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Time of influence as a decision variable in game-theoretic models of opinion dynamics in social groups

1.2.3. Theoretical informatics, cybernetics

DISSERTATION

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of Candidate of Physical and Mathematical Sciences

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Contents

Introduction

Relevance of thesis topic

People exchange their thoughts and opinions for a long time beginning from face-to-face communication and letters to using nowadays online platforms. We all are involved in social interactions, such as discussions of popular topics on social media, voters' decision-making during political elections, and small-scale interactions in daily life. The social environment in which people live, their interactions with others, media influence on them, and many other factors can affect people's thoughts and opinions, thus creating a complex and rich network. In this network, dissemination, acceptance, and transformation of opinions are not only the result of individual behaviors, but also the reflection of group dynamics. In this dynamic process, people may reach a consensus, or there may be a disagreement. We will study this phenomenon and establish mathematical models of the process called opinion dynamics.

The study of opinion dynamics covers several disciplinary fields, including social sciences, computer sciences, mathematics, and physics, and it is used to understand and analyze a number of complex social phenomena. The results of our research on opinion dynamics can provide important references for policy makers, helping them to better understand the trends of social and public opinions, and formulate policies to be in line with public needs and expectations. Researchers can also analyze the influential and key nodes among individuals in social networks, which can play an important role in information dissemination, thus helping to effectively guide public opinions and for reaching the target opinion. In opinion dynamics models, we also study the patterns of changes in networks to better identify the paths of information spreading in the network, and provide real-time information spreading strategies for epidemic prevention and control, disaster warning, etc. In addition, the study of opinion dynamics can help companies to better understand consumer's attitudes and preferences, and they can develop more targeted marketing strategies to increase product sales and brand awareness. Enterprises can also predict consumer's behavior and purchase intentions to optimize product design, pricing and promotion strategies.

In summary, opinion dynamics modeling is conducted to understand the process of individual strategy adjustment. This means that opinion formation modeling is sensitive to new ideas and may change over time. In order to find a better model to describe the evolutionary process of opinion dynamics, many researchers tend to use game theory in their studies. Game theory is a mathematical area to study the interaction of individuals, and combined with the study of opinion dynamics, it helps to deepen analysis of information transmission among people. This integrated approach enables mathematical modeling to be based on the characteristics of the problem and finding appropriate solutions. For example, in product marketing, choosing the right time for advertising and promotional activities is crucial for product sales. Other domains such as social media platforms, political elections, and social movements face similar timing problems. In addition, considering the influence of other individuals and their number also affects the change of individual opinions. Therefore, analyzing the impact of social network structure on opinion dynamics is also one of my research priorities.

This thesis focuses on modeling of opinion dynamics in small social networks. In the proposed models, we consider the aspects of "right time" choice to validate and influence social network when influencers solve the cost minimization problems.

We hope that our analysis and research can provide some ideas to solve the real-world problems.

Overview of the results in this area

According to the different aims of social development, opinion dynamics modeling has many classifications. For example, they are categorized into continuous opinion [11, 20] and discrete opinion [15, 28] cases taking into account the form of opinions. Opinion dynamics models can be studied in a macro or micro way taking into account the complexity of social networks. Macro models are generally used to describe the evolution of opinions in large social networks, which can selectively simplify the rules of evolution, such as the Ising model [55, 73], and the voter model [14, 47, 70]. In contrast, micro-models are suitable for small social networks, which describe opinion evolution from an individual's perspective, e.g., the DeGroot model [19]. Microscopic models are also suitable for explaining large social networks, which can describe interactions between individuals in more detail. Opinion dynamics originated from French's paper [25], which goes from simple assumptions about interpersonal relationships to many complex phenomena related to groups, and has been used in the study of sociology and psychology. Later G. Ray Funkhouser formally proposed the dynamics of public opinion [27], revealing that the news in media plays an important role in shaping public opinions and perceived reality. Based on French's research, DeGroot further suggested his model of reaching consensus, which made his model the most classic and fundamental in the field. It was the first to use the Markov process theory to model the dynamics of opinions in social networks.

The DeGroot model is applicable to different scenarios, which can lead to a number of corresponding variants, such as the bounded confidence model taking into account the fact that an individual's opinion is influenced only by the opinions of other people whose opinions do not differ from that of an individual by more than a certain confidence level in the opinion updating rule. Among bounded confidence models, the Deffuant-Weisbuch model [18] and the Hegselmann-Krause model [46] are the most representative. They are quite similar, but differ in opinion updating rules, with the former focusing on two-by-two interactions between individuals and the latter one focuses on inter-group interactions. The Friedkin-Johnsen model [26, 69] proposes a modification of the DeGroot model assuming that individuals are partially stubborn, i.e., that individuals will hold on to their initial opinions due to certain factors.

The models in [67, 50] are used to explore multidimensional opinion dynamics in social networks. Both models have open-loop information structures, and the former considers the impact of such factors as individual interactions, recalcitrance, and the speed of opinion change on reaching consensus and disagreement, while the latter focuses on strategic individuals with recalcitrant strategies and proposes a distributed implementation of the Nash/worst-case equilibrium solution. By studying group behavior [66], factors affecting opinion formation in multilayer networks can be analyzed, in particular, race, and ideology may affect network aggregation. The work [6] focuses on similarity of opinion attributes, the multiple-population mean-field game (MPMFG) is proposed, and the model can be used to estimate and predict the effects of people's behaviors of different groups or populations on other groups' beliefs and opinions, and the validity of the theory is demonstrated in analyzing the dynamics of multiple-population opinions.

The opinion maximization model can be used to design suitable algorithms for information spreading in specific groups. The work [44] considers the realism of dynamic changes (previous studies, where opinions remain unchanged after node activation, do not correspond to real scenarios where opinions change over time), the algorithm improves spreading rational opinions. In [36], the authors provide practical and effective algorithms for forming favorable opinions for targeted users forms overall favorable opinions for specific items of information. The research [57] combines opinion estimation and influence propagation in the marketing process to maximize the number of users who know about the product through seed user exposure, the proposed active learning framework improves overall opinion propagation in social networks. The interplay between social relationships, individual stubbornness, and opinion evolution is considered in [5] and it is demonstrated that opinion dynamics converge to consensus under certain conditions, but requires reasonable assumptions about changes in social relationships and individual beliefs. Paper [68] explores consensus formation in social networks using a dynamic non-cooperative game model where each member minimizes a cost function representing its motivation.

Nowadays, researchers are interested in examining and modeling of social networks over time. There are a large number of network participants and also external forces trying to influence these participants. The social network participants are usually called agents, and the external forces influencing their opinions are called players. The emergence of multi-agent systems [21] promises to address the complexity of social networks. It serves as a tool to decompose and assign complex problems to agents, who use the collected information to make decisions. These systems are widely used to model the opinion dynamics. Models are designed to describe the exchange of information among participants in a social network. Over time, a participant's information can be influenced by his neighbors or by external forces. It is important to understand how users in the network update their opinions based on their neighbors' opinions and the global opinion structure that is implied when users update their opinions interactively. It is worth mentioning that the paper [16] proposing the Biased Voter model studying the update of users' opinions based on the opinions of their neighbors. It also gives a preliminary theory about the convergence and structure of opinions in the whole network. However, in real life, consensus is rarely reached in actual opinion dynamics, so how do people form their own opinions? The paper [12] studies a sociological model by interpreting repeated averaging as the best response dynamics in the underlying game with natural payoffs, and examines the cost of disagreement for these models relative to the social optimum. Assume that each agent exchanges views with all his neighbors in each round. In particular, in the Hegselmann-Krause model, determining each agent's neighbors requires access to the opinions of all agents. However, this assumption can be questionable in modern society. Therefore, the paper [24] considers the convergence properties of dynamic views with local interactions and limited information exchange. They do this through a generic model where agents update their opinions rounded to a weighted average of the opinions in their neighborhood. Two variants of the Hegselmann-Krause model are considered, one accompanied by a fixed social network, and the other by a neighborhood of agents determined by random sampling in each round. The determination of interaction rules is important when studying opinion dynamics and consensus shapes. The paper [64] inspired by gas dynamics theory proposes a model of opinion dynamics in a multi-agent system consisting of two classes of agents. Since understanding of public opinion change patterns is often neglected, in paper [54] the authors propose a multi-subject model that can be used to identify some mechanisms that cause opinion change. Public opinion change in this model is a process of individual opinion shifts triggered by changes in the opinions of highly correlated subjects.

The mean-field game theory has been extensively used in the study of opinion dynamics. It was originated in [53] and it is characterized by the fact that the strategy choice of a single agent is influenced by the mass behavior of other agents. It has also been applied to many different fields, including economics, physics, biology, networks, and production engineering (see [1, 8, 9, 42]). For the situation when multiple social groups interact, the authors of [10] study emulation, imitation, and herding behavior in this phenomenon. The paper presents a detailed analysis of the mean-field game results for the polymorphic boundary and L^2 boundary cases on control and perturbation.

