rate corresponds to $alive = 49$. Thus, out of 50 populations, 96% of the populations survive in this scenario.

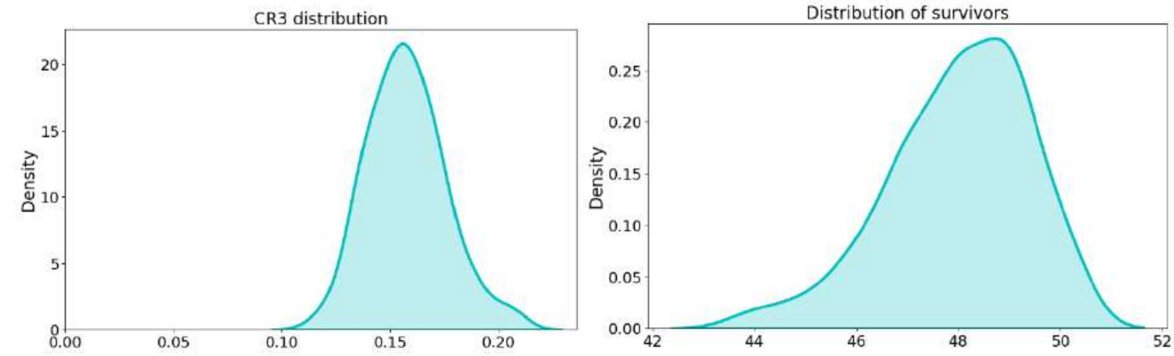


Fig. 31. A – Distribution density of the concentration index
B – Density of distribution of surviving populations

Scenario No2. Competition of n populations under varying intensity of environmental factors. Number of populations $n = 50$, number of experiments $N = 500$. The coefficients are generated in a similar way, except for ε:

$$\varepsilon \in [0.01, 150]$$

This scenario examines the dependence of the concentration index on the intensity of competition, as well as the dependence of the survival rate of populations on the intensity of competition. Figure 32 (A) shows the dependence of the concentration coefficient on the environmental factor. In this model, the competition intensity threshold is very high for a low concentrated environment. This model is suitable for describing weakly competing communities and communities with simultaneous coexistence of species. This indicates a stable competitive environment.

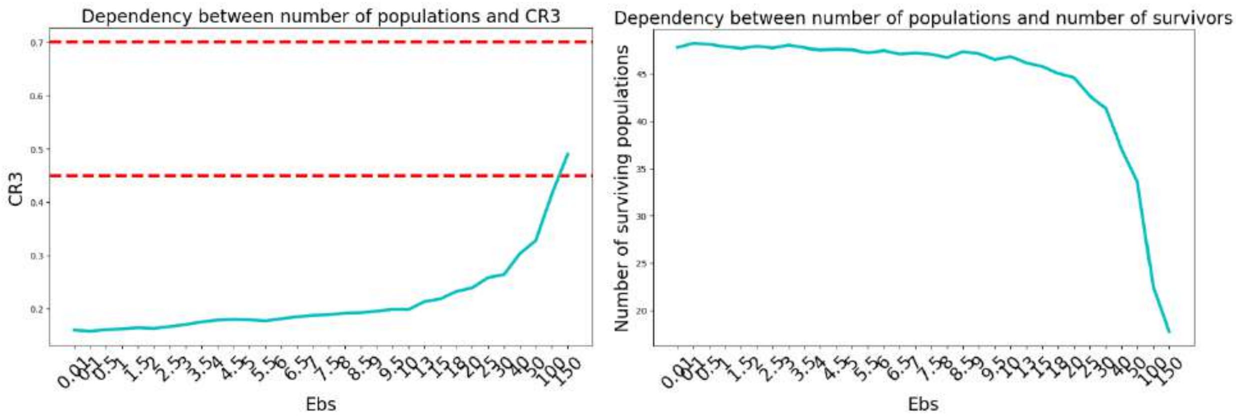


Fig. 32. A – Dependence of the concentration coefficient on the environmental factor
B – Dependence of the number of surviving populations on environmental factors

Figure 32 (B) shows the dependence of the number of surviving populations on the environmental factor. When $\varepsilon < 20$, the survival rate is stable and lies in the range [45, 47] or 90 – 94%. When $\varepsilon \geq 20$ there is a sharp decrease in the survival curve. The lower bound for the model is 10 population, which is 20%.

Figure 33 shows the dependence of population numbers on environmental factors. It is difficult to identify patterns in the data presented in the figure, but it can be argued that the population size does not change and the population remains average in size.

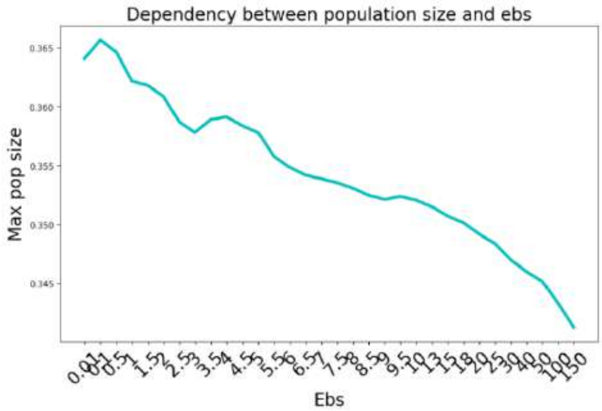


Fig. 33. Dependence of population numbers on environmental factors

Scenario No3. Dynamic simulation modeling. Number of populations $n = 50$, number of experiments $N = 500$. At the first step, 1 population is considered, at the second step, 2 populations, and so on until 50 populations are considered. At each step, 500 experiments are carried out and the average value is selected. In this interaction scenario, in each experiment a solution to system (12) is found for k populations with similar generation of coefficients.

$$\mu_i = rand_{0.03,0.1}, K_i = rand_{0.1}, \alpha_i = rand_{0.1}, u_{i0} = rand_{0.01}$$

$$b_{ij} = matrix(rand_{0.1}), \gamma_{ij} = matrix(rand_{0.1}), \varepsilon = 0.003,$$

$$\mu_s = rand_{0.03,0.1}, K_s = rand_{0.1}, i = 1, \dots, k$$

This scenario examines the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations in a similar manner relative to the previous scenario. Figure 34 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure stable competitive environmental conditions, the required number of competitors must be at least 14 within the model. Figure 34 (B) shows the dependence of the number of surviving populations on the number of populations. A linear relationship is observed.

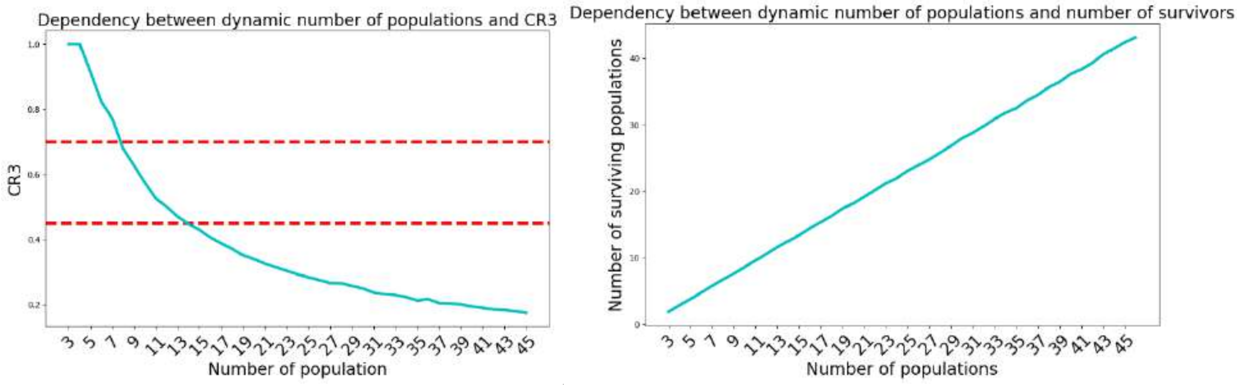


Fig. 34. A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

Threshold values for the number of subjects to determine the concentration of the environment are given in Table 17.

Table 17. Threshold values for the number of subjects

Concentrated environment	Subject Number Thresholds
Low	$k > 14$
Moderate	$k \in [8, 14]$
High	$k < 8$

3.1.6. Operational competition model

Single-population operational model. An alternative mathematical model is proposed, where resource (13) acts as a separate differential equation. Unlike the interference model, in this model there is no direct interaction between populations. In this model, competition occurs due to resource consumption. The mathematical model (13) involves one population u, which consumes the resource S.

$$\begin{cases} \dfrac{du}{dt} = \mu u\left(\dfrac{s}{b+s} - u\right) \\ \dfrac{ds}{dt} = \mu_s s(1-s) - \gamma\mu\dfrac{us}{b+s} \end{cases} \tag{13}$$

where u – population size; μ – specific population growth rate; s – amount of resource; γ – resource processing or consumption ratio; $f(s) = \frac{s}{b+s}$ – a function characterizing the influence of the amount of resource on the rate of its consumption; μ_s – specific resource growth rate. All constants are positive.

The Jacobian matrix looks like:

$$J = \begin{pmatrix} \mu\left(\dfrac{s}{b+s} - u\right) - \mu u & \dfrac{\mu u b}{(b+s)^2} \\ -\dfrac{\gamma\mu s}{b+s} & \mu_s(1-2s) - \dfrac{\gamma\mu u s}{(b+s)^2} \end{pmatrix}$$

Stationary points are:

1. u=0, s=0. At this stationary point, the Jacobian matrix $J = \begin{pmatrix} 0 & 0 \\ 0 & \mu_s \end{pmatrix}$ has a positive eigenvalue. Therefore, the stationary point is unstable.
2. u=0, s=1. At this stationary point, the Jacobian matrix $J = \begin{pmatrix} \mu\frac{1}{b+1} & 0 \\ -\frac{\gamma\mu}{b+1} & -\mu_s \end{pmatrix}$ has a positive eigenvalue. Therefore, the stationary point is unstable.
3. $u = \frac{s}{b+s}$, $f(s) = \mu_s(1-s) - \gamma\mu\frac{s}{(b+s)^2}$.

The nontrivial stationary value s is found as the root of the equation:

$$1 - \frac{s}{\mu_s} - \gamma\mu\frac{s}{\mu_s(b+s)^2} = 0 \tag{14}$$

Equation (14), depending on the parameter values, can have one or three roots. Since stationary points 1 and 2 are unstable and the solution does not go to infinity, it means that one of these three points will be stable according to the Lagrange criterion.

Operational model involving two populations. Expanding model (12) to two populations, we obtain a system of differential equations (15):

$$\begin{cases} \dfrac{du_1}{dt} = \mu_1 u_1 \left(\dfrac{S}{S+b_1} - u_1\right) \\ \dfrac{du_2}{dt} = \mu_2 u_2 \left(\dfrac{S}{S+b_2} - u_2\right) \\ \dfrac{dS}{dt} = -\gamma_1 \mu_1 u_1 \dfrac{S}{S+b_1} - \gamma_2 \mu_2 u_2 \dfrac{S}{S+b_2} + \mu_s S\left(1 - \dfrac{S}{K}\right) \end{cases} \tag{15}$$

where u_1, u_2 – volume of the first and second populations; μ_1, μ_2 – specific growth rate of the first and second populations; S – amount of resource; K – medium capacity (maximum amount of resource that can be produced); γ_1, γ_2 – resource processing or consumption ratio; $f(s) = \frac{S}{b+S}$ – a function characterizing the influence of the amount of resource on the rate of its consumption; μ_s – specific resource growth rate. All constants are positive. If the resource is not replenished, then in the third equation $\mu_s = 0$.

The Jacobian matrix looks like:

$$J = \begin{pmatrix} \mu_1\left(\dfrac{s}{b_1+s} - 2u_1\right) & 0 & \dfrac{u_1\mu_1 b_1}{(b_1+s)^2} \\ 0 & \mu_2\left(\dfrac{s}{b_2+s} - 2u_2\right) & \dfrac{u_2\mu_2 b_2}{(b_2+s)^2} \\ -\dfrac{\gamma_1\mu_1 s}{b_1+s} & -\dfrac{\gamma_2\mu_2 s}{b_2+s} & -\dfrac{\gamma_1\mu_1 b_1 u_1}{(b_1+s)^2} - \dfrac{\gamma_2\mu_2 b_2 u_2}{(b_2+s)^2} + \mu_s\left(1 - 2\dfrac{s}{K}\right) \end{pmatrix}$$

Stationary points are:

1. $u_1 = 0, u_2 = 0, s = 0$. At this stationary point, the Jacobian matrix $J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_s \end{pmatrix}$ has a positive eigenvalue. Therefore, the stationary point is unstable.

2. $u_1 = 0, u_2 = 0, s = K$. At this stationary point, the Jacobian matrix

$$J = \begin{pmatrix} \mu_1 \frac{K}{b_1 + K} & 0 & 0 \\ 0 & \mu_2 \frac{K}{b_2 + K} & 0 \\ -\frac{\gamma_1 \mu_1 K}{b_1 + K} & -\frac{\gamma_2 \mu_2 K}{b_2 + K} & -\mu_s \end{pmatrix}$$

has a positive eigenvalue. Therefore, the stationary point is unstable.

3. $u_1 = 0, u_2 = \frac{s}{s+b_2}, f(s) = -\gamma_2 \mu_2 \frac{s}{(s+b_2)^2} + \mu_s \left(1 - \frac{s}{K}\right)$. At this stationary point, the Jacobian matrix

$$J = \begin{pmatrix} \mu_1 \left(\frac{s}{b_1 + s}\right) & 0 & 0 \\ 0 & -\mu_2 \frac{s}{b_2 + s} & \frac{s\mu_2 b_2}{(b_2 + s)^3} \\ -\frac{\gamma_1 \mu_1 s}{b_1 + s} & -\frac{\gamma_2 \mu_2 s}{b_2 + s} & -\frac{\gamma_2 \mu_2 b_2 s}{(b_2 + s)^3} + \mu_s \left(1 - 2\frac{s}{K}\right) \end{pmatrix}$$

has a positive eigenvalue. Therefore, the stationary point is unstable.