Combining opinion dynamics with optimization theory leads to some new results, for example, the linear-quadratic optimal control problem or game [61] focuses on the effect of different communication structures on information formation and the minimization of control costs. The study [7] analyzes the optimization problem of information control and proposes a reduction of a certain class of dynamic control problems to the typical problem of controllability study and design of linear discrete control systems. Some works on opinion dynamics are also formulated as discrete-time linear-quadratic problems or games [48, 58]. The opinion dynamics models with cooperative and noncooperative influencers are considered in [71, 72]. The mean-field optimal control problems are studied in [2, 22, 23]. Among many existing models, we found that external factors play an extremely important role in reaching consensus among the agents. The purpose of paper [77] is to investigate whether external factors influence the members to reach consensus in social networks. Controlling all participants' opinions around a predetermined level, the cases of one-player or two-player games were considered, and the optimal control and equilibrium were found using Bellman's equations. There are several studies considering the influence of average opinion in the society on individuals' opinions with limited capabilities of validated opinions in a linear-quadratic optimal control problem [32, 33]. The papers [31] and [30] consider linear-quadratic optimization problems in the opinion dynamics modeling. The former paper focuses on validating agents' opinions in the terminal time of a finite-time horizon. The latter paper focuses on the optimization problem when the difference between agents' opinions and socially desirable opinions are taken into account for a given number of validation periods.

Opinion dynamics modeling can play an important role in addressing the issue of information spreading in social networks. In this context, this research addresses the multifaceted theme of competitive opinion dynamics, and a phenomenon of increasing prominence in diverse social scenarios, such as online marketing, advertising, promotions, voting [13, 40, 52], etc. The overarching objective is to delve into the intricate mechanisms that underlie competition and opinion diffusion, which are pivotal factors in shaping opinions within connected communities. A scenario in which two centers of influence compete for the agents' attention within a network is considered in [62]. The analysis is limited with constraints imposed by a network structure. They find the necessary conditions of the Nash equilibrium and steady state for a given state dynamics. In another aspect, the SI_1SI_2S model is introduced to control dissemination of two opinions [41]. This model presents open-loop Nash control strategies that empower campaigners to actively monitor opinion spread and adapt their strategies in response. An innovative perspective is offered through the introduction of the cost-effective competition (CEC) problem [59]. A multi-objective optimization approach (MOCEC) is developed to aim at achieving more votes with minimized recruitment costs. Unlike the DeGroot model, this research [45] introduces an innovative dimension accounting for both individual competition and switching topologies within a social network. The analysis reveals whether the structure of the network topology (balanced or not) affects opinions to reach consensus. The paper [60] examines a problem of influence maximization in a social network where two players compete by means of dynamic targeting strategies. The authors obtained some elements for the characterization of equilibrium strategies through a model analysis. A game-theoretic model for competitive information dissemination in social network is proposed in [78]. It is shown that the speed of information spreading is influenced by characteristics of individuals. In the paper [39], the authors investigate the idea of keeping a scalar opinion of every agent above a predetermined ferment level over a finite time horizon. They obtain the optimal control trajectory with the turnpike property by using Pontryagin's maximum principle.

Influencers or media centers use various methods to control opinions of the social network members on the given topic, and then they try to keep the opinions closer to the desired level, this process can be modeled as a dynamic game of competition for agents' opinions [29, 49, 51].

Goals of the thesis

The main goal of this work is to study the opinion dynamics in small social networks in a non-cooperative and game-theoretic settings, taking the two aspects: (i) an influencer or player can optimally choose the periods in which the opinions of the agents in the network are taken under consideration in her objective function, (ii) an influencer or player can optimally choose the periods in which she controls (puts some impulse to) the agents' opinions, and these two sets of periods may be the same or different depending on the purpose of modeling. I am willing to find the corresponding optimal strategies, optimal state trajectories in the optimization problems, and equilibrium strategies and corresponding state trajectories for the given settings. Experiments are set up to discuss the efficiency of each influencing factor. The three sections of this thesis answer specific research questions:

Question 1: What is the optimal set of significant periods which should be taken into account into the objective function of the influencer?

Question 2: When would it be an optimal set of periods for a player to control for agents' opinions?

Question 3: What is the effect of competition between influencers on opinion dynamics and what are their costs when the number of players is more than one? What are the equilibrium outcomes?

Main tasks

In order to carefully examine the opinion dynamics optimal control and competitive models with the specific restrictions on influencers' or players' behavior and to answer the research questions given above, I formulate the following tasks of my work:

1. For small social groups, we first consider the effect of time validations on their members' opinions. We assume that participants are engaged in the social network using different models of choosing the periods: 1) the player assumes the agent's opinion in the last time is significant and we include the term with the agents' opinions in her functional, no other opinions are included in the functional, and 2) the player can choose the set of periods being significant ones and take them into her objective functional. The task is to find the optimal solution, i.e. the optimal set of periods minimizing the player's costs.

2. We consider the effect of choosing the set of periods to control agents' opinions. We study three scenarios: 1) the player includes all periods in her objective functional but she can choose the set of a limited number of periods in which she controls the agents' opinions; 2) the player chooses a set of a limited number of periods in which she can control agents and exactly these periods are included in the player's objective functional; and 3) the player can choose two different sets of periods (both of limited sizes) at which she controls opinions and validates opinions but does not control them. The task is to find the optimal policy and the optimal state trajectory under these restrictions on player's strategies.

3. The effect of competition between players when they both influence agents' opinions is considered. I increase the number of players while keeping the rules for opinion evolution unchanged. We extend the scenarios involved in the first two chapters. This task is to find the Nash equilibrium and corresponding state trajectories in considered games.

Scientific novelty

This thesis builds on the results of opinion dynamics research and proposes several new models with restrictions on players' or influencers' behavior. We analyze in detail the factors that affect solutions of the formulated optimization problems, including validation sets, sets of periods to control opinions, and the competition between players. These refinements help decision makers to make their solutions of the actual problems arising in social networks.

In this thesis, we consider the development of optimal or equilibrium strategies by decision makers in the absence of cooperation. Time is a key factor influencing opinion dynamics. We examine at what time players validate and control agents' opinions from a time perspective to minimize their costs. First, players do not need to monitor the opinion updating process all the time, which may greatly reduce the costs. Second, players make decisions or influence network members at the right periods, which can significantly improve the efficiency of influencers.

Finally, for different factors, we conducted comparative analysis and controlled variable experiments in the experimental section. The experimental results show that it is significant that the theorem provides the necessary conditions to find the optimal and equilibrium solutions. All models have Matlab code for faster computational screening of results. All theoretical results and program codes used in this thesis are produced by the author of this work.

Research methods

In this thesis, I used main concepts and methods of game theory (the Nash equilibrium, Pareto optimality, and normal-form games) and dynamic game theory (the Nash equilibrium in open-loop strategies, the Euler-equation approach, and Pontryagin's maximum principle), optimization theory, and basic concepts of mathematical programming. The comparative analysis is applied in the experimental part of my thesis.

Theoretical and practical significance

The theoretical results in this work further improve and extend the applications of average-oriented opinion dynamics from non-cooperative game theory perspective. Compared with research that requires a large amount of real data, such as deep learning and artificial intelligence, the models we built for simulations and analysis can reflect the real situation relatively accurately. This is because in reality, people's opinion data are often difficult to obtain, and a lot of them are ambiguous. Moreover, it is often difficult to analyze these data directly. Instead, modeling the problem by simplifying the interaction rules is more likely to provide reliable theoretical references for decision makers.

In Chapter 1, the models of opinion dynamics with two types of constraints on player's behavior are proposed. The models are applicable to real-world situations because influencers in reality do not validate the society opinion all the time, but rather find some specific periods significant in order to understand how opinions are evolving. I characterized the player's optimal behavior in these cases.

In Chapter 2, the issue of decision making with time restrictions is mainly considered. In the new models presented in this chapter, the player can choose the limited number of periods to validate opinions and to control them. I formulated the optimization problems for the influencer or player and characterized her optimal behavior in various cases of restrictions on controls.

Chapter 3 extends the models presented in Chapters 1 and 2 and the results on a multi-player case to better adapt for real world. The research in this chapter provides more options and guidance for practical decision making and helps decision makers to better cope with various scenarios. In this chapter, I characterized the Nash equilibria for two-player games in all examined cases.

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Brief description of the thesis structure

The thesis consists of an introduction, three chapters, conclusion, a list of references, and appendices. Each chapter begins with a description of the model and formulation of the problem, then theoretical results with their proofs are provided. The results including numerical simulations illustrating the theory are presented at the end of each chapter. The thesis consists of 124 pages (128 pages in Russian), including 19 tables and 38 figures. The bibliography is organized in the alphabetical order and contains 78 citations.