4. $u_2 = 0, u_1 = \frac{s}{s+b_1}, f(s) = -\gamma_1 \mu_1 \frac{s}{(s+b_1)^2} + \mu_s \left(1 - \frac{s}{K}\right)$. At this stationary point, the Jacobian matrix

$$J = \begin{pmatrix} -\mu_1 \frac{s}{b_1 + s} & 0 & \frac{s\mu_1 b_1}{(b_1 + s)^3} \\ 0 & \mu_2 \frac{s}{b_2 + s} & 0 \\ -\frac{\gamma_1 \mu_1 s}{b_1 + s} & -\frac{\gamma_2 \mu_2 s}{b_2 + s} & -\frac{\gamma_1 \mu_1 b_1 s}{(b_1 + s)^3} + \mu_s \left(1 - 2\frac{s}{K}\right) \end{pmatrix}$$

has a positive eigenvalue. Therefore, the stationary point is unstable.

5. $u_1 = \frac{s}{s+b_1}, u_2 = \frac{s}{s+b_2}, f(s) = -\gamma_1 \mu_1 \frac{s}{(s+b_1)^2} - \gamma_2 \mu_2 \frac{s}{(s+b_2)^2} + \mu_s \left(1 - \frac{s}{K}\right)$.

The only nontrivial solution here is the equation $f(s) = 0$. For s=0 the right-hand side is positive; for s=∞, the right-hand side is negative, which means that this equation will have at least one solution. Similar to the previous model, we find that one of the roots of the equation is a stationary point.

Figure 35 shows the change in the number of two populations at the initial moment of populations on a non-renewable trophic resource for the case $\mu_1 = 2$, $\mu_2 = 2$, $b_1 = 1$, $b_2 = 1$, $\gamma_1 = 1.4$, $\gamma_2 = 1.6$, $\mu_s = 1$, $K = 10$, $u_0^1 = 0.001$, $u_0^2 = 0.002$, $S_0 = 1$. The coexistence of populations is observed.

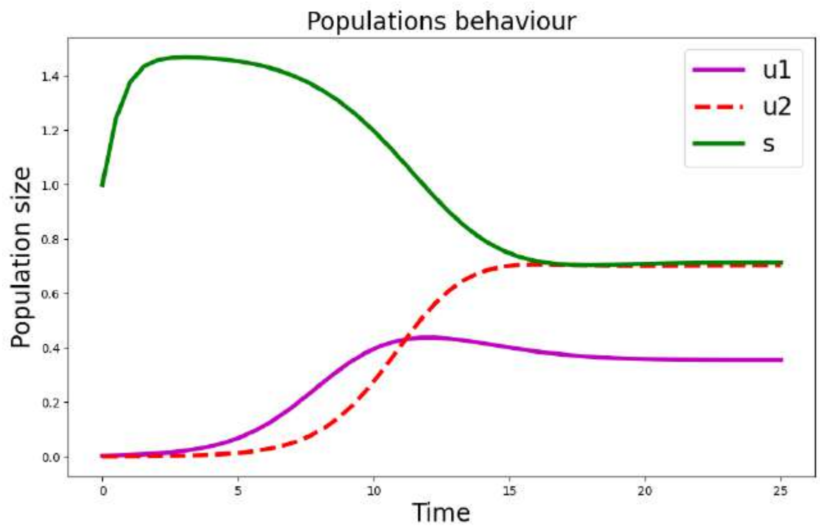


Fig. 35. Changes in the size of two populations

Operational model involving n populations. It is possible to construct a similar model (16) for n populations, which will allow one to estimate the boundaries of changes in concentration indices depending on n. According to (13) we have:

$$\begin{cases} \dfrac{du_i}{dt} = \mu_i u_i \left(\dfrac{s}{s+b_i} - \dfrac{u_i}{K_i}\right) \\ \dfrac{ds}{dt} = -\sum_{i=1}^{n} \gamma_i \mu_i u_i \dfrac{s}{s+b_i} + \mu_s s\left(1 - \dfrac{s}{K}\right) \end{cases} \quad (16)$$

where u_i – volume of the i- th population; μ_i – specific growth rate of the i- th population; s – amount of resource; K_i, K – medium capacity (maximum amount of resource that can be produced); γ_i – resource recycling rate; $f(s) = \frac{s}{s+b_i}$ - a function characterizing the influence of the amount of resource on the rate of its consumption; μ_s – specific resource growth rate. All constants are positive.

Also suppose:

1. $\frac{\partial u_m}{\partial t} = \mu_m u_m$ – this is the rate of increase in population volume (product output) if the resource is unlimited;

2. $\left(\frac{S}{b_m+S}-\frac{u_m}{K_m}\right)$ – it is the proportion of possibilities for developing population volumes (what can be produced) on a limited resource if the population size in question is u_m.

Scenario No1. Competition of n populations under conditions of low environmental factors. Next, to set the boundaries of the indices, we generate constants and initial data for n populations as follows:

$$\mu_i = rand_{0.03,0.1},\ K_i = rand_{0.1},\ \alpha_i = rand_{0.1},$$

$$u_{i0} = rand_{0.01},\ b_{ij} = matrix(rand_{0.1}),\ \gamma_{ij} = matrix(rand_{0.1})$$

$$\varepsilon = 0.003,\ \mu_s = rand_{0.03,0.1},\ K_s = rand_{0.1},$$

where $rand_1$ corresponds to a random rational number in the interval $[0,1]$, $rand_{0.01}$ corresponds to a random rational number in the interval $[0, 0.01]$, $matrix(rand_1)$ is an $n \times n$ matrix with random rational numbers in the interval $[0,1]$. This generation is chosen in order to avoid singularity on the right side of the system of differential equations.

The distribution density of the concentration index CR_3 and the survival rate of populations is constructed. Figure 36 (A) shows the density distribution of the concentration coefficient CR_3 for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.2$, which indicates a low concentration of the competitive environment. Figure 36 (B) shows the density distribution of the population survival rate for 500 experiments. The average survival rate corresponds to $alive = 49$. Thus, out of 50 populations, 98% of the populations survive in this scenario.

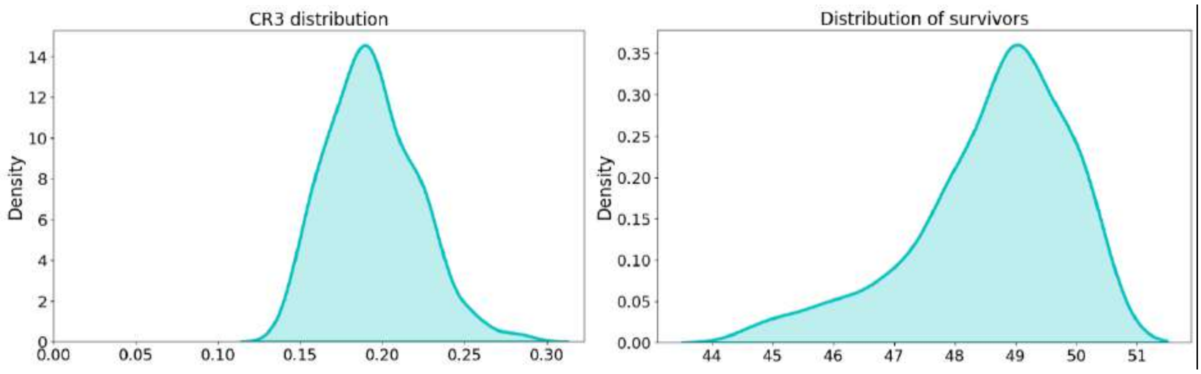


Fig. 36. A – Distribution density of the concentration index
B – Density of distribution of surviving populations

Scenario No3. Dynamic simulation modeling. Number of populations $n = 50$, number of experiments $N = 500$. At the first step, 2 populations are considered, at the second step, 3 populations, and so on up to 50 populations. At each step, 500 experiments are carried out and the average value is selected. In this interaction scenario, in each experiment, a solution to system (16) is found for k populations with similar generation of coefficients.

This scenario examines the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations in a similar manner relative to the previous scenario. Figure 37 (A) shows the dependence of the concentration coefficient on the total number of populations.

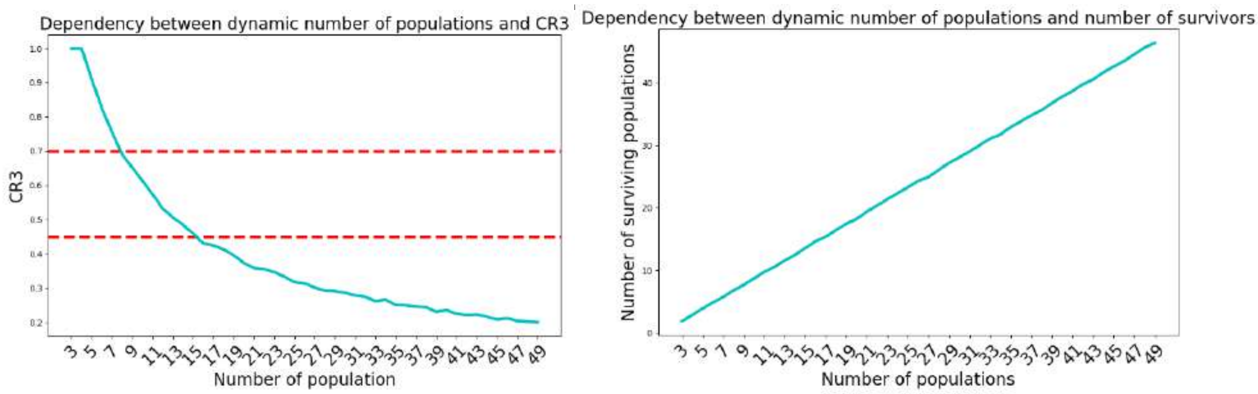


Fig. 37. A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

That is, in order to ensure healthy competitive environmental conditions, the required number of competition participants must be at least 15 within the model. Figure 37 (B) shows the dependence of the number of surviving populations on the number of populations. The survival rate is 100%, indicating that no population is dying within this model. That is, simultaneous coexistence of populations is observed. Threshold values for the number of subjects to determine the concentration of the environment are given in Table 18.

Table 18. Threshold values for the number of subjects

Concentrated environment	Subject Number Thresholds
Low	$k > 15$
Moderate	$k \in [8, 15]$
High	$k < 8$

As the number of subjects increases, the concentration index decreases, which is confirmed by theory (Chapter 1, 1.2.2, p. 22). The obtained results correlate with the results getting with the proportional criterion (Chapter 2, 2.10, p. 46), which also proves the validity of the developed operational model.

3.1.7. Mathematical model of "passive" avoidance of competition

Models of competition discussed above suggested that populations directly interact with each other or through the resource. Next, models suitable for describing weakly competitive or non-competing environmental communities that seek to escape competition by finding their environmental niche will be considered. Biological populations can coexist in the same territory or disperse in ecological niches depending on:

- the presence of mutated species;
- the ability of the population to use the available resource economically and fully;
- the ability of the population to respond poorly to external factors.

In this model, it is assumed that the avoidance of competition takes place "naturally":

$$\begin{cases} \dfrac{du_1}{dt} = u_1(1 - u_1 - \gamma_1 e^{-\alpha_1 u_1} u_2) \\ \dfrac{du_2}{dt} = \gamma u_2(1 - u_2 - \gamma_2 e^{-\alpha_2 u_2} u_1) \end{cases} \qquad (17)$$

In the first equation, a function $e^{-\alpha_1 u_2}$, is introduced, which decreases with the growth of u_2. That is, it is assumed that an increasing number of the second population leads to a natural escape from competition. A nontrivial stationary point in which $0 < u_1 < 1, 0 < u_2 < 1$, is located as the solution of a system of equations:

$$\begin{cases} 1 - u_1 - \gamma_1 e^{-\alpha_1 u_1} u_2 = 0 \\ 1 - u_2 - \gamma_2 e^{-\alpha_2 u_2} u_1 = 0 \end{cases} \qquad (18)$$

The function $f(z) = 1 - \alpha e^{-\beta_0} z$ at the point $z = 0$ is positive ($f(0) = 1$) and is decreasing ($f'(0) = -\alpha$), and at the point $z = 1$ will take positive values, if the inequality $\alpha e^{-\beta} < 1$. The extremum of the function $f(z)$ is reached at the point $z = \frac{1}{\beta}$.

If $\beta < 1$ at the interval [0,1] is held, it will be a monotonically decreasing function. Therefore, if the inequalities $\gamma_1 e^{-\alpha_1} < 1, \gamma_2 e^{-\alpha_2} < 1, \alpha_1 < 1, \alpha_2 < 1$ the system of equations (21) will have a unique solution $0 < u_1 < 1, 0 < u_2 < 1$.

If the inequality $\beta > 1$ is fulfilled, the function $f(z)$ will have at least an interval [0, 1]. Therefore, when $\alpha_1 > 1$ or $\alpha_2 > 1$, the system of equations (17), depending on the values of the constants included in equations (18), can have two solutions satisfying the conditions $0 < u_1 < 1, 0 < u_2 < 1$. The eigenvalues of the Jacobi matrix on the right side of the equations (17) are the roots of the polynomial:

$$\lambda^2 + (u_1 + \gamma u_2)\lambda + \gamma u_1 u_2 (1 - \gamma_1 \gamma_2 e^{-\alpha_1 u_2 - \alpha_2 u_1}(1 - \alpha_1 u_2)(1 - \alpha_2 u_1) = 0.$$

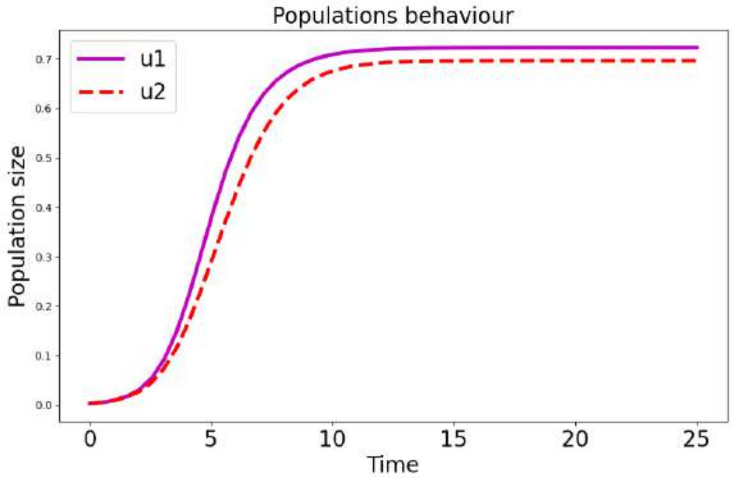


Fig 38: Dependence of functions u_1 and u_2 on time

At the same time $\gamma_1 e^{-\alpha_1} < 1, \gamma_2 e^{-\alpha_2} < 1, \alpha_1 < 1, \alpha_2 < 1$ the free term of this polynomial is positive its cort will have negative real parts and, accordingly, the nontrivial stationary point will be stable. Figure 38 shows the behavior of the model (17) over time.

An example of passive care is the change in internal metabolism during evolution. In Begon's work [57] there is an experiment with populations of Lebistes reticulates, which confirmed this assumption. Some members of this species were stained during the experiment. It was later discovered that Lebistes reticulates with unusual coloration were not attacked by predators. For n populations, the passive avoidance model (17) will look like (19):

$$\frac{du_i}{d\tau} = \mu_i u_i \left(1 - u_i - \varepsilon \sum_{j=1, j\neq i}^{n} \gamma_i \, e^{-\alpha_i u_i} u_j\right) \qquad (19)$$

where ε is the environmental factor.

Scenario No1. Competition of n ***populations in low environmental factor conditions.*** Next, to set the bounds of the indices, we generate constants and initial data for n populations as follows:

$$\mu_i = rand_{0.03,0.1}, \gamma_i = rand_{0.1}, \alpha_i = rand_{0.1}, \varepsilon = 0.003,$$

where $rand_1$ corresponds to a random rational number in the interval $[0, 1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.07$, which indicates a low concentration of the competitive environment. The survival rate corresponds to 50 populations, i.e., in the passive model with low competition intensity, all populations survive.

Scenario No2. Competition of n populations under conditions of variation in the intensity of the environmental factor. Number of populations $n = 50$, number of experiments $N = 100$. The coefficients are generated in the same way, except for ε:

$$\varepsilon \in [0.01, 150]$$

This scenario examines the dependence of the concentration index and population survival on the intensity of competition.

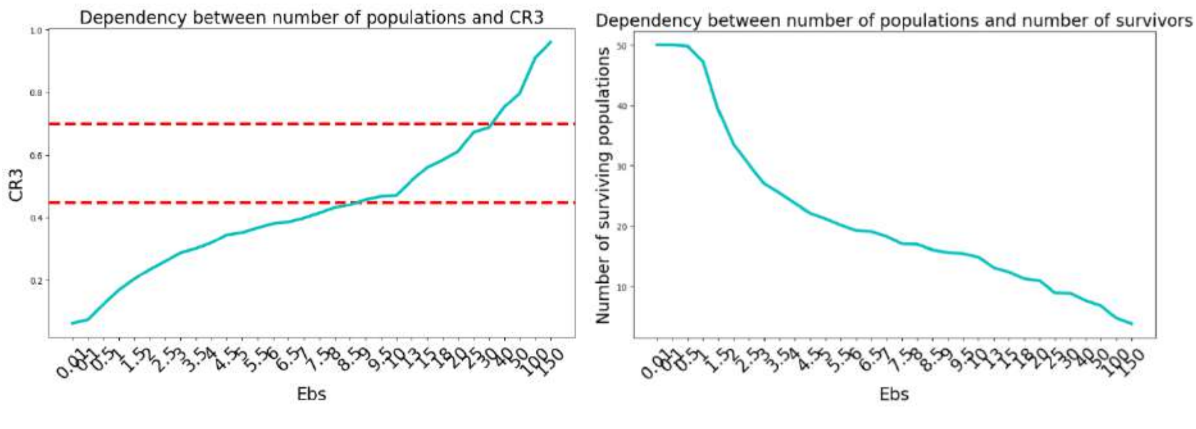


Fig 39. A – Dependence of the concentration factor on the environment factor
B – Dependence of the number of surviving populations on the environmental factor

Figure 39 (A) shows the dependence of the concentration coefficient on the intensity of competition. It can be assumed that the threshold of competition intensity in the model is sufficiently high for a low concentration environment. The model is suitable for describing weakly competitive communities or communities that seek to find an ecological niche. Figure 39 (B) shows the dependence of the number of surviving populations on the intensity of competition. The intensity of competition $\varepsilon = 0.5$ is the threshold value for low intensity. At $\varepsilon > 0.5$, there is a decrease in the survival rate - the survival rate decreases from 50 to 5, or from 100% to 2%. The higher the intensity of competition, the lower the population survival rate. The dependence of the environmental factor and survival rate on the concentration of the environment is shown in Table 19.

Table 19. Threshold values of environmental factor and survival rate

Concentration of the environment	Threshold values of the environment factor	Survival thresholds
Low	$\varepsilon < 9$	$alive > 18$
Moderate	$9 < \varepsilon < 25$	$alive \in [10, 18]$
High	$\varepsilon > 25$	$alive < 10$

Figure 40 shows the dependence of population size on the environmental factor. The number of populations decreases as the intensity of competition increases, but the population remains large.

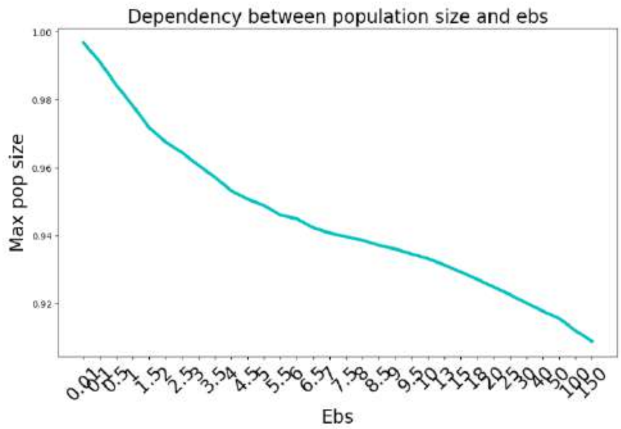


Fig. 40. Dependence of population size on environment factor

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (19) for k populations with similar generation of coefficients.

$$\mu_i = rand_{0.03,0.1}, \gamma_i = rand_{0.1}, \alpha_i = rand_{0.1}, \varepsilon = 0.003, i = 1 \dots k$$

In this scenario, the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations, is considered in the same way as the previous scenario. Figure 41 (A) shows the dependence of the concentration coefficient on the total number of populations. That is, to ensure the competitive environment, the required number of participants in the competition should be at least 14 subjects within the framework of the model. Figure 41 (B) shows the dependence of the number of surviving populations on the number of populations. There is a linear dependence.

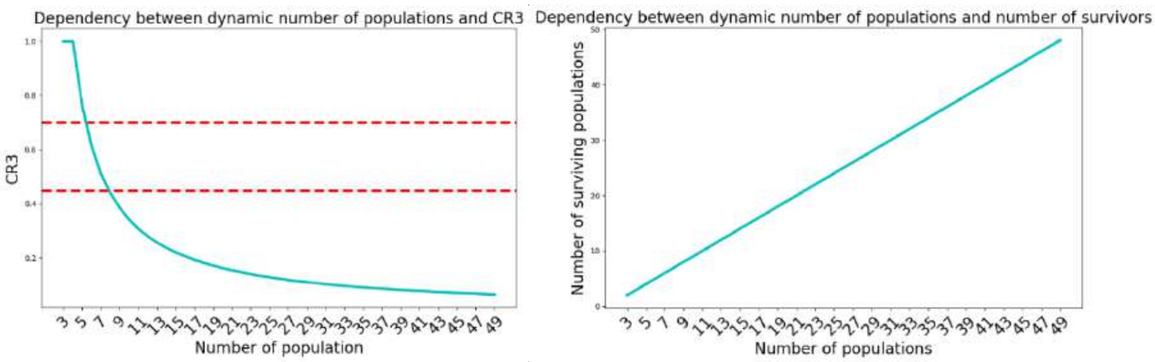


Fig41: A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 20.

Table 20. Threshold values of the number of subjects

Concentration of the environment	Subject threshold values
Low	$k > 8$
Moderate	$k \in [5, 8]$
High	$k < 5$

3.1.8. Mathematical model of seasonal competition

Biological and ecological systems are characterized by varying fluctuations. These fluctuations may depend on natural conditions, biological rhythms, geophysical rhythms, periodicity of offspring, physical abilities of individuals, and other causes. That is, the intensity of competitive relationships can depend on time. Examples of such systems include fluctuations in the heart muscle or fluctuations in the concentration of calcium in cells. Economic systems are also characterized by seasonal purchases. Fluctuations in biological populations can be considered in the model (20) by introducing time-dependent functions $\varphi_1(t)$ and $\varphi_2(t)$:

$$\begin{cases} \frac{du_1}{dt} = u_1(1 - u_1 - \gamma_1\varphi_1(t)u_2) \\ \frac{du_2}{dt} = \gamma u_2(1 - u_2 - \gamma_2\varphi_2(t)u_1) \\ \varphi_1(t) = \frac{1}{1 + |a_1|}(1 - \alpha_1 cos\omega_1 t) \\ \varphi_2(t) = \frac{1}{1 + |a_2|}(1 - \alpha_2 cos\omega_2 t) \end{cases} \tag{20}$$

where $|a_1| \leq 1$ and $|a_2| \leq 1$.

The initial conditions are $u_1(0) = 1$ and $u_2(0) = 1$. This corresponds to the case when the competitive relationship begins at the initial moment of time at the maximum values of u_1 and u_2. At the assumed values of $\gamma_1 = 1.1$ and $\gamma_1 = 1.3$ in the case of $\omega_1 = 0$ and $\omega_2 = 0$, as follows from the analysis of equations (20), the simultaneous existence of two populations is impossible. Figure 42 shows the changes in the numbers of the two populations over time.

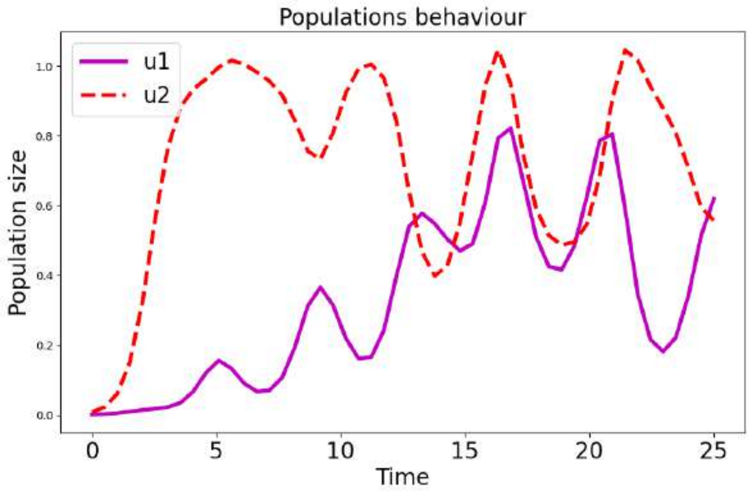


Fig. 42. Dependence of functions u_1 and u_2 on time

In the economy seasonality of competition can depend on such reasons as the influence of psychological or emotional factors, the influence of popularity on demand of certain goods, weather phenomena [107-108]. There are also many examples in the biological system. For example, in Siberia, shrews, lemmings and mice live together. Reproduction of Siberian shrews (about 10 species) begins at the end of April - early May, and ends at the end of August. Lemming begins to reproduce in April and stops at the end of June. The field mouse begins to reproduce in mid-April and stops in September. In October and November, there is a periodic outbreak of development of diatomic algae [109-111].

Reproduction of all small mammals begins when they are small numbered. The trophic resource (seeds, leaves of plants, insects) is the same for all, but its maturation also depends on time. Therefore, competition for the trophic resource during the season is shifted in time. As a result, there is an inter-species asynchronous change in the number of populations - the maximums and minimums of seasonal numbers in all species are shifted in time [112].

In nature, population numbers fluctuate. Due to the size of the population range, the number of individuals in the population can also change significantly. Thus, insects and small plants in open spaces, the number of individuals in populations can reach hundreds of thousands and millions of individuals. On the contrary, animal and plant populations may also be relatively small. There are periodic and non-periodic variations in the number of populations. The first are committed during the season or several years, the second are the outbreaks of mass reproduction of some pests of useful plants, with violations of habitat conditions (droughts, unusually cold or warm winters, too rainy growing seasons) unforeseen migrations to new habitats. Periodic and non-periodic variations in the number of populations due to biotic and abiotic environmental factors inherent in all populations are called population waves.

For some ecological communities, the rule of oscillation (cyclicity) is valid: no population is in absolute equilibrium of the number of individuals. Often, in addition to seasonal changes in the number of individuals, there are periodic fluctuations caused by factors external to the population, and oscillations related to their own (internal) dynamic changes in the population. In addition, there is a generalized maximum size rule for population density fluctuations. E. Odum formulates this rule as follows: "There are certain upper and lower limits for the size of the population which are observed in nature or which could theoretically exist for any long period of time" [113].