Chapter 1 focuses on the use of different time patterns by players to influence agents' opinions. In Section 1.1, we describe a small social network model where players validate opinions only in the last time over an arbitrary horizon, introduce the necessary terminology, and present theorems and proofs. In Section 1.2, we show numerical simulations. In Section 1.3, we describe the model where players validate opinions in restricted periods, the theorems and their proofs are also provided there. In Section 1.4, we set up experiments with control variables to compare the effect of target opinions and the number of validation periods on opinion dynamics, respectively. Section 1.5 summarizes the results of Chapter 1.

Chapter 2 focuses on opinion dynamics under three control scenarios. In Section 2.1, we describe a model of opinion dynamics with players controlling their opinions in limited number of periods, providing corresponding theorems and their proofs. In Section 2.2, the model where a player both validates and controls opinions in the same periods is described, the notation and opinion updating rules in the model are briefly described, and theorems and proofs are provided. In Section 2.3, the opinion dynamics when validating and controlling periods are represented by two sets without overlapping are considered. We explain the meaning of the notation and the opinion updating rules and prove the theorems. Section 2.4 provides numerical simulations and analyses of all the models of this chapter, and, in particular, sets up comparative experiments in Sections 2.4.2 and 2.4.3 to check the difference between the best and the worst results in terms of the corresponding costs. Finally, Section 2.5 summarizes the chapter.

Chapter 3 focuses on modeling noncooperative opinion dynamics with competition between players. This chapter mainly extends the models presented in Chapters 1 and 2 in a two-player case, so Section 3.1 is an extension of the model of Section 1.1. Section 3.2 is an extension of the model of Section 1.2. Section 3.3 is an extension of the model of Section 2.1. The first three sections of this chapter provide the corresponding model descriptions, theorems and proofs. In Section 3.4, numerical simulations of each model are shown and, in particular, comparative experiments are conducted and presented in Section 3.3. They are testing the validity of the resulting Nash equilibrium. We conclude the chapter with Section 3.5.

The conclusion of the thesis contains a brief description of the results obtained in the work.

Results submitted for defense

1. Models of opinion dynamics with different scenarios of restrictions on players'

or influencers' behavior in small social networks are presented. The restrictions are as follows: (i) the player assumes the agent's opinion at the last time significant and includes the term with the agents' opinions at the terminal time in her functional, no other opinions are considered in the functional, and (ii) the player can choose the set of periods to include into the set of significant ones and take them into her objective functional. For all these models, the optimization problems are formulated for a one-player case. The necessary conditions for the optimal strategies are found. A series of numerical simulations is conducted to test the results and to make conclusions about parameter influence on optimal strategies.

2. Models of opinion dynamics with different scenarios of restrictions on players' or influencers' controls in small social networks are presented. The restrictions are as follows: (i) the player can choose the periods when she controls opinions, and the size of this set is limited, and (ii) the player can choose the set of periods to control the opinions but they should be different from the periods when she is validating these opinions among agents. For all these models, the optimization problems are formulated and necessary conditions for the optimal strategies are proved. A series of numerical simulations is conducted to represent theoretical results.

3. The necessary condition for the Nash equilibria and corresponding state trajectories for several models described in Items 1 and 2 are found assuming the presence of two players influencing the network agents and competition on the opinions between them.

Main scientific results

1. The study proposes and implements a solution for selecting optimal opinion verification moments in finite time, see paper [33] in the bibliography

(personal contribution is at least 80%).

2. Equilibria dynamic games modeling competition when players optimize influence and reduce the costs are found, see paper [35] in the bibliography (personal contribution is at least 100%).

3. In the case when a player influences an agent's opinion at specific time periods in order to bring the community's opinion closer to her target opinion and to minimize costs, the optimal solution is found, see paper [34] in the bibliography (personal contribution is at least 100%).

4. In the case when players can exert influence on agents only for the limited number of periods and minimize costs by choosing moments of control, the optimal solution is found, see paper [32] in the bibliography (personal contribution is at least 80%).

5. In the case when a center of influence chooses the limited number of validation moments and at the same time it influences the agents' opinions at these moments, the optimal solution of the problem when to influence the agents is solved, see paper [30] in the bibliography (personal contribution is at least 80%).

6. The minimization of the costs of influence on agents' opinions making it closer to the target opinion when only terminal time period is validated is solved, optimal solution is found, see paper [31] in the bibliography (personal contribution is at least 80%).

7. The game of competition of two players minimizing their costs on influence of agents' opinions when there is the limited number of validation time moments is examined, the Nash equilibrium is found in open-loop strategies, see paper [29] in the bibliography (personal contribution is at least 100%).

Verification of results

The main results obtained in the thesis were presented at the International Conferences “Game Theory and Management” (Saint Petersburg, 2021, 2022); International Scientific Conference on “Control Processes and Stability” (Saint Petersburg, 2021, 2022), at the seminars of Department of Mathematical Game Theory and Statistical Decisions of Saint Petersburg State University.

Publications

Based on the results of the thesis, the following works were published: [29, 30, 31, 32, 33, 34, 35]. The papers [34] and [35] are published in the journals recommended by the Higher Attestation Commission of the Russian Federation. The works [30, 32, 33] are indexed in Scopus and Zentralblatt MATH.

Research publications

1. Gao J. Two-player opinion control game with limited observation moments // Control Processes and Stability. 2022. Vol. 9. N. 1. P. 464-468.

2. Gao J., Parilina E. M. Opinion Control Problem with Average-Oriented Opinion Dynamics and Limited Observation Moments // Contributions to Game Theory and Management. 2021. Vol. 14. P. 103-112.

3. Gao J., Parilina E. M. Average-oriented opinion dynamics with the last moment of observation // Control Processes and Stability. 2021. V. 8, N. 1. P. 505-509.

4. Gao J., Parilina E. M. Optimal control in a multiagent opinion dynamic system // Contributions to Game Theory and Management. 2022. Vol. 15. P. 51-59.

5. Gao J., Parilina E. M. Opinion dynamics in multiagent systems with optimal choice of opinion verification moments // Doklady Mathematics, 2023. Vol.

108. Suppl. 1. P. S75-S85.

6. Gao J. Controlled opinion formation in multiagent systems with constraints on control set // In: Petrosyan, L.A., Mazalov, V., Zenkevich, N.A. (eds) Frontiers of Dynamic Games. GTA 2022. Trends in Mathematics. Birkhauser, Cham. 2024. P. 27-42.

7. Gao J. Competition on agents' opinions in average-oriented opinion small dynamic systems with limited number of controls // Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes. 2024. Vol. 20. N. 3. P. 402-413.

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Chapter 1

Average-oriented opinion dynamics in social groups with limitations on validating periods

1.1 The case when opinions in terminal period are significant

We consider a small social network consisting of two agents. The agents can influence each other, and their opinions change over time, which is discrete and finite. Define the opinion of agent 1 at time t by $x_1(t) \in \mathbb{R}, t = 0, \ldots, T$ and the opinion of agent 2 at time t by $x_2(t) \in \mathbb{R}, t = 0, \ldots, T$. Vector $(x_1(t), x_2(t))$ represents the state variable at time t.

We also assume that there is a player who can influence agent 1 at any time $t = 0, \ldots, T-1$ with the control variable $u(t)$. The player does not influence the opinion of agent 1 at time T. The player has a target opinion s and is interested in converging the agents' opinions to this target opinion at the termination time, i.e. he is willing to minimize the difference between target opinion and opinions of the agents in the social network at the terminal time. We could also assume that the player can have no option to validate the agents' opinions along the time line.

The dynamics of two agents 1 and 2 opinions is given by

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right) + u(t), \tag{1.1}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right) \tag{1.2}$$

with initial condition

$$x_1(0) = x_1^0, x_2(0) = x_2^0 \tag{1.3}$$

where $a_1, a_2 \in \mathbb{R}^+$.

The minimized functional for the player is

$$J(u) = \sum_{t=0}^{T-1} \delta^t \left(cu^2(t)\right) + \delta^T \left((x_1(T)-s)^2 + (x_2(T)-s)^2\right). \tag{1.4}$$

Theorem 1.1. *Let $\{u^*(t) : t = 0,\dots,T-1\}$ be the optimal strategy minimizing functional (1.4) subject to initial conditions (1.3) and state dynamics (1.1) and (1.2), and $\{(x_1^*(t), x_2^*(t)) : t = 0,\dots,T\}$ be the corresponding state trajectory, then the optimal strategy $u^*(t), t = 0,\dots,T-1$ is*

$$u^*(t) = z^*(t+1) - Az^*(t)$$

and corresponding optimal state trajectory $(x_1^*(t), x_2^*(t)), t = 1,\dots,T$ satisfy equations:

$$\begin{cases} cA\delta^2 z(t+1) - Bz(t) + Cz(t-1) - Acz(t-2) = 0, \\ \quad t = 2,\dots,T-1, \\ -\left(cA - \frac{a_2\delta}{2}\right) z(T) + \left(A^2c + \frac{c}{\delta}\right) z(T-1) - A\frac{c}{\delta} z(T-2) \\ \quad = -\delta a_2 (x_2(T) - s), \\ \frac{c}{\delta}(z(T) - Az(T-1)) + z(T) + x_2(T) - s = 0, \\ x_2(t+1) = x_2(t) + \frac{a_2}{2} z(t), \quad t = 1,\dots,T-1, \end{cases} \tag{1.5}$$

where $z^*(t) = x_1^*(t) - x_2^*(t), A = 1 - \frac{a_1+a_2}{2}, B = cA\delta + c\delta + cA^2\delta$, and $C = c + cA\delta + cA^2\delta$.