Economic communities are also cyclical. Economic cyclicality refers to periods of growth followed by a decline in economic activity. Changes in economic activity have implications for the well-being of the general population as well as for private institutions. Often economic cycles are tracked by changes in economic indicators, for example GDP [114]. Cyclicality of economic processes can also be manifested in Kondratiev waves, which are hypothetical cyclical events in a modern global economy. This phenomenon is closely linked to the life cycle of technology. It is believed that the wave period varies from 40 to 60 years, the cycles consist of alternating intervals of high sectoral growth and intervals of relatively slow growth [115-116]. For n populations, the competition seasonality model (21) looks like:

$$\begin{cases} \dfrac{du_i}{dt} = u_i(1 - u_i - \varepsilon \sum\limits_{j=1, j\neq i}^{n} \gamma_i \, \varphi_i(t) u_j) \\ \varphi_i(t) = \dfrac{1}{1 + a_i}(1 - \alpha_i cos\omega_i t) \end{cases} \qquad (21)$$

where ε – the environmental factor.

Scenario No1. Competition of n populations in low environmental factor conditions. Next, to set the bounds of the indices, we generate constants and initial data for n populations as follows:

$$\mu_i = rand_{0.03,0.1}, \gamma_i = rand_{0.1}, \alpha_i = rand_{0.1}, \omega_i = rand_{0.1}, \ \varepsilon = 0.003,$$

where $rand_1$ corresponds to a random rational number in the interval $[0, 1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.07$, which indicates a low concentration of the competitive environment. The survival rate corresponds to 50 populations, i.e., in the low-intensity seasonal competition model, all populations survive.

Scenario No2. Competition of n populations under conditions of variation in the intensity of the environmental factor. Number of populations $n = 50$, number of experiments $N = 500$. The coefficients are generated in the same way, except for ε:

$$\varepsilon \in [0.01, 150]$$

In this scenario, we consider the dependence of the concentration index and the dependence of the survival rate of the populations on the environmental factor.

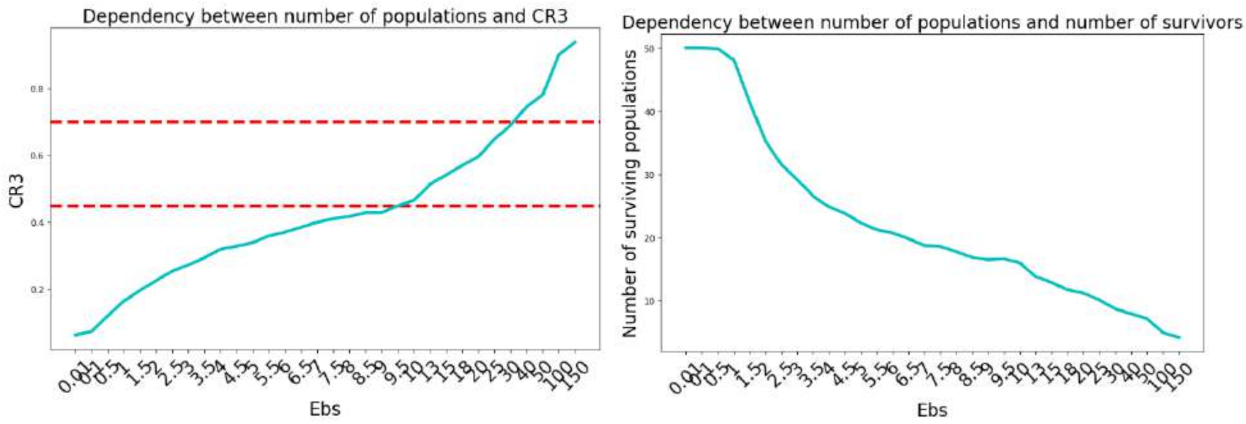


Fig. 42. A – Dependence of the concentration factor on the environment factor
B – Dependence of the number of surviving populations on the environmental factor

Figure 42 (A) shows the dependence of the concentration factor on the environment factor. It can be assumed that the threshold of the environmental factor in the model is sufficiently high for a low concentrated environment. The model is suitable for describing weakly competitive communities or communities with species coexistence. This suggests a stable competitive environment. Figure 42 (B) shows the dependence of the number of surviving populations on the environmental factor. The medium factor $\varepsilon = 1$ is the threshold value for low intensity. At $\varepsilon > 1$, there is a decrease in the survival rate - the survival rate decreases from 50 to 5, or from 100% to 2%. The higher the intensity of competition, the lower the population survival rate. The dependence of the environmental factor and survival rate on the concentration of the environment is shown in Table 21.

Table 21. Threshold values of environmental factor and survival rate

Concentration of the environment	Threshold values of the environment factor	Survival thresholds
Low	$\varepsilon < 9.5$	$alive > 15$
Moderate	$9.5 < \varepsilon < 30$	$alive \in [10,15]$
High	$\varepsilon > 30$	$alive < 10$

Figure 43 shows the dependence of population size on the environmental factor. The number of populations decreases as the intensity of competition increases, but the size of the population does not change and the population remains large.

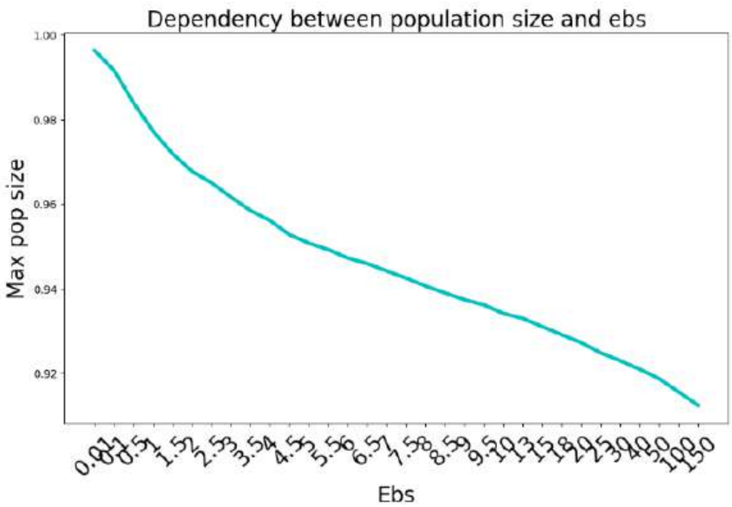


Fig. 43. Dependence of population size on environment factor

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (21) for k populations with similar generation of coefficients.

$$\mu_i = rand_{0.03,0.1}, \gamma_i = rand_{0.1}, \alpha_i = rand_{0.1},$$
$$\omega_i = rand_{0.1}, \ \varepsilon = 0.003, \ i = 1, \dots, k$$

In this scenario, we consider the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations. Figure 44 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a stable competitive environment, the required number of competitors should be at least 14 within the model. Figure 44 (B) shows the dependence of the number of surviving populations on the number of populations. There is a linear dependence.

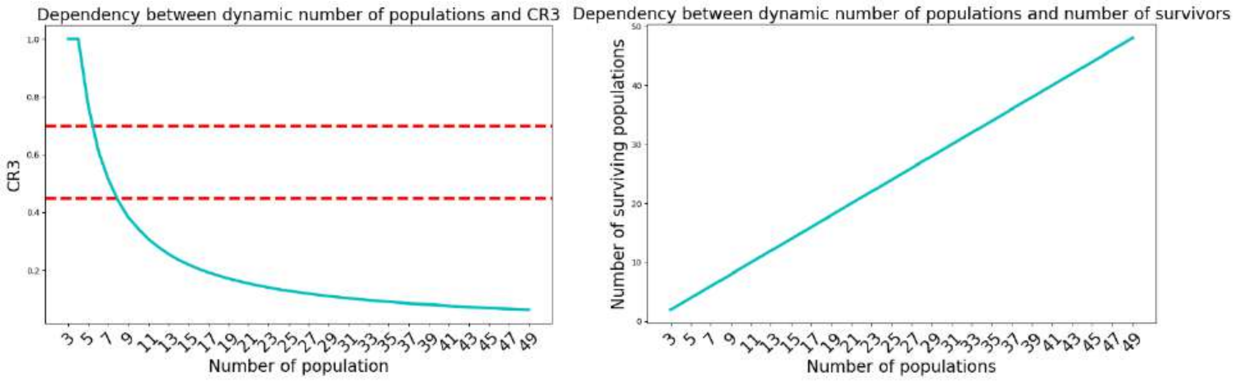


Fig. 44. A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 22.

Table 22. Threshold values of the number of subjects

Concentration of the environment	Subject threshold values
Low	$k > 8$
Moderate	$k \in [5, 8]$
High	$k < 5$

3.1.9. Two-chamber model of the dynamics of the number of a single population

In an ecosystem, population movements within its habitat are most common. Suppose two groups of individuals of the same population exist on two adjacent habitats with different biophysical properties (birth rate, mortality, intraspecies competition). At the same time, individuals transit from one habitat to another. A model of interaction between two such groups of the same population is described by a system of equations (22). This model is suitable for describing mobile and frequently moving populations.

$$\begin{cases} \dfrac{dN_1}{dt} = F_1(N_1) - v_1N_1 + v_2N_2 \\ \dfrac{dN_2}{dt} = F_2(N_2) + v_1N_1 - v_2N_2 \end{cases} \tag{22}$$

where v_1 –the proportion of individuals moving from the first range to the second, and v_2 from the second to the first. v_1 and v_2 are constrained: $v_1 < 1$ and $v_2 < 1$. The system of equations (22) by replacing the variables $u_1 = \frac{N_1}{K_1}, u_2 = \frac{N_2}{K_2}$ is reduced to a system of equations (23):

$$\begin{cases} \dfrac{du_1}{dt} = f_1(u_1) - v_1u_1 + av_2u_2 \\ \dfrac{du_2}{dt} = f_2(u_2) + \dfrac{1}{a}v_1u_1 - v_2u_2 \end{cases} \tag{23}$$

where $a = \frac{K_2}{K_1}$, and functions$f_1(u) = \frac{1}{K_1}F_1(N_1), f_2(u) = \frac{1}{K_2}F_2(N_2),$ satisfy the following conditions: $f_1(0) = 0, f_2(0) = 0, f_1(1) = 0, f_2(1) = 0.$

Special points of the system of equations (23) will be $u_1 = 0, u_2 = 0$ and solutions of the system of algebraic equations:

$$\begin{cases} f_1(u_1) - v_1u_1 + av_2u_2 = 0 \\ f_2(u_2) + \dfrac{1}{a}v_1u_1 - v_2u_2 = 0 \end{cases} \tag{24}$$

It follows from equations (24) that the stationary points are equal:

$$f_1(u_1) + af_2(u_2) = 0 \tag{25}$$

Equity (25), since $f_1(u) < 0$ and $f_2(u) < 0$ at $u > 1$, can only be achieved if one population is greater than one and the other is less than one. If $v_1 = v_2$ and media capacitances are the same, then the system of equations (23) satisfies the solution $u_1 = u_2 = 1$ regardless of the type of functions $f_1(x)$ and $f_2(x)$.

In the absence of transitions, both groups of populations exist independently of each other. The stable stationary point in this case is $u_1 = u_2 = 1$. When the migration occurs, over time there will be a transition to a new equilibrium position, defined as the solution of a system of equations (24). At small values v_1 and v_2 from the expansion into a series of left parts of the equation (24) in the first approximation with precision to the values of the second order of smallness, it can be assumed that:

$$u_1 = 1 - \frac{1}{\mu_1}(v_1 - av_2), u_2 = 1 + \frac{1}{a\mu_2}(v_1 - av_2)$$

$$\mu_1 = -f_1'(1), \mu_2 = -f_2'(1)$$

It follows that $v_1 = v_2$ should increase the number of the area whose capacity is smaller. At the same medium capacity $(a = 1)$, the number of the range, the proportion of individuals moving from which there are more, will decrease. The eigenvalues of the Jacobi matrix on the right side of the system of equations (23) are the roots of the characteristic polynomial:

$$\lambda^2 - (f_1' - v_1 + f_2' - v_2)\lambda + f_1'f_2' - f_1'v_2 - f_2'v_1 = 0,$$

where derivatives are calculated at stationary points:

$$f_1' = \frac{\partial f_1(u_1)}{\partial u_1}, f_2' = \frac{\partial f_2(u_2)}{\partial u_2}$$

The stationary points, which are the solution of the system of equations (23), will be stable if the inequalities are fulfilled:

$$f_1' - v_1 + f_2' - v_2 < 0, \qquad 0 < f_1'f_2' - f_1'v_2 - f_2'v_1$$

These inequalities are met if at stationary points $f_1' < 0,\ f_2' < 0$.

At the first stationary point, the characteristic polynomial will have both roots with negative real parts if the inequalities are simultaneously fulfilled:

$$\mu_1\left(\frac{v_1}{\mu_1} - 1\right) + \mu_2\left(\frac{v_2}{\mu_2} - 1\right) > 0, \qquad \frac{v_1}{\mu_1} + \frac{v_2}{\mu_2} < 1$$

If the second inequality is fulfilled, and it is satisfied only if $\frac{v_1}{\mu_1} < 1, \frac{v_2}{\mu_2} < 1$ the first one cannot be executed simultaneously. Therefore, at least one root of the characteristic polynomial will have a positive real part and therefore the trivial solution is unstable. For the case where $f_1(x) = f_2(x), v_1 = v_2, a = 1$, the stable stationary point is $u_1 = 1, u_2 = 1$.

A two-chamber model of a single population considering a limited trophic resource can be represented by a system of equations:

$$\begin{cases} \frac{du_1}{dt} = \mu_1 u_1 (\frac{S_1}{b_1 + S_1} - \frac{u_1}{K_1}) - v_1 u_1 + v_2 u_2 \\ \frac{du_2}{dt} = \mu_2 u_2 \left(\frac{S_2}{b_2 + S_2} - \frac{u_2}{K_2}\right) + v_1 u_1 - v_2 u_2 \\ \frac{dS_2}{dt} = \mu_{s1} S_1 \left(1 - \frac{S_1}{S_{10}}\right) - \alpha_1 \frac{S_1}{b_1 + S_1} u_1 \\ \frac{dS_1}{dt} = \mu_{s2} S_2 \left(1 - \frac{S_2}{S_{20}}\right) - \alpha_2 \frac{S_2}{b_2 + S_2} u_2 \end{cases} \tag{26}$$

In this model, it is assumed that individuals in the population can move from one chamber to another, the trophic resource in each chamber is immobile, is replenished with local specific velocities μ_{s1} and μ_{s2},, and the media capacitance of trophic resources S_{10} and S_{20}, K_1 and K_2 are media capacitances for the first and second populations, and μ_1 and μ_2 their local specific growth rates. Parameters $b_1, b_2, \alpha_1, \alpha_2$characterizing the rate of consumption of trophic resource by populations and the rate of its removal. In Figure 45 for the case $\mu_1 = 2, \mu_2 = 2,\ b_1 = 1,\ b_2 = 1, \gamma_1 = 1.4, \gamma_2 = 1.6, \mu_s = 1, K = 10, u_0^1 = 0.001,\ u_0^2 = 0.002,\ S_0 = 1$ changes in functions u_1 and u_2 in time.

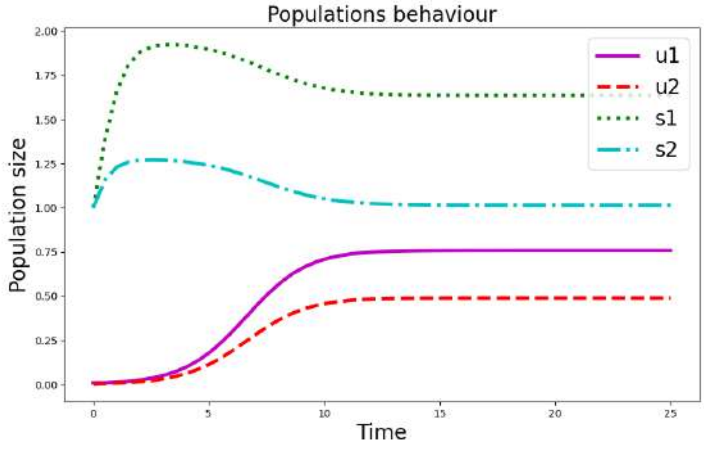


Fig. 45. Change in the number of two populations

The degree of isolation of the neighboring populations of the species varies greatly. In some cases, they are sharply divided by uninhabitable territory, and clearly localized in space, or vice versa, continuous settlement of species of vast areas. The boundaries between populations in such cases are hardly identified. However, since co-dwelling individuals come in contact with each other more often than with other parts of the range, populations in remote areas can be considered different populations. Within the same species there may be populations with both distinct and illegible boundaries.

The two-chamber model of a single population, considering the limited trophic resource, can be extended to n chambers:

$$\begin{cases} \dfrac{du_l}{dt} = \mu_l u_l (\dfrac{S_l}{b_l + S_l} - \dfrac{u_l}{K_l}) - v_l u_l + v_k u_k \\ \dfrac{du_k}{dt} = \mu_k u_k \left(\dfrac{S_k}{b_k + S_k} - \dfrac{u_k}{K_k}\right) + v_l u_l - v_k u_k \\ \qquad \cdots \\ \dfrac{dS_k}{dt} = \mu_{sl} S_l \left(1 - \dfrac{S_l}{S_{l0}}\right) - \alpha_l \dfrac{S_l}{b_l + S_l} u_l \\ \dfrac{dS_l}{dt} = \mu_{sk} S_k \left(1 - \dfrac{S_k}{S_{k0}}\right) - \alpha_k \dfrac{S_k}{b_k + S_k} u_k \\ \qquad \cdots \end{cases} \tag{27}$$

We consider n chambers, where n is even. Random non-intersecting pairs of interacting cameras are generated. The same behavior of moving through cells and finding one's niche will be observed in the model for m populations.

Scenario No1. Competition in n chambers in conditions of low environmental factor. Next, to set the bounds of the indices, we generate constants and initial data for n populations as follows:

$$\mu_i = rand_{0.03,0.1}, K_i = rand_{0.1}, \alpha_i = rand_{0.1}, u_{i0} = rand_{0.01},$$

$$b_i = rand_{0.1}, v_i = rand_{0.1}, \ \mu_s = \mu_i = rand_{0.03,0.1}, S_{i0} = rand_{0.1},$$

where $rand_1$ corresponds to a random rational number in the interval $[0, 1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 46 (A) shows the distribution density of the CR_3 concentration coefficient for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.13$, which indicates a low concentration of the competitive environment. Figure 46 (B) shows the population survival density distribution for 500 experiments. The median survival rate is $alive = 50$. That is, it is most likely that the entire population will survive in this scenario. This can also be interpreted as the fact that all individuals in a single population are evenly distributed among the cells, that is, all individuals find their ecological niche. The humpiness of the curves can be related to the element of random movements across the cameras.

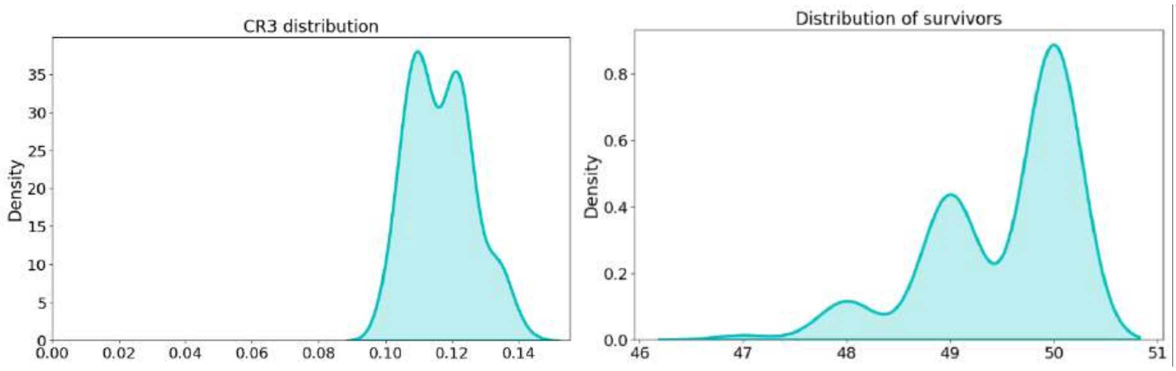


Fig. 46. A – Density of concentration index distribution
B – Density of distribution of surviving populations

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (27) for k populations with similar generation of coefficients.

$$\mu_i = rand_{0.03,0.1}, K_i = rand_{0.1}, \alpha_i = rand_{0.1},$$

$$u_{i0} = rand_{0.01}, b_i = rand_{0.1},, \nu_i = rand_{0.1}$$

$$\mu_i = rand_{0.03,0.1},, S_{i0} = rand_{0.1}, i = 1 \dots k$$

In this scenario, we consider the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations. Figure 47 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a stable competitive environment, the required number of competitors should be at least 13 within the model. Figure 46 (B) shows the dependence of the number of surviving populations on the number of populations. With 50 chambers, all subgroups of the population survive.

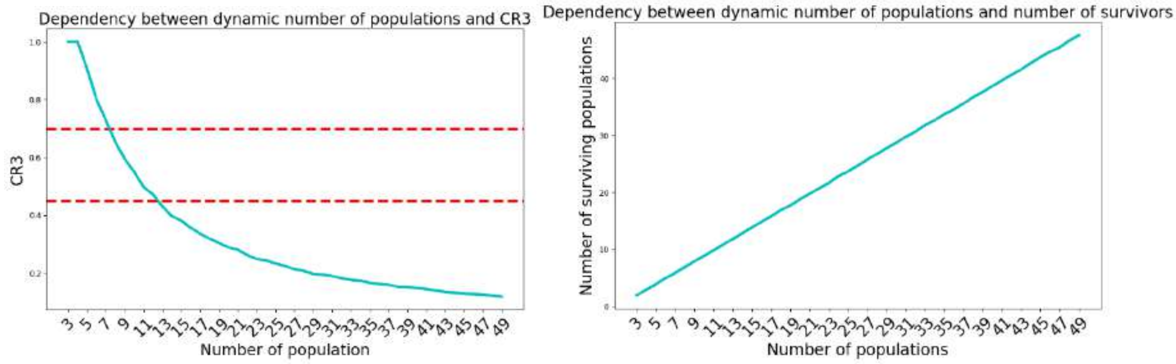


Fig. 47. A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 23.

Table 23. Threshold values of the number of subjects

Concentration of the environment	Subject threshold values
Low	$k > 13$
Moderate	$k \in [8, 13]$
High	$k < 8$

§ 2. Spatial mathematical models of competition

3.2.1. Mathematical model of a heterogeneous habitat

All biological systems are active distributed systems. Biological and economic communities are not homogeneous. For example, in biocoenoses, depending on the climatic conditions in different areas, different types of trees grow, and their density depends on the temperature gradient. In the economic sphere, this is manifested in different traditions, diverse resources and territorial industrial centers of different capacities. All species live in territories, in spatially bounded habitats with their own climatic conditions and trophic resources. For spatially inhomogeneous areas it is necessary to consider the heterogeneity of spatial distribution of individuals on the territory [117]. When building mathematical models of such areas the apparatus of partial differential equations is used. The main hypothesis in these models suggests that individuals in the territory move randomly.

Natural resources are also randomly distributed. Similarly, there was the settlement of human beings in the territory of the Earth and, consequently, the creation of industry. In mathematical models, in the first approximation, let's assume that initial distribution of resources in the territory occurs randomly. As the territory in which the competition takes place, consider the linear area. In spatiotemporal biological systems it is possible to propagate impulses and waves of excitation, formation of stationary spatially inhomogeneous distributions of substances and other phenomena of self-organization, which have been named autowave processes [118].

Let's consider the problem of species distribution in the territory rich in food resource. At any point the line reproduction of the species is described by the equation $f(x) = x(1 - x)$. To the left of zero the territory is occupied by the species, to the right – the territory is unoccupied. At an initial point in time, the view begins to propagate to the right.

This model will look like:

$$\frac{\partial x}{\partial t} = f(x) + \frac{D\partial^2 x}{\partial r^2}$$

In these equations, r – the coordinate, x – the linear density of the population, t – time, D – the parameter characterizing the mobility of individuals, $f(x)$ – is the trophic function characterizing the behavior of the population. For $t > 0$ the wave propagates in this case. This wave is the result of:

- Random movement of individuals in the population.
- The reproduction that describes the function $f(x)$.

The wave front moves to the right and reaches a certain limit. Wave velocity can be found using the formula:

$$\lambda = 2\sqrt{D}f'(0)$$

For the predator-prey system in unlimited space, the "running and chasing" waves will propagate. For limited space, autowaves (stationary spatially inhomogeneous structures) will take place [119]. In linear systems, diffusion leads to equalization of population concentration in the territory. If there is a non-linear interaction between species, then there will be steady state instability, resulting in the formation of the above-mentioned autowaves. A condition for the emergence of dissipative structures is the difference of diffusion coefficients.

Interacting populations on a segment. Examples of linear habitats include canals, roads, and shorelines of lakes and rivers [120-121]. A mathematical model of two populations interacting on a segment for the case of a segment is represented by a system of two evolutionary equations:

$$\begin{cases} \frac{\partial u_1}{dt} = D_1 \frac{\partial^2 u_1}{dx^2} + f_1(u_1, u_2) \\ \frac{\partial u_2}{dt} = D_2 \frac{\partial^2 u_2}{dx^2} + f_2(u_1, u_2) \end{cases} \quad (28)$$

In these equations, x – the coordinate, t – the time, $u_1 = u_1(t, x)$, $u_2 = u_2(t, x)$ – linear population densities, D_1, D_2 – parameters characterizing the mobility of individuals,

$f_1(u_1, u_2)$, $f_2(u_1, u_2)$ – trophic functions, characterizing interactions between populations.

The initial and boundary conditions must be added to the system of equations (28) for the case of a segment of length l. The value of functions $u_1 = u_1(t, x)$, $u_2 = u_2(t, x)$ is specified as the initial conditions at the start of time: when $t = 0 \rightarrow u_1(x) = u_{10}(x), u_2(x) = u_{20}(x)$.

Two options are considered as boundary conditions:

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0, \qquad \left.\frac{\partial u_2}{\partial x}\right|_{x=0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 \tag{29}$$

and

$$u_1|_{x=0} = \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0, \qquad u_2|_{x=0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 \tag{30}$$

The condition of turning to zero functions u_1 and u_2 on the boundary of the segment corresponds to the impossibility of existence of a population at this point, and the zeroing condition of the derivatives $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ (filling condition [122]) allows for free growth of populations. The total numbers of the first and second populations ($M_1(t)$ and $M_2(t)$) on the segment at time t are calculated using the formulas:

$$M_1 = \int_0^l u_1(t, x)dx\,, M_2 = \int_0^l u_2(t, x)dx.$$

For two competing populations, the functions $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ must satisfy the following conditions for $u_1 = 0$ and $u_2 = 0$:

$$f_1(u_1, u_2) = 0, f_2(u_1, u_2) = 0, \frac{\partial f_1}{\partial u_1} > 0, \frac{\partial f_2}{\partial u_2} > 0$$

As predator-prey models, two models with trophic functions are considered below:

$$f_1(u_1, u_2) = u_1 - u_1 u_2, f_2(u_1, u_2) = -\gamma u_2 + u_1 u_2 \tag{31}$$

and

$$f_1(u_1, u_2) = u_1(1 - u_1) - u_1 u_2, f_2(u_1, u_2) = -\gamma u_2 + u_1 u_2 \tag{32}$$

The local model follows from (25) with $D_1 = 0$ and $D_2 = 0$:

$$\begin{cases} \frac{du_1}{dt} = f_1(u_1, u_2) \\ \frac{du_2}{dt} = f_2(u_1, u_2) \end{cases} \tag{33}$$

The stationary points of this system of equations are found as a solution to the system of algebraic equations:

$$f_1(u_1, u_2) = 0, f_2(u_1, u_2) = 0.$$

Stability of solutions. Stationary points of the system of equations (33) (let $u_1 = u_1^*, u_2 = u_2^*$) will be stable if the eigenvalues of the Jacobi matrix:

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

$$f_{11} = \frac{\partial f_1(u_1^*, u_2^*)}{\partial u_1}, f_{12} = \frac{\partial f_1(u_1^*, u_2^*)}{\partial u_2}$$

$$f_{21} = \frac{\partial f_2(u_1^*, u_2^*)}{\partial u_1}, f_{22} = \frac{\partial f_2(u_1^*, u_2^*)}{\partial u_2}$$

at stationary points will have negative real parts. At the zero-stationary point, given the conditions (29) imposed on the functions $f_1 = f_1(u_1, u_2)$ and $f_2 = f_2(u_1, u_2)$, both eigenvalues will be positive and, accordingly, this stationary point will be unstable.