Proof. We consider a new variable $z(t)$

$$z(t) = x_1(t) - x_2(t), t = 0, \cdots, T.$$

From state equations (1.1), (1.2) substituting $z(t)$, we obtain the new state equations:

$$z(t+1) = Az(t) + u(t), \tag{1.6}$$

$$x_2(t+1) = x_2(t) + \frac{a_2}{2} z(t) \tag{1.7}$$

with initial condition

$$z(0) = x_1^0 - x_2^0, x_2(0) = x_2^0,$$

where $A = 1 - \frac{a_1+a_2}{2}$.

We find expression of $u(t)$ from (1.6):

$$u(0) = z(1) - Az(0)$$

$$u(t) = z(t+1) - Az(t) = q(z(t), z(t+1)), t = 0, \cdots, T-1.$$

Then we substitute expression to $\sum_{t=0}^{T} \delta^t g_t(x(t), x(t+1))$, we can get

$$J(z, x_2) = c(z(1) - Az(0))^2 + \sum_{t=1}^{T-1} \delta^t \left[c(z(t+1) - Az(t))^2 \right]$$
$$+ \delta^T \left[(z(T) + x_2(T) - s)^2 + (x_2(T) - s)^2 \right].$$

Minimizing $J(z, x_2)$ under equations (1.6)-(1.7), we form the Lagrange function

$$L(z, x_2, k) = J(z, x_2) + \sum_{t=1}^{T-1} k_t \left(x_2(t+1) - x_2(t) - \frac{a_2}{2} z(t) \right).$$

The first order conditions should be $\frac{\partial L(z,x_2)}{\partial z(t)} = 0, t = 1, \ldots, T$ and $\frac{\partial L(z,x_2)}{\partial x_2(t)} = 0, t = 1, \ldots, T$.

Using Euler equation approach, we get

$$\frac{\partial J(z, x_2)}{\partial z(t)} = 2c(z(t) - Az(t-1)) - 2A\delta c \{z(t+1) - Az(t)\},$$

$$\frac{\partial J(z,x_2)}{\partial x_2(t)} = 0, \forall t \neq T,$$

$$\frac{\partial J(z,x_2)}{\partial x_2(T)} = \delta^T \left\{2\left(z(T) + x_2(T) - s\right) + 2\left(x_2(T) - s\right)\right\},$$

$$\begin{cases} -\frac{cA}{\delta} z(t-1) + z(t)\left(\frac{c}{\delta} + cA^2\right) - cAz(t+1) = \frac{a_2}{4} k_t \delta^{-t}, \\ \quad t = 1, \dots, T-1, \\ \frac{c}{\delta}\left(z(T) - Az(T-1)\right) + z(T) + x_2(T) - s = 0, t = T, \\ k_{t-1} - k_t = 0, t = 2, \dots, T-1, \\ \delta^t \left[2z(t) + 4\left(x_2(t) - s\right)\right] + k_{t-1} = 0, t = T \end{cases} \tag{1.8}$$

with initial condition $z(0) = x_1^0 - x_2^0$, $x_2(0) = x_2^0$.

Excluding k_t from system (1.8) we can get

$$\begin{cases} cA\delta^2 z(t+1) - Bz(t) + Cz(t-1) - Acz(t-2) = 0, \quad t = 2, \dots, T-1, \\ -\left(cA - \frac{a_2\delta}{2}\right) z(T) + \left(A^2 c + \frac{c}{\delta}\right) z(T-1) - A\frac{c}{\delta} z(T-2) = -\delta a_2\left(x_2(T) - s\right), \\ \frac{c}{\delta}\left(z(T) - Az(T-1)\right) + z(T) + x_2(T) - s = 0, \end{cases}$$

where $B = cA\delta + c\delta + cA^2\delta$, and $C = c + cA\delta + cA^2\delta$.

The theorem is proved. □

1.2 Numerical examples for Section 1.1

Example 1.1. *Let* $a_1 = 0.2, a_2 = 0.6, \delta = 1, c = 2$ *and initial opinions be* $x_1(0) = 0.7, x_2(0) = 0.2$. *For time horizon* $T = 10$ *and target opinion* $s = 0.4$, *the optimal state and optimal control trajectories are presented in Table 1.1. The optimal value for functional (1.4) is 0.068.*

For the same parameters and duration $T = 10$ *we introduce optimal state and strategy trajectory on Figures 1.1 and 1.2, respectively.*

Table 1.1: Optimal control trajectories and state.

t	0	1	2	3	4	5
$x_1(t)$	0.7	0.4714	0.4159	0.4024	0.3991	0.3984
$x_2(t)$	0.2	0.35	0.3864	0.3953	0.3974	0.3979
$z(t)$	0.5	0.1214	0.0295	0.0071	0.0017	0.0005
$u(t)$	-0.1786	-0.04334	-0.0106	-0.00256	-0.00052	-0.0004
t	6	7	8	9	10	
$x_1(t)$	0.398	0.3984	0.3976	0.3984	0.3989	
$x_2(t)$	0.3981	0.398	0.3982	0.398	0.3981	
$z(t)$	-0.0001	0.0004	-0.0006	0.0004	0.0008	
$u(t)$	0.00046	-0.00084	0.00076	0.00056		

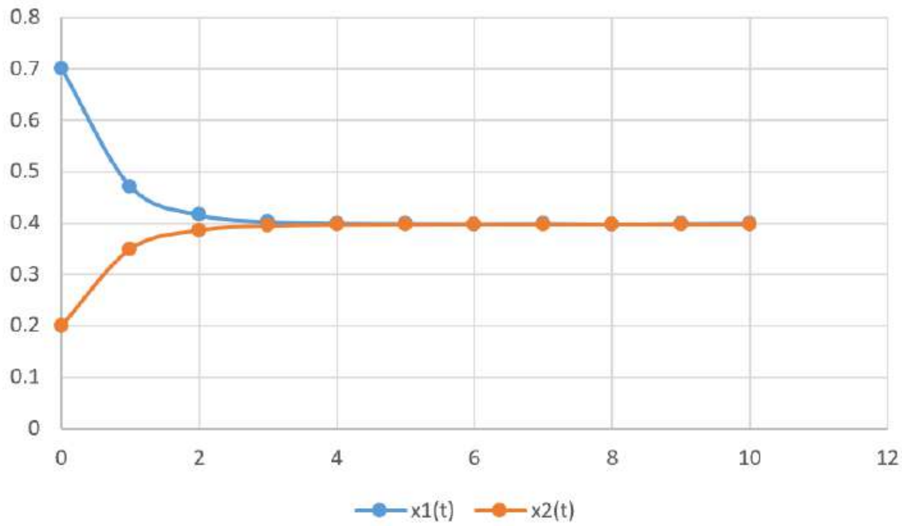

Figure 1.1: State trajectories (blue - $x_1(t)$, red - $x_2(t)$).

From Figure 1.1, it can be seen that the opinions of both agents gradually approach the target opinion under the player's influence. From figure 1.2 it can be seen that the player's influence on the agents also tends to be stable.

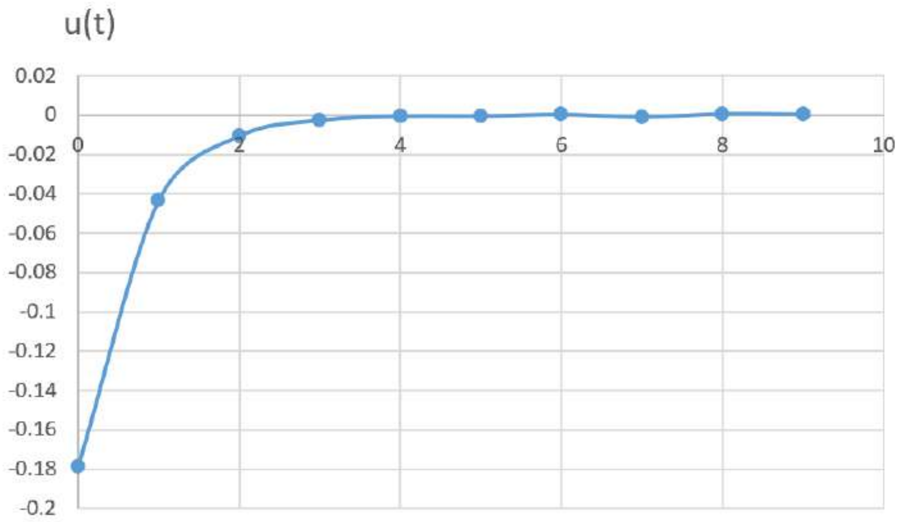

Figure 1.2: Strategy trajectory $u(t)$.

Example 1.2. *Let $a_1 = 2, a_2 = 6, \delta = 1, c = 2$, and initial opinions be $x_1(0) = 0.5, x_2(0) = 0.2$. For time horizon $T = 10$ and target opinion $s = 0.8$, the optimal state and control trajectories are presented in Table 1.2. The optimal value of player's functional (1.4) is 1.499.*

Table 1.2: Optimal control and state trajectories.

t	0	1	2	3	4	5
$x_1(t)$	0.5	0.9738	0.7323	0.7847	0.8171	0.7735
$x_2(t)$	0.2	1.1	0.7215	0.7539	0.8463	0.7588
$z(t)$	0.3	-0.1262	0.0108	0.0308	-0.0292	0.0147
$u(t)$	0.7738	-0.3678	0.0632	0.0632	-0.0729	0.0411

t	6	7	8	9	10
$x_1(t)$	0.7998	0.7918	0.7908	0.796	0.7978
$x_2(t)$	0.8028	0.7938	0.7878	0.7967	0.7945
$z(t)$	-0.003	-0.002	0.003	-0.0007	0.0033
$u(t)$	-0.011	-0.003	0.0083	0.0012	

We introduce optimal state and strategy trajectories on Figures 1.3 and 1.4, respectively.

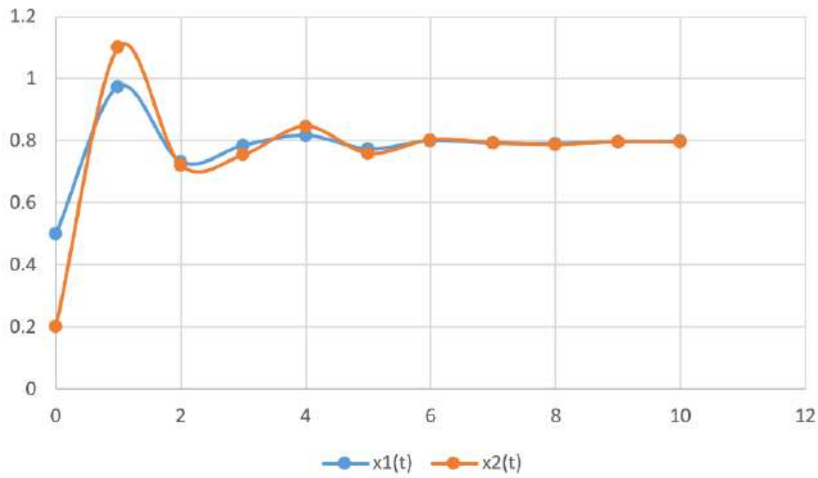

Figure 1.3: State trajectories (blue - $x_1(t)$, red - $x_2(t)$).

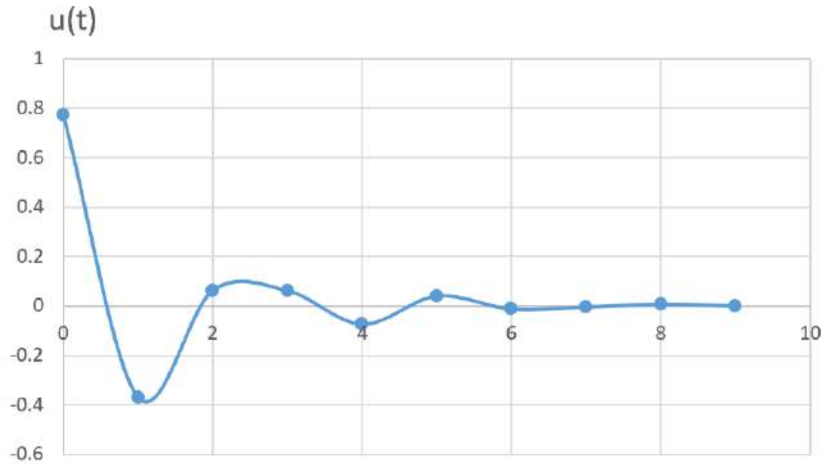

Figure 1.4: Strategy trajectory $u(t)$.

On Figure 1.3 we see that the opinions of both agents under the influence of the player gradually approach the target opinion at the terminal time. From Figure 1.4 it can be seen that the player's influence on the agent also tends to be stable.

1.3 The case when opinions in limited number of periods are validated

We consider a small social network with 2 agents. The opinion of agent 1 is denoted by x_1 and the opinion of agent 2 is x_2. The agents can influence each other, and their opinions change over time. We assume that the opinions change over discrete and finite time. Define the opinion of agent 1 at time t by $x_1(t) \in \mathbb{R}, t = 0, \ldots, T$ and the opinion of agent 2 at time t by $x_2(t) \in \mathbb{R}, t = 0, \ldots, T$. Vector $(x_1(t), x_2(t))$ represents the state variable at time t.

We also assume that there is a player in the social network and denote his influence on agent 1 at time $t = 0, \ldots, T-1$ by $u(t)$. The player does not influence the opinion of agent 1 at time T. The dynamics of the agent's opinion depend on the current state and player's control. Agent 1's next opinion is composed of agent 1's current opinion, the social average opinion and the player's influence. Agent 2's next opinion is composed of agent 2's current opinion and the average social opinion. The dynamics of their opinions are defined by the following equations:

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right) + u(t), \tag{1.9}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right) \tag{1.10}$$

with initial condition

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0. \tag{1.11}$$

In the above equation, $a_1 > 0, a_2 > 0$ denote the agent's beliefs about the average social opinion, respectively.

We assume that the player can only monitor the level of opinion along the state trajectory at some periods. The number of periods to put in the functional is k and it is fixed, $1 \leq t_1 < t_2 < \cdots < t_k \leq T-1$, where $k < T$ is known but which periods are chosen is unknown. The player's target opinion is $s \in \mathbb{R}$. The player aims to choose the periods $1 \leq t_1 < t_2 < \cdots < t_k \leq T-1$ to minimize his costs, which are the following:

$$\begin{aligned} J(u) = &\sum_{t=0}^{T-1} \delta^t \left(cu^2(t)\right) + \sum_{j=1}^{k} \delta^{t_j} \left((x_1(t_j) - s)^2 + (x_2(t_j) - s)^2\right) \\ &+ \delta^T \left((x_1(T) - s)^2 + (x_2(T) - s)^2\right), \end{aligned} \tag{1.12}$$

where $\delta \in (0,1]$ is the discount factor and $c > 0$ is the player's cost per unit level of influence.

Theorem 1.2. *Let $\{u^*(t) : t = 0, \ldots, T-1\}$ be the optimal strategy minimizing functional (1.12) subject to initial conditions (1.11) and state dynamics (1.9) and (1.10), and $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ be the corresponding state trajectory, then the optimal strategy $u^*(t), t = 0, \ldots, T-1$ is*

$$u^*(t) = z^*(t+1) - Az^*(t)$$

and corresponding optimal state trajectory $\left(x_1^*(t), x_2^*(t)\right), t = 1, \ldots, T$ satisfying the following equations:

$$\begin{cases} Ac\delta z(t+1) + Bz(t-1) - Cz(t) \\ \quad + \frac{Ac}{\delta} z(t-2) = 0, t = 2, \ldots, T-1, \\ Dz\left(t\right) - Ez\left(t-1\right) - A\delta cz\left(t+1\right) + \frac{Ac}{\delta} z\left(t-2\right) \\ \quad = \left(a_2 - \delta\right)\left(x_2\left(t\right) - s\right) + x_2\left(t-1\right) - s, t = t_j, j = 2, \cdots, k, \\ c\left(z\left(t\right) - Az\left(t-1\right)\right) + \delta\left(z\left(t\right) + x_2\left(t\right) - s\right) = 0, t = T, \\ \left(\frac{c}{\delta} + A^2 c\right) z\left(T-1\right) - \frac{Ac}{\delta} z\left(T-2\right) - \left(Ac - \frac{a_2}{2}\right) z\left(T\right) \\ \quad = a_2\left(x_2\left(T\right) - s\right), \\ x_2\left(t+1\right) = x_2\left(t\right) + \frac{a_2}{2} z\left(t\right), t = 1, \ldots, T-1, \end{cases} \tag{1.13}$$

where $z^*(t) = x_1^*(t) - x_2^*(t)$, $A = 1 - \frac{a_1 + a_2}{2}$, $B = Ac + \frac{c}{\delta} - A^2 c$, $C = A^2\delta c + Ac - c$, $D = c - A^2\delta c + \delta + Ac - \frac{a_2}{2}$ and $E = Ac - \frac{c}{\delta} + A^2 c - 1$.