Stationary solutions of the system of equations (28) will also be solutions of the system of equations (33) under boundary conditions (29), if $u_{10}(x) = u_1^*,\ u_{20}(x) = u_2^*$ as initial conditions. Let there be a solution close to it, along with this solution:

$$u_1(t, x) = u_1^* + \delta u_1(t, x),\ u_2(t, x) = u_2^* + \delta u_2(t, x)$$

such that $\delta u_1(t, x)$ and $\delta u_2(t, x)$ are small compared to u_1^* and u_2^* values. Then the equations for $\delta u_1(t, x)$ and $\delta u_2(t, x)$ follow with the accuracy of the second-order small values from the system of equations (33):

$$\begin{cases} \frac{\partial \delta u_1}{\partial t} = D_1 \frac{\partial^2 \delta u_1}{\partial x^2} + f_{11}\delta u_1 + f_{12}\delta u_2, \\ \frac{\partial \delta u_2}{\partial t} = D_2 \frac{\partial^2 \delta u_2}{\partial x^2} + f_{21}\delta u_1 + f_{22}\delta u_2, \end{cases} \tag{34}$$

Solutions of these linear equations satisfying boundary conditions (29) are searched in the form of trigonometric series:

$$\delta u_1 = \sum_{k=0}^{\infty} A_k(t)\cos\,(k\pi x/l)\,, \delta u_2 = \sum_{k=0}^{\infty} B_k(t)\cos\,(k\pi x/l).$$

The coefficients$A_k(t)$ abd $B_k(t)$ must satisfy the equations $\left(\mu_k = \frac{k\pi}{l}, k = 0,1,2,\dots\right)$:

$$\begin{cases} \dfrac{dA_k}{dt} = -D_1\mu_k^2 A_k + f_{11}A_k + f_{12}B_k, \\ \dfrac{dB_k}{dt} = -D_2\mu_k^2 A_k + f_{21}A_k + f_{22}B_k \end{cases} \quad (35)$$

The solution of these equations is stable if the eigenvalues of the Jacobi matrix are:

$$J_k = \begin{pmatrix} -D_1\mu_k^2 + f_{11} & f_{12} \\ f_{21} & -D_2\mu_k^2 + f_{22} \end{pmatrix} \quad (36)$$

right side of these equations will have a negative real part for all $k = 0,1,2,\dots$.

For the case of trophic functions (31), the stationary point of the system of equations (33) $u_1 = \gamma, u_2 = 1$ is the center, since the eigenvalues of the Jacobi matrix will be $\lambda_{1,2} = \pm\sqrt{\gamma}$. For the case of the system of equations (35), the Jacobi matrix (36):

$$J_k = \begin{pmatrix} -D_1\mu_k^2 & -\gamma \\ 1 & -D_2\mu_k^2 \end{pmatrix}$$

has eigenvalues with negative real part for all $k = 0,1,2,\dots$, and for $k = 0$ eigenvalues coincide with the eigenvalues of the Jacobi matrix of the system of equations (33). Therefore, in the neighborhood of this equilibrium position $A_0(t)$ and $B_0(t)$ will change according to the harmonic law, and $A_k(t)$ and $B_k(t)$ for $k = 0,1,2,\dots$ will eventually go to zero. Accordingly, the general solution of the system of equations will eventually cease to depend on the spatial coordinate.

For trophic functions (32), the stationary point of the equations (33) is $u_1 = \gamma, u_2 = 1 - \gamma$. The eigenvalues of the Jacobi matrix at this stationary point, since the inequalities of $0 < \gamma < 1$, are fulfilled, will be a pair of complexly conjugate roots $\lambda_{1,2} = \frac{1}{2}(-\gamma \pm i\sqrt{\gamma(4-\gamma)}$ and, respectively, The stationary point is a steady focus.

The eigenvalues of the Jacobi matrix (35) of the system of equations (36):

$$J_k = \begin{pmatrix} -D_1\mu_k^2 - \gamma & -\gamma \\ 1 & -D_2\mu_k^2 \end{pmatrix}$$

for all $k = 0,1,2,\ldots$ will have negative real parts. Therefore, in case of small deviations from the equilibrium position, the solution of the system of equation (34) will tend to the homogeneous solution $u_1(t,x) = \gamma,\ u_2(t,x) = 1 - \gamma$. For the case of boundary conditions (30) the stationary system of equations (28) is satisfied with the solution:

$$u_1(x) = 0, u_2(x) = 0.$$

The solution of the equations for perturbations (34), satisfying the boundary conditions (30), in the vicinity of this stationary solution is represented in the form of a trigonometric series:

$$\delta u_1 = \sum_{k=0}^{\infty} A_k(t)\sin\mu_k x\,, \delta u_2 = \sum_{k=0}^{\infty} B_k(t)\mu_k \sin\mu_k x \left(\mu_k = \frac{\left(k - \frac{1}{2}\right)\pi}{l}\right).$$

Jacobi Matrix:

$$J_k = \begin{pmatrix} -D_1\mu_k^2 + |f_{11}| & f_{12} \\ f_{21} & -D_2\mu_k^2 - |f_{22}| \end{pmatrix} \tag{37}$$

of right parts of the equations (35) will have eigenvalues:

$$\lambda_1 = -D_2\mu_k^2 - |f_{22}| \text{ и } \lambda_1 = -D_1\mu_k^2 + |f_{11}|.$$

The first eigenvalue is negative, and the second eigenvalue is negative, for all $k = 1,2,\ldots$ if the condition $D_1\left(\frac{\pi}{2l}\right)^2 > \frac{\partial f_1}{\partial u_1}$ is satisfied. If this inequality is satisfied, the trivial solution is stable.

3.2.2. Bazykin-Volterra mathematical model

Examples of linear habitats are pipelines, roadsides, forests. The mathematical model of competition of two types (3) on a segment is represented by a system of two evolutionary equations:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1 f_1(u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + f_2(u_1, u_2), \end{cases} \tag{38}$$

In these equations, x – the coordinate, t – the time, $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ – linear population densities, D_1 and D_2 – parameters characterizing the mobility of individuals. As initial conditions, the value of functions $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ at the start of time: when $t = 0$ $u_1(x) = u_{10}(x)$, $u_2(x) = u_{20}(x)$. Two options are considered as boundary conditions for the case of a segment of length l:

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0, \qquad \left.\frac{\partial u_2}{\partial x}\right|_{x=0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 \tag{39}$$

and

$$u_1|_{x=0} = \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0, \qquad u_2|_{x=0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 \tag{40}$$

The condition of turning to zero functions u_1 and u_2 on the boundary of the segment corresponds to the impossibility of existence of a population at this point, and the zeroing condition of the derivatives $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ (filling condition) allows for free growth of populations. The total numbers of populations $M_1(t)$ and $M_2(t)$ at time t are calculated using the following formulas:

$$M_1 = \int_0^l u_1(t, x)dx\,, M_2 = \int_0^l u_2(t, x)dx.$$

Let's consider the model of competition according to Volterra in partial derivatives:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(1 - u_1) - \gamma_1 u_1 u_2, \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \gamma u_2(1 - u_2) - \gamma_2 u_1 u_2 \end{cases} \tag{41}$$

The system of equations (41) under boundary conditions (39) is satisfied by functions independent of the spatial coordinate. Therefore, stability of solutions of equations (41) under boundary conditions (39) will coincide with stability of solutions of equations (3). Therefore, the stability of solutions satisfying only boundary conditions (40) will be investigated below. In the stationary case of a system of equations (41) under boundary conditions (40) the trivial solution $u_1 = 0, u_2 = 0$ is satisfied. The perturbation of this equilibrium state is represented as $u_1 = \delta u_1,\ u_2 = \delta u_2$, where δu_1 and δu_2 are small compared to the unit of magnitude:

$$0 \le \delta u_1 = 1, 0 \le \delta u_2 = 1.$$

Then the equations (41) with precision to the values of the second order of smallness are reduced to the form:

$$\begin{cases} \frac{\partial \delta u_1}{\partial t} = D_1 \frac{\partial^2 \delta u_1}{\partial x^2} + u_1 \delta, \\ \frac{\partial \delta u_2}{\partial t} = D_2 \frac{\partial^2 \delta u_2}{\partial x^2} + \gamma u_2 \delta. \end{cases} \quad (42)$$

The solution of these equations satisfying the boundary conditions (40) is represented in the form of trigonometric series:

$$\delta u_1 = \sum_{k=0}^{\infty} A_k(t) \sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l}, \qquad \delta u_2 = \sum_{k=0}^{\infty} B_k(t) \sin\left(k\pi - \pi/2\right)\frac{x}{l}.$$

In this case, the decomposition coefficients shall satisfy the equations:

$$\begin{cases} \frac{dA_k}{dt} = \left(1 - D_1\left(\left(k - \frac{1}{2}\right)\frac{\pi}{l}\right)^2\right)A_k, \\ \frac{dB_k}{dt} = \left(\gamma - D_2\left(\left(k - \frac{1}{2}\right)\frac{\pi}{l}\right)^2\right)B_k. \end{cases} \quad (k = 1,2,\dots)$$

It follows that when the inequality $\left(\frac{2l}{\pi}\right)^2 < D_1$ all coefficients of A_k will be decreasing functions of time and, accordingly, the solution $u_1 = 0$ will be stable. And when the inequality is fulfilled:

$$\gamma\left(\frac{2l}{\pi}\right)^2 < D_2$$

It means that when individuals are highly mobile, small populations in the model under consideration die. Biology and economics are not homogeneous. In biology, for example, different types of trees grow in different areas depending on the climatic conditions. In economics, this manifests itself in different traditions, resources, and so on.

Numerical solution. It is not possible to construct an analytical solution of the nonlinear equations (41). Therefore, various methods of approximation of equations (41) or their solutions are used [123]. The finite-difference approximation of equations and variational methods, based on the representation of the solution in the form of a linear combination of analytic functions, have become most popular. Numerical solution of equations (41) satisfying boundary conditions (40) on the segment is searched in the form of sum of trigonometric functions [124]:

$$u_1 = \sum_{k=1}^{n} A_k(t)\sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l}, u_2 = \sum_{k=1}^{n} B_k(t)\sin(k\pi - \pi/2)\frac{x}{l}. \tag{43}$$

The system of functions $sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l}$ $(k = 1,2,\dots)$ satisfies the boundary conditions (40), is complete and orthogonal on the segment $[0, l]$. After substitution of expressions (43) into equations (41), multiplying the last variable by $sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l}$ $(k = 1,2,\dots)$ and subsequent integration along the interval $[0, l]$ a system of ordinary differential equations for the coefficients A_k and B_k $(k = 1,2,\dots)$:

$$\begin{cases} \dfrac{dA_k}{dt} = -D_1\left(\dfrac{k\pi - \frac{\pi}{2}}{l}\right)^2 A_k + \dfrac{2}{l}\displaystyle\int_0^l u_1(1 - u_1 - \gamma_1 u_2)\sin\dfrac{\left(k\pi - \frac{\pi}{2}\right)x}{l}dx, \\ \dfrac{dB_k}{dt} = -D_2\left(\dfrac{k\pi - \frac{\pi}{2}}{l}\right)^2 B_k + \dfrac{2}{l}\displaystyle\int_0^l \gamma u_2(1 - u_2 - \gamma_2 u_1)\sin\dfrac{\left(k\pi - \frac{\pi}{2}\right)x}{l}dx \end{cases} \tag{44}$$

Initial conditions for functions $A_k(t)$ and $B_k(t)(k = 1,2,\dots,n)$ are determined from the relations:

$$\begin{cases} A_k(0) = \dfrac{2}{l}\displaystyle\int_0^l u_{10}(x)\sin(k\pi - \pi/2)\dfrac{x}{l}dx, \\ B_k(0) = A_k(0) = \dfrac{2}{l}\displaystyle\int_0^l u_{20}(x)\sin(k\pi - \pi/2)\dfrac{x}{l}dx. \end{cases}$$

For one expansion term $(n = 1)$ in (43) for a unit length segment $(l = 1)$ the coefficients $A_1(t)$ and $B_k(t)$ satisfy the equations:

$$\begin{cases} \frac{dA_1}{dt} = -\frac{\pi^2}{4} D_1 A_1 + A_1 - \frac{8}{3\pi} A_1 (A_1 + \gamma_1 B_1), \\ \frac{dB_1}{dt} = -\frac{\pi^2}{4} D_2 B_1 + \gamma B_1 - \frac{8}{3\pi} \gamma B_1 (\gamma_2 A_1 + B_1). \end{cases} \tag{45}$$

The Jacobi matrix on the right side of these equations:

$$J = \begin{pmatrix} 1 - \frac{\pi^2}{4} D_1 - \frac{8}{3\pi}(2A_1 + \gamma_1 B_1) & -\frac{8}{3\pi}\gamma_1 A_1 \\ -\frac{8}{3\pi}\gamma\gamma_2 B_1 & \gamma - \frac{\pi^2}{4} D_2 - \frac{8}{3\pi}\gamma(\gamma_2 A_1 + 2B_1) \end{pmatrix}.$$

The system of equations (45) has four stationary points:

1. $\boldsymbol{A_1 = 0, B_1 = 0}$. At this stationary point, the eigenvalues

$$\lambda_1 = 1 - \frac{\pi^2}{4} D_1, \lambda_2 = \gamma - \frac{\pi^2}{4} D_2$$

   of Jacobi matrices are negative, if the equations are $\frac{4}{\pi^2} < D_1$ and $\frac{4\gamma}{\pi^2} < D_2$. In this case, the trivial stationary point will be stable. That is, with high mobility, both populations can die.