Proof. We represent a new variable $z\left(t\right)$ as

$$z\left(t\right) = x_1\left(t\right) - x_2\left(t\right), \; t = 0, \ldots, T.$$

From state equations (1.9), (1.10) taking into account expression of $z\left(t\right)$, we obtain the new state equations:

$$z\left(t+1\right) = Az\left(t\right) + u\left(t\right), \tag{1.14}$$

$$x_2(t+1) = x_2(t) + \frac{a_2}{2} z(t), \tag{1.15}$$

with initial condition

$$z(0) = x_1^0 - x_2^0, x_2(0) = x_2^0,$$

where $A = 1 - \frac{a_1 + a_2}{2}$.

We find an expression of $u(t)$ from (1.14) and obtain

$$u(t) = z(t+1) - Az(t), \tag{1.16}$$

with

$$u(0) = z(1) - Az(0).$$

Substitute these expressions into $\sum_{t=0}^{T} \delta^t g_t(x(t), x(t+1))$, we can rewrite the functional in the following form:

$$\begin{aligned} J(z, x_2) =& c(z(1) - Az(0))^2 + \sum_{t=0}^{T-1} \delta^t \left[c(z(t+1) - Az(t))^2 \right] \\ &+ \sum_{j=1}^{k} \delta^{t_j} \left[(z(t_j) + x_2(t_j) - s)^2 + (x_2(t_j) - s)^2 \right] \\ &+ \delta^T \left[(z(T) + x_2(T) - s)^2 + (x_2(T) - s)^2 \right]. \end{aligned}$$

To minimize $J(z, x_2)$ under condition given by equations (1.15) and (1.16), we form the Lagrange function

$$L(z, x_2, k) = J(z, x_2) + \sum_{t=1}^{T-1} k_t \left(x_2(t+1) - x_2(t) - \frac{a_2}{2} z(t) \right).$$

The first-order conditions are $\frac{\partial L(z, x_2, k)}{\partial z(t)} = 0, t = 1, \ldots, T$ and $\frac{\partial L(z, x_2, k)}{\partial x_2(t)} = 0$, $t = 1, \ldots, T$.

First, we find the derivatives and get

$$\begin{aligned} \frac{\partial J(z, x_2)}{\partial z(t)} =& \delta^{t-1} 2c(z(t) - Az(t-1)) - \delta^t 2Ac(z(t+1) - Az(t)), \\ & t = 1, \ldots, T-1, t \neq t_j, \end{aligned}$$

$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^{t-1}2c\left(z(t) - Az(t-1)\right) - \delta^t 2Ac\left(z(t+1) - Az(t)\right)$$

$$+\,\delta^t 2\left(z(t) + x_2(t) - s\right), t = t_j, j = 1,\ldots,k,$$

$$\frac{\partial J(z,x_2)}{\partial x_2(t)} = 0, t = 1,\ldots,T-1, t \neq t_j,$$

$$\frac{\partial J(z,x_2)}{\partial x_2(t)} = \delta^t\left[2\left(z(t) + x_2(t) - s\right) + 2\left(x_2(t) - s\right)\right], t = t_j, j = 1,\ldots,k,$$

Second, we write the systems of the first-order conditions that are

$$\begin{cases} \frac{c}{\delta}\left(z(t) - Az(t-1)\right) - Ac\left(z(t+1) - Az(t)\right) = \frac{a_2}{4}k_t\delta^{-t}, \\ \quad t = 1,\ldots,T-1, t \neq t_j, \\ \frac{c}{\delta}\left(z(t) - Az(t-1)\right) - Ac\left(z(t+1) - Az(t)\right) \\ \quad +\left(z(t) + x_2(t) - s\right) = \frac{a_2}{4}k_t\delta^{-t}, t = t_j, j = 1,\cdots,k, \\ c\left(z(t) - Az(t-1)\right) + \delta\left(z(t) + x_2(t) - s\right) = 0, t = T, \\ k_{t-1} - k_t = 0, t = 1,\ldots,T-1, t \neq t_j, \\ \delta^t\left[2z(t) + 4\left(x_2(t) - s\right)\right] - k_t + k_{t-1} = 0, t = t_j, j = 2,\ldots,k, \\ \delta^t\left[2z(t) + 4\left(x_2(t) - s\right)\right] + k_{t-1} = 0, t = T, \end{cases} \tag{1.17}$$

with initial conditions $z(0) = x_1^0 - x_2^0$, $x_2(0) = x_2^0$.

Excluding k_t from system (1.17), finally we obtain the system of equations

$$\begin{cases} Ac\delta z(t+1) + Bz(t-1) - Cz(t) - \frac{Ac}{\delta}z(t-2) = 0, t = 2,\ldots,T-1, \\ Dz(t) - Ez(t-1) - A\delta cz(t+1) + \frac{Ac}{\delta}z(t-2) \\ \quad = (a_2-1)\delta\left(x_2(t) - s\right) + x_2(t-1) - s, t = t_j, j = 2,\cdots,k, \\ c\left(z(t) - Az(t-1)\right) + \delta\left(z(t) + x_2(t) - s\right) = 0, t = T, \\ \left(\frac{c}{\delta} + A^2c\right)z(T-1) - \frac{Ac}{\delta}z(T-2) - \left(Ac - \frac{a_2\delta}{2}\right)z(T) = a_2\left(x_2(T) - s\right), \end{cases}$$

where $B = Ac + \frac{c}{\delta} + A^2c$, $C = Ac - c - A^2\delta c$, $D = c + A^2\delta c + \delta + Ac - \frac{a_2\delta}{2}$ and $E = Ac + \frac{c}{\delta} + A^2c + 1$.

The theorem is proved. □

1.4 Numerical simulations for Section 1.3

1.4.1 Experiment Description

We initially set up 6 sets of experiments. The parameters $a_1, a_2, \delta, c, x_0, y_0, T$ were the same in all experiments. In Experiments 1-3, given the same target opinion and varying only the number of validation periods, we compare changes in state trajectories and strategy trajectories. We then do not change the number of validation periods in Experiments 1-3 and only change the target opinion for each experiment, which produces Experiments 4-6. We compare changes in state trajectories and strategy trajectories.

Experiment 1 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.4$ and the number of validation periods $k = 3$. By combining, we get 84 cases to choose three periods.

Experiment 2 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.4$ and the number of validation periods $k = 4$. By combining, we get 126 cases for four validation periods.

Experiment 3 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.4$ and the number of validation periods $k = 5$. By combining, we get 126 cases for five periods.

Experiment 4 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.5$ and the number of validation periods $k = 3$. By combining, we get 84 cases to choose three periods.

Experiment 5 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.5$ and the number of validation periods $k = 4$. By combining, we get 126 cases for four periods.

Experiment 6 Set the initial parameters $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$. Given the target opinion $s = 0.5$ and the number of validation periods $k = 5$. By combination, we obtain 126 cases for five periods.

1.4.2 Experimental flowchart

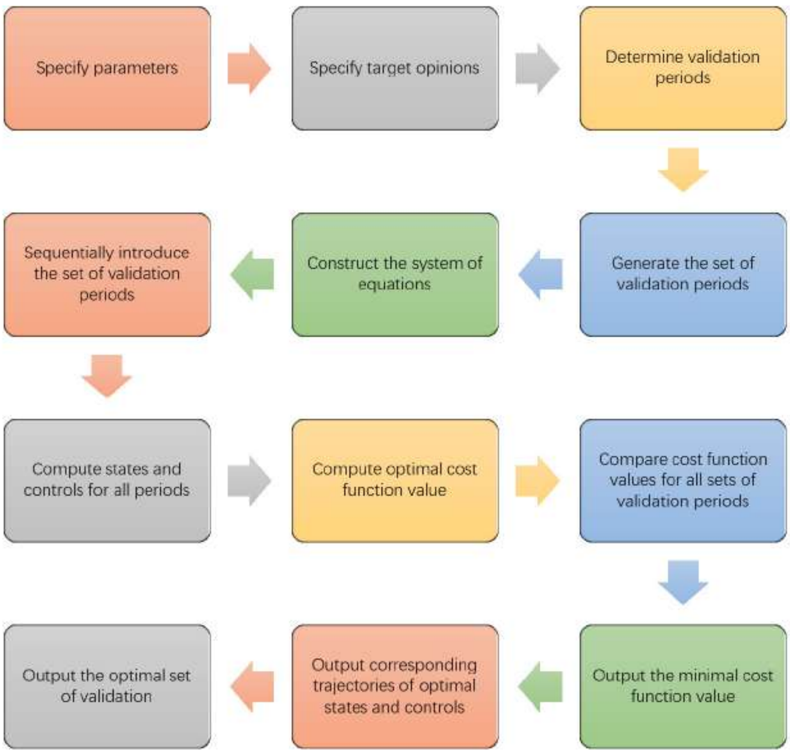

Figure 1.5: Experimental flowchart.

Detailed steps

1. Set parameters: $a_1 = 0.2; a_2 = 0.8; \delta = 1; c = 0.1; x_1(0) = 0.9; x_2(0) = 0.1; T = 10$.

2. Set target: $s = 0.4$ (e.g., experiments 1, 2, 3) or $s = 0.5$ (e.g., experiments 4, 5, 6).

3. Fix the number of validation periods: $k = 3$ (e.g., experiments 1 and 4) or 4 (e.g., experiments 2 and 5) or 5 (e.g., experiments 3 and 6).

4. Generate validation periods sets: three periods are randomly selected from periods 1,...,9 to form a set $\{t_1, t_2, t_3\}$, such as $\{1, 2, 3\}$, $\{2, 3, 4\}$, and 84 sets of periods can be obtained. From periods 1,...,9, the four periods are randomly selected to form validation period set $\{t_1, t_2, t_3, t_4\}$, such as $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, and 126 sets of periods can be obtained. From periods $1, \ldots, 9$, there are five periods which are randomly selected to form a set $\{t_1, t_2, t_3, t_4, t_5\}$, such as $\{1, 2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6\}$, and 126 sets can be obtained.

5. Build a system of equations: input system (1.13) in Theorem 1.2 to form a system of equations.

6. Input validation period sets in order: Substitute the selected sets into system (1.13) respectively.

7. Obtain any time state and control: each validation period set is calculated in system (1.13), which can output $0, 1, 2, \ldots, 9, 10$ periods corresponding to the state and control.

8. Calculate the cost function: put $0, 1, \ldots, 9$ periods corresponding to the control, the validation period set corresponding to the state and the end state are substituted into the cost function (1.12), and the result is obtained.

9. Compare the cost values of all validation period sets: In each of the six experiments, compare all the cost values of each experiment.

10. Output the minimum cost value: output the minimum cost value of each experiment.

11. Output the corresponding state and control: According to the validation period set corresponding to the minimum cost value, output this set corresponding to the state and control.

12. Get the optimal validation period set: the set corresponding to the minimum cost value is the optimal one, and output the set.

1.4.3 Experimental results

In this section we represent the optimal state, control trajectories and player' s cost corresponding to each case, we obtain the following results.

Example 1.3. *We start with* $k = 3$*. Solving minimization problem of (1.12) with respect to periods* $1 \leq t_1 < t_2 < t_3 \leq 9$*. The player sets the number of validation periods 2, 3, 4 and the target opinion being* $s = 0.4$*. The optimal state and control trajectories are presented in Table 1.3. The optimal cost for the player is 0.016417.*

In this run, the player obtains the optimal state trajectory and optimal control by choosing periods 2, 3, 4 within the total time $T = 10$*, in Figures 1.6 and 1.7 respectively.*

Table 1.3: Optimal state trajectory and control

t	0	1	$t_1 = 2$	$t_2 = 3$	$t_3 = 4$	5
$x_1(t)$	0.9	0.4252	0.3977	0.4001	0.4045	0.4061
$x_2(t)$	0.1	0.42	0.4221	0.4123	0.4075	0.4063
$z(t)$	0.8	0.0052	-0.0244	-0.0122	-0.0029	-0.0001
$u(t)$	-0.3948	-0.027	0.000003	0.0032	0.0013	-0.0016
t	6	7	8	9	10	
$x_1(t)$	0.4045	0.4056	0.4112	0.3974	0.3999	
$x_2(t)$	0.4062	0.4055	0.4056	0.4078	0.4036	
$z(t)$	-0.0017	0.0001	0.0056	-0.0105	-0.0038	
$u(t)$	0.001	0.0055	-0.0133	0.0014		

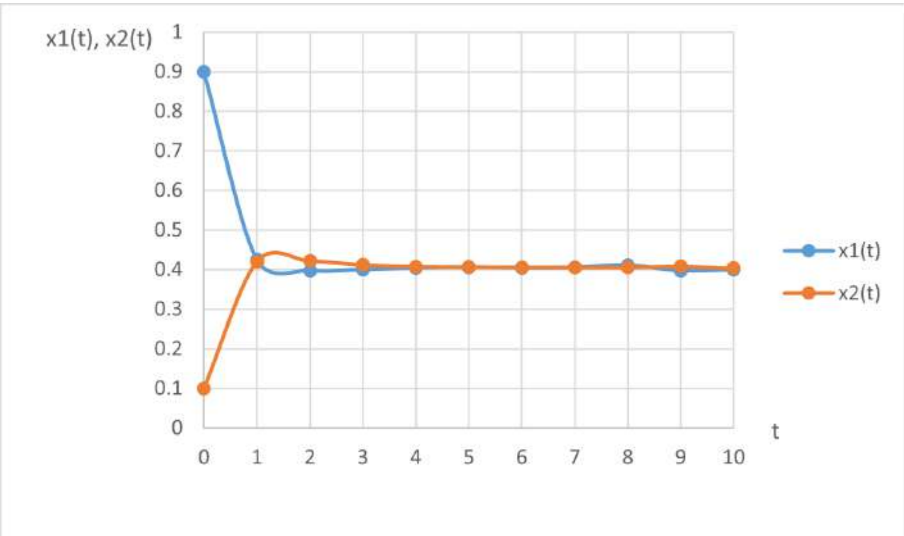

Figure 1.6: Optimal state trajectories, validation periods are 2, 3, 4 (blue — $x_1(t)$, red — $x_2(t)$).

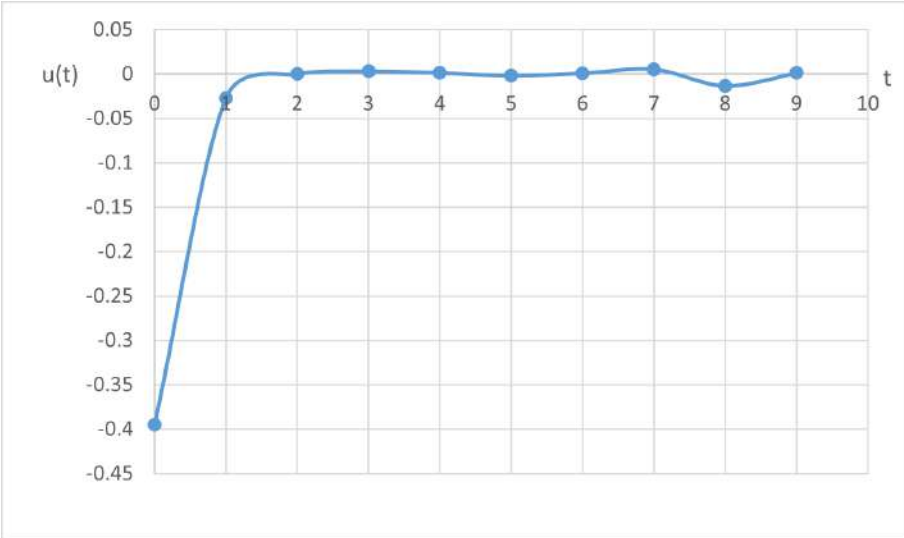

Figure 1.7: Optimal strategy trajectory $u(t)$.

Example 1.4. *Let $k = 4$. His target opinion is $s = 0.4$. The solution of the problem is to choose validation periods as 2, 3, 4, 5 periods. The optimal state and control trajectories are presented in Table 1.4. The optimal cost for the player is 0.016413. We should mention that the cost for $k = 3$ are a bit larger than for $k = 4$. It can be easily explained by additional term in function (1.12) in case of this example.*

We also notice that when we add one more period, in the optimal solution of the problem the new additional period $t = 5$ is added to the set $\{2, 3, 4, 5\}$. But the latter set remains the same. The optimal state and control trajectories are presented in Figures 1.8 and 1.9 respectively.