2. $\boldsymbol{A_1 = 0, B_1 = \frac{3\pi}{8}\left(1 - \frac{\pi^2}{4\gamma} D_2\right)}$. This stationary point will be stable if the eigenvalues of the Jacobi matrix are negative:

$$\lambda_1 = 1 - \frac{\pi^2}{4} D_1 - \gamma_1 \left(1 - \frac{\pi^2}{4\gamma} D_2\right), \lambda_2 = -\left(\gamma - \frac{\pi^2}{4} D_2\right).$$

3. $\boldsymbol{A_1 = \frac{3\pi}{8}\left(1 - \frac{\pi^2}{4\gamma} D_1\right), B_1 = 0}$. This stationary point will be stable if the eigenvalues of the Jacobi matrix are negative:

$$\lambda_1 = -\left(1 - \frac{\pi^2}{4} D_1\right), \lambda_2 = \gamma - \frac{\pi^2}{4} D_2 - \gamma\gamma_2 (1 - \frac{\pi^2}{4} D_1.$$

4. $\boldsymbol{A_1 = \frac{1}{1-\gamma_1\gamma_2} \frac{3\pi}{8}\left(1 - \gamma_1 - \frac{\pi^2}{4} D_1 + \frac{\pi^2}{4\gamma} D_2\right)}$,

   $$\boldsymbol{B_1 = \frac{1}{1-\gamma_1\gamma_2} \frac{3\pi}{8}\left(1 - \gamma_2 - \frac{\pi^2}{4\gamma} D_2 + \frac{\pi^2}{4} D_1\right)} \tag{46}$$

If the inequality $\gamma_1\gamma_2 < 1$ is satisfied with positive values of A_1 and B_1 the second and third stationary points will be unstable. The eigenvalues of the Jacobi matrix in the considered stationary point are the roots of the quadratic equation:

$$\lambda^2 + \frac{8}{3\pi}(A_1 + \gamma B_1)\lambda + \left(\frac{8}{3\pi}\right)^2 \gamma(1 - \gamma_1\gamma_2)A_1B_1 = 0.$$

The roots of this equation under the equations $\gamma_1 < 1$ and $\gamma_2 < 1$ will have negative real parts. That is, this stationary point is stable if $A_1 > 0$ and $B_1 > 0$. In this case, the second and third stationary points will be unstable. The total number of the population on the segment is calculated using the formulas – $M_1 = \frac{2A_1}{\pi}$ and $M_2 = \frac{2B_1}{\pi}$. As can be seen from (46), an increase in the mobility of a single population (parameters D_1 or D_2) leads to a decrease in its total number in the segment and an increase in the total number of the competing population.

Thus, the conditions for the existence of the fourth stationary point, provided that $\gamma_1 < 1$ and $\gamma_2 < 1$, are determined by a system of inequalities:

$$\frac{\pi^2}{4}D_1 < 1 \, \frac{\pi^2}{4\gamma}D_2 < 1$$

$$0 < \frac{4}{\pi^2}(1 - \gamma_1) - D_1 + \frac{1}{\gamma}D_2$$

$$0 < \frac{4}{\pi^2}(1 - \gamma_2) - \frac{1}{\gamma}D_2 + D_1$$

In the coordinate system (D_1, D_2) these inequalities define the range of values of the parameters D_1 and D_2, in which the fourth stationary point is realized and stable. Analysis of the behavior of the solution with a larger number of terms in representations (45) is not possible without the use of numerical methods. The Cauchy problem for the system of equations (45) was solved using the finite difference method.

In some literature, the Volterra competition model is analyzed for stability, but simulation is not carried out [125]. The model (41) for n populations, considering the environmental factor, will have the form (47):

$$\frac{\partial u_i}{\partial t} = D_i \frac{\partial^2 u_i}{\partial x^2} + \mu_i u_i \left(1 - u_i - \varepsilon \sum_{j=1, j\neq i}^{n} \beta_{ij} u_j\right) \quad i = 1, \dots, n \qquad (47)$$

Scenario No1. Competition of n populations in low environmental factor conditions. In this case, the parameter generation for the model (41) is as follows:

$$D_i = rand_{0,001},\ \mu_i = rand_{0.03,0.1},\ \beta_i = 1 + rand_{0,1}$$

where $rand_{0,001}$ corresponds to a random rational number in the interval $[0, 0.001]$, $rand_{0,1}$ corresponds to a random rational number in the interval $[0, 0.1]$. As the initial conditions, a 10x10 grid is generated, where the distribution of populations begins according to the formula:

$$Grid = \left|\frac{\sin(x)}{x} - r_1\right| * \left|\frac{\sin(y)}{y} - r_2\right|,$$

where $r_1 = rand_1, r_2 = rand_1, rand_1 -$ random integer from interval [0,1]. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 48 shows the distribution density of the concentration coefficient CR_3 for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.15$, which indicates a low concentration of the competitive environment. The median survival rate is $alive = 50$.

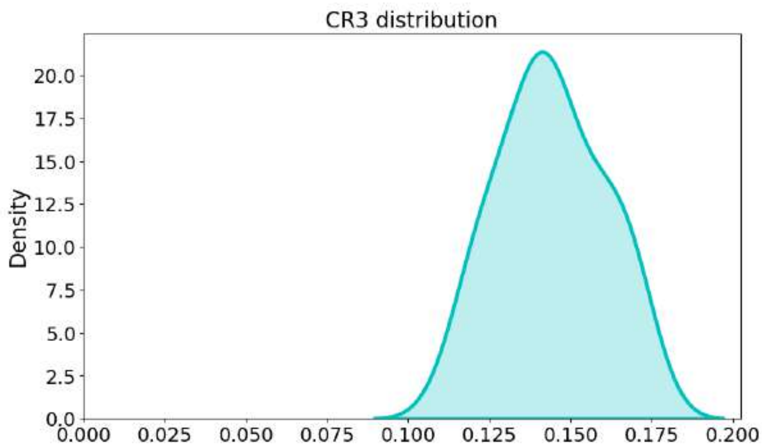


Fig. 48. Density of concentration index distribution

Scenario No2. Competition of n populations under conditions of variation in the intensity of the environmental factor. Number of populations $n = 50$, number of experiments $N = 500$. The coefficients are generated in the same way, except for ε:

$$\varepsilon \in [0.01, 150].$$

This scenario examines the dependence of the concentration index on the intensity of competition, as well as the dependence of population survival on the intensity of environmental factor. Figure 49 (A) shows the dependence of the concentration coefficient on the intensity of environmental factor. Figure 49 (B) shows the dependence of the number of surviving populations on the environmental factor. The medium factor $\varepsilon = 8$ is the threshold value for low intensity. The higher the intensity of environmental factor, the lower the population survival rate.

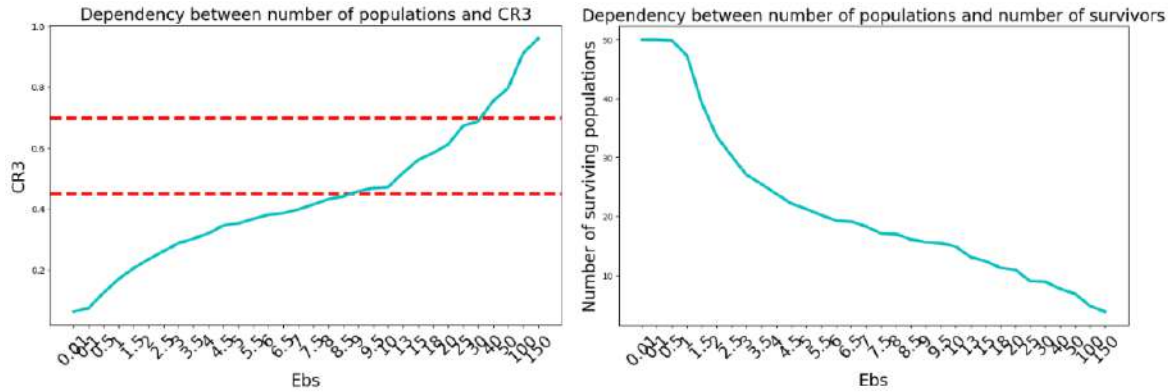


Fig. 49. A – Dependence of the concentration index on the environmental factor
B – Dependence of the number of surviving populations on the environmental factor

The dependence of the thresholds and survival rates on the concentration of the medium is shown in Table 24.

Table 24. Threshold values of environmental factor and survival rate

Concentration of the environment	Threshold values of the environment factor	Survival thresholds
Low	$\varepsilon < 9$	$alive > 19$
Moderate	$9 < \varepsilon < 25$	$alive \in [10, 19]$
High	$\varepsilon > 25$	$alive < 10$

Figure 50 shows the dependence of population size on the environmental factor. The number of populations decreases as the intensity of competition increases, but the size of the population does not change much and the population remains large.

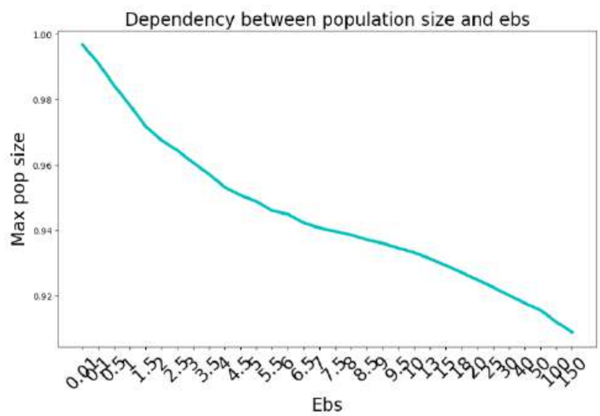


Fig. 50. Dependence of population size on environment factor

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (47) for k populations with similar generation of coefficients.

$$D_i = rand_{0,001},\ \mu_i = rand_{0.03,0.1},\ \beta_i = 1 + rand_{0,1},\ i = 1 \dots k$$

In this scenario, the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations, is considered in the same way as the previous scenario. Figure 51 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a competitive environment, the required number of competitors should be at least 16 entities within the model. Figure 51 (B) shows the dependence of the number of surviving populations on the number of populations.

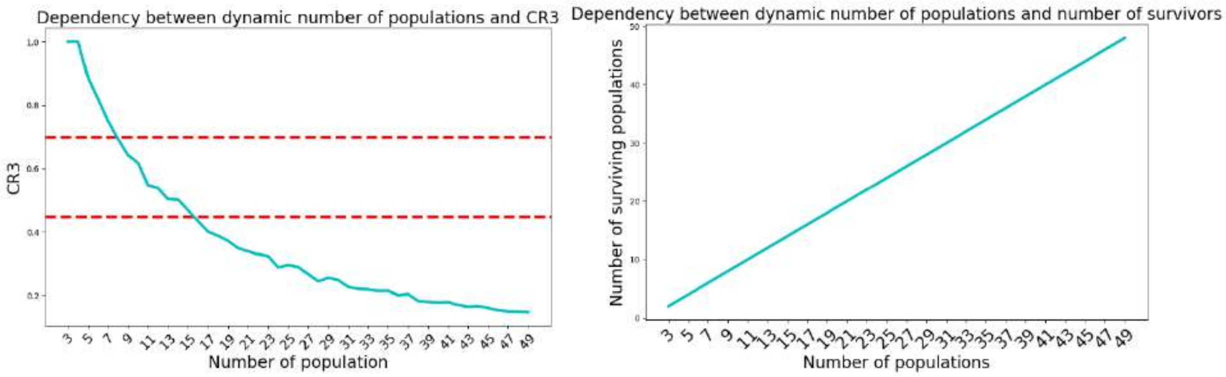


Fig. 51. A – Dependence of the concentration coefficient on the total number of populations
B – Dependence of survival rate on the total number of populations

3.2.3. Competition on a segment with a meeting on a resource

This model (48) is a resource model. In the proposed model, competition begins only at the moment when species meet on a given resource.

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1 \frac{\partial^2 u_1}{\partial x^2} + \mu_1 u_1 \left(1 - \frac{u_1}{K_1}\right) - a_{12} u_1 u_2 \frac{S}{b_1 + S}, \\ \frac{\partial u_2}{\partial t} = D_2 \frac{\partial^2 u_2}{\partial x^2} + \mu_2 u_2 \left(1 - \frac{u_2}{K_2}\right) - a_{12} u_1 u_2 \frac{S}{b_1 + S}, \\ \frac{\partial S}{\partial t} = -\gamma_1 \frac{S}{b_1 + S} u_1 - \gamma_2 \frac{S}{b_2 + S} u_2 + \mu_3 S(1 - \frac{S}{K}) \end{cases} \tag{48}$$

That is, in the framework of this model, competition arises when two populations meet and the resource begins to be consumed. For n populations, the model will take the form of:

$$\begin{cases} \frac{\partial u_i}{\partial t} = D_i \frac{\partial^2 u_i}{\partial x^2} + \mu_i u_i \left(1 - \frac{u_i}{K_i}\right) - u_i \sum_{j=1}^{n} a_{ij} u_j \frac{S}{b_j + S}, \\ \frac{\partial S}{\partial t} = -\gamma \sum_{i=1}^{n} u_i \sum_{j=1}^{n} a_{ij} u_j \frac{S}{b_j + S} + \mu_s S(1 - \frac{S}{K_s}) \end{cases} \tag{49}$$

Scenario No1. Competition of n populations in low environmental factor conditions. In this case, the parameter generation for the model (49) is as follows:

$$D_i = rand_{0,001}, \mu_i = rand_{0.03,0.1}, a_i = rand_{0,1},$$

$$\gamma_i = rand_{0,1}, b_i = rand_{0,1}$$

where $rand_{0,001}$ corresponds to a random rational number in the interval $[0, 0.001]$, $rand_{0,1}$ corresponds to a random rational number in the interval $[0, 0.1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 52 (A) shows the distribution density of the concentration coefficient CR_3 for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.21$, which indicates a low concentration of the competitive environment. Figure 52 (B) shows the population survival density distribution for 500 experiments. The median survival rate is $alive = 50$. Thus, out of 50 populations, 100% of the populations in this scenario survive.

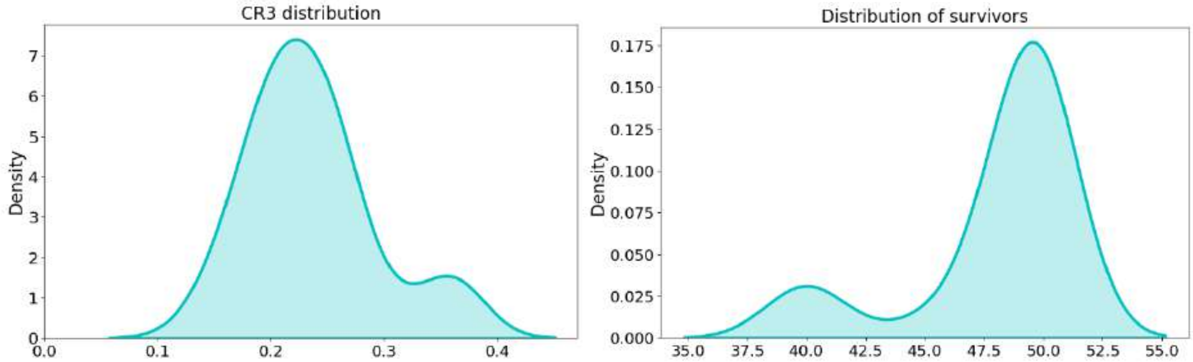


Fig. 52. A – Density of concentration index distribution
B – Density of distribution of surviving populations

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (49) for k populations with similar generation of coefficients.

$$D_i = rand_{0,001}, \mu_i = rand_{0.03,0.1}, a_i = rand_{0,1},$$

$$\gamma_i = rand_{0,1}, b_i = rand_{0,1} i = 1 \dots k$$

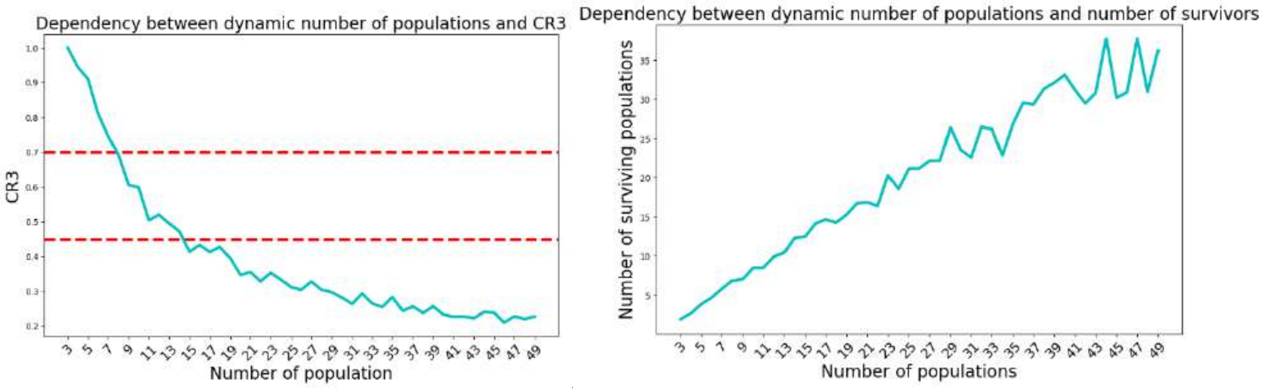


Fig. 53. A – Dependence of the concentration index on the total number of populations
B – Dependence of survival rate on the total number of populations

In this scenario, the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations, is considered in the same way as the previous scenario. Figure 53 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a competitive environment, the required number of competitors should be at least 20 participants within the model. Figure 53 (B) shows the dependence of the number of surviving populations on the number of populations. Threshold values of the number of subjects for determining the concentration of the medium are given in Table 25.

Table 25. Threshold values of the number of subjects.

Concentration of the environment	Subject threshold number
Low	$k > 15$
Moderate	$k \in [8, 15]$
High	$k < 8$

3.2.4. Operational competition model

Similar to the local operational model of competition, the space-time operational model (50) is considered:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + \mu_1 u_1 \left(\dfrac{s}{s+b_1} - u_1\right) - \beta_1 f(u_1, u_2), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \mu_2 u_2 \left(\dfrac{s}{s+b_2} - \dfrac{u_2}{K_2}\right) - \beta_2 f(u_1, u_2), \\ \dfrac{\partial S}{\partial t} = D_3 \dfrac{\partial^2 s}{\partial x^2} - \gamma_1 \mu_1 \dfrac{S}{b_1 + S} u_1 - \gamma_2 \mu_2 \dfrac{S}{b_2 + S} u_2 + \mu_S S(1 - \dfrac{S}{K}) \end{cases} \quad (50)$$

The functions $\frac{s}{s+b_1}$ and $\frac{s}{s+b_2}$ characterize the share of the consumed resource of each population separately. The functions $\beta_1 f(u_1, u_2)$ and $\beta_2 f(u_1, u_2)$ characterize the influence of populations on each other through direct contact or external factors. It is proposed that the function $f(u_1, u_2)$ can have the form:

$$f(u_1, u_2) = \frac{u_1 u_2}{(\alpha_1 + u_1)(\alpha_2 + u_2)}$$

If $\beta_1 = 0, \beta_2 = 0,$ then:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + \mu_1 u_1 \left(\dfrac{s}{s+b_1} - u_1\right), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \mu_2 u_2 \left(\dfrac{s}{s+b_2} - u_2\right), \\ \dfrac{\partial S}{\partial t} = D_3 \dfrac{\partial^2 s}{\partial x^2} - \gamma_1 \mu_1 \dfrac{S}{b_1 + S} u_1 - \gamma_2 \mu_2 \dfrac{S}{b_2 + S} u_2 + \mu_S S(1 - \dfrac{S}{K}) \end{cases} \quad (51)$$

In this model, competition occurs only on the trophic resource s. For n populations, the model (50) will take the form (51):

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = D_i \dfrac{\partial^2 u_i}{\partial x^2} + \mu_i u_i \left(\dfrac{s}{s+b_i} - u_i\right) \\ \dfrac{ds}{dt} = D_s \dfrac{\partial^2 s}{\partial x^2} - \displaystyle\sum_{i=1}^{n} \gamma_i \mu_i u_i \dfrac{s}{s+b_i} + \mu_s s\left(1 - \dfrac{s}{K}\right) \end{cases} \quad (52)$$

Scenario No1. Competition of $\mathbf{n}$ populations in low environmental factor conditions. In this case, the parameter generation for the model (52) is as follows:

$$D_i = rand_{0,001}, \mu_i = rand_{0.03,0.1}, a_i = rand_{0,1},$$
$$\gamma_i = rand_{0,1}, b_i = rand_{0,1}$$

where $rand_{0,001}$ corresponds to a random rational number in the interval $[0, 0.001]$, $rand_{0,1}$ corresponds to a random rational number in the interval $[0, 0.1]$, $rand_{0.03,0.1}$ corresponds to a random rational number in the interval $[0.03, 0.1]$. The distribution density of the CR_3 concentration index and the survival rate of the populations is constructed. Figure 54 shows the distribution density of the concentration coefficient CR_3 for 500 experiments. The probabilistic concentration index for 500 experiments corresponds to $CR_3 = 0.15$, which indicates a low concentration of the competitive environment. The median survival rate is $alive = 50$.

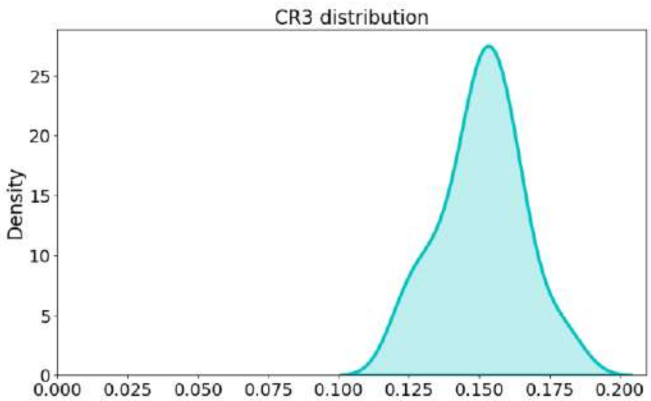


Fig. 54. Density of concentration index distribution CR_3

Scenario No3. Dynamic simulation. Number of populations $n = 50$, number of experiments $N = 500$. In the first step, 2 populations are considered, in the second step, 3 populations and so on up to 50 populations. At each step, 500 experiments are performed and the average value is selected. In this interaction scenario, in each experiment there is a solution of the system (52) for k populations with similar generation of coefficients.

This scenario considers the dependence of the concentration index on the total number of populations, as well as the dependence of species survival on the total number of populations in the same way as the previous scenario. Figure 55 (A) shows the dependence of the concentration coefficient on the total number of populations. To ensure a competitive environment, the required number of competitors should be at least 15 entities within the model. Figure 55 (B) shows the dependence of the number of surviving populations on the number of populations.

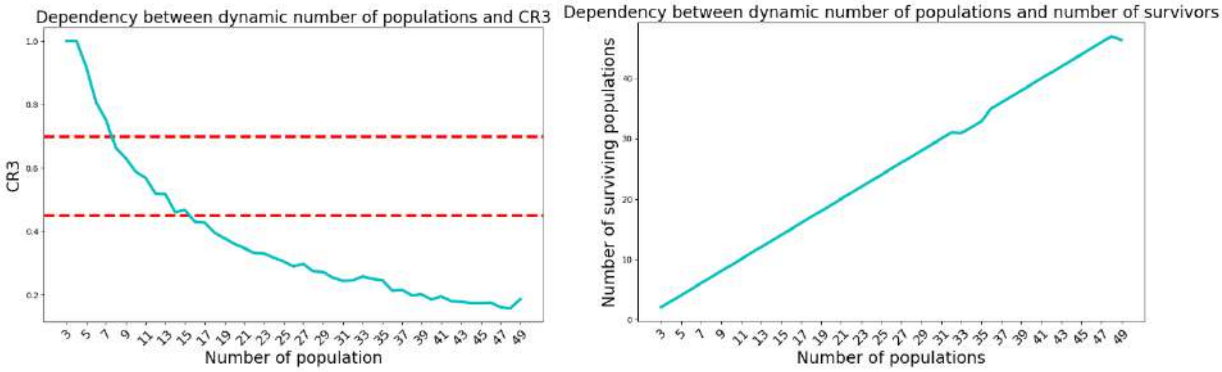


Fig. 55. A – Dependence of the concentration index on the total number of populations,
B – Dependence of survival rate on the total number of populations

Threshold values of the number of subjects for determining the concentration of the medium are given in Table 26.

Table 26. Threshold values of the number of subjects

Concentration of the environment	Subject threshold number
Low	$k > 15$
Moderate	$k \in [7, 15]$
High	$k < 7$

Conclusion

The present work is devoted to the problem of analysis of competitive environment in biological and economic systems. Theoretical analysis of the global food market showed that the existing criteria are not sufficient to draw an unambiguous conclusion about the market situation and the existing indices of assessing the level of market competition need to be redefined. Analysis of existing models of competitive relations showed that most models do not consider key indicators of competition, such as change of habitat, niche determinancy, hierarchical relations, trophic resource. In the course of the dissertation research the following key results were obtained:

- The global food market has been analyzed in terms of 120 items, 12 product categories and 190 economic entities. The analysis of Faostat statistical data showed that during the competition most economic entities remain on the market in the studied time interval of 60 years, that is, the share of the countries that have left the market is low.
- On the basis of the statistical analysis of the food market, three criteria for assessing the market state and measuring the level of competition of economic entities have been developed - group, correlation and share criteria. The group criterion allows assessing the product category and revealing monopolistic relationships, the correlation criterion assesses the state of the market, the share criterion determines the core of the market.
- 11 new local and spatial models for interference and operational competition were developed, considering the environmental factor, time factor, trophic resource and other factors.
- The model for n populations was also developed for all the proposed models. Simulations were also carried out. By means of simulation modeling, the principle of competitive exclusion of Gause was analyzed, threshold values of environment factor and number of competing entities for the corresponding level of competition were determined.

- The principle of competitive exclusion of Gause has not been confirmed for the models reviewed. The most likely cause of population death is the influence of the external environment on the ecosystem. Some of the results are consistent with some positions of the theory of neutralism.
- In all models, the stationary state of the system was analyzed and the stability was proved. With a small environment factor, the stationary point is also stable due to continuity.
- Stability of trivial states is analyzed for spatial systems. Analytical and quantitative results for distributed models are consistent with those of local models.

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