Table 1.4: Optimal state trajectory and control.

t	0	1	$t_1 = 2$	$t_2 = 3$	$t_3 = 4$	$t_4 = 5$
$x_1(t)$	0.9	0.4246	0.3965	0.3979	0.4002	0.4026
$x_2(t)$	0.1	0.42	0.4219	0.4117	0.4062	0.4038
$z(t)$	0.8	0.0046	-0.0253	-0.0139	-0.0059	-0.0012
$u(t)$	-0.3954	-0.0276	-0.0012	0.001	0.0018	-0.0005
t	6	7	8	9	10	
$x_1(t)$	0.4022	0.4028	0.4057	0.3987	0.3999	
$x_2(t)$	0.4033	0.4029	0.4029	0.404	0.4019	
$z(t)$	-0.0011	-0.00002	0.0028	-0.0053	-0.0019	
$u(t)$	0.0006	0.0028	-0.0067	0.0007		

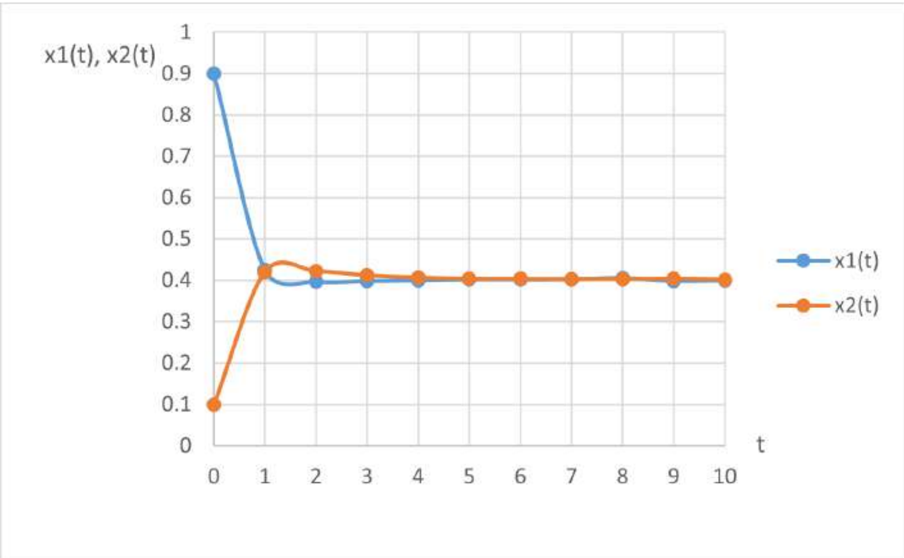

Figure 1.8: Optimal state trajectories, validation periods are 2, 3, 4, 5 (blue — $x_1(t)$, red — $x_2(t)$).

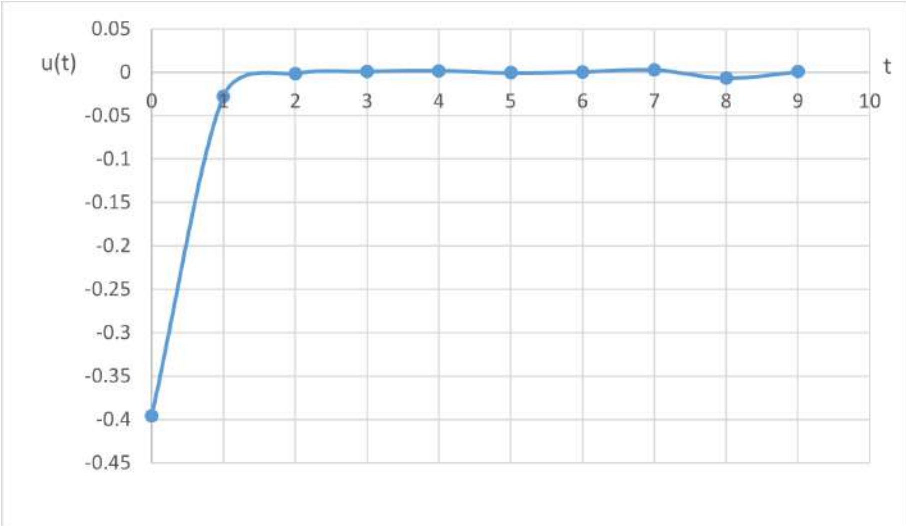

Figure 1.9: Optimal strategy trajectory $u(t)$.

Example 1.5. *Let $k = 5$. The player' s target opinion is $s = 0.4$. The solution of the problem is to choose 2, 3, 4, 5, 9 periods as validation periods. The optimal state and control trajectories are presented in Table 1.5. The optimal cost for the player is 0.016405. We mention that the costs for $k = 4$ are also a bit larger than for $k = 5$.*

Table 1.5: Optimal state trajectory and control

t	0	1	$t_1 = 2$	$t_2 = 3$	$t_3 = 4$	$t_4 = 5$
$x_1(t)$	0.9	0.4245	0.3963	0.3974	0.3993	0.4009
$x_2(t)$	0.1	0.42	0.4218	0.4116	0.4059	0.4033
$z(t)$	0.8	0.0045	-0.0255	-0.0142	-0.0066	-0.0023
$u(t)$	-0.3955	-0.0278	-0.0015	0.0005	0.001	0.0006

t	6	7	8	$t_5 = 9$	10
$x_1(t)$	0.4018	0.4045	0.399	0.3987	0.3999
$x_2(t)$	0.4023	0.4021	0.4031	0.4015	0.4004
$z(t)$	-0.0005	0.0024	-0.004	-0.0027	-0.0005
$u(t)$	0.0026	-0.0052	-0.0007	0.0009	

Noticing that when we add one more period in the optimal solution of the problem, the new additional period $t = 9$ is added to the set $\{2, 3, 4, 5, 9\}$. The optimal state trajectory and control trajectory are presented in Figures 1.10 and 1.11 respectively.

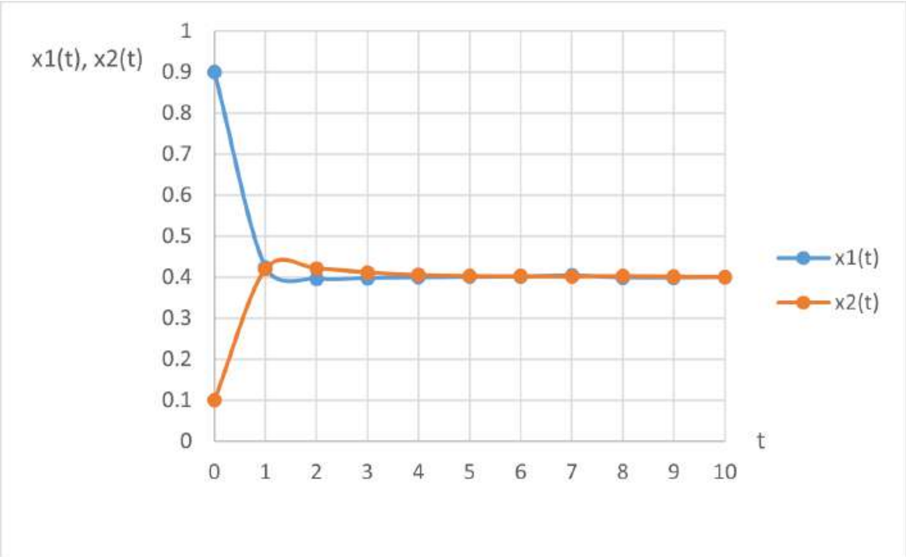

Figure 1.10: Optimal state trajectories, validation periods are 2, 3, 4, 5, 9 (blue — $x_1(t)$, red — $x_2(t)$).

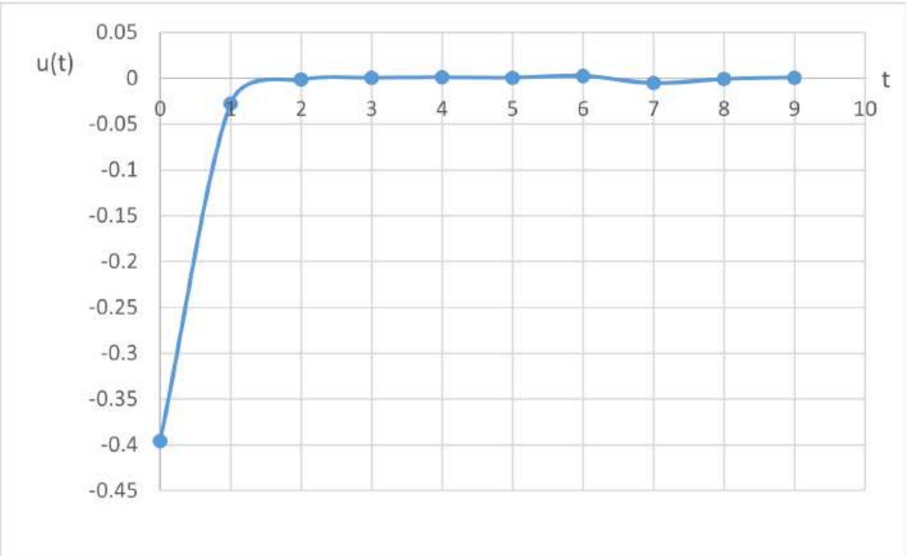

Figure 1.11: Optimal strategy trajectory $u(t)$.

From examples 1.3-1.5, we find that when the target opinion $s = 0.4$, the 2 agents can reach consensus quickly under the influence of the player. Player's control tends to stabilize after period 2. There is no significant effect on the agents' opinions approaching the target opinion after each increase of one validation point. The corresponding cost value also slightly reduces.

Example 1.6. *Let* $k = 3$*, the player choose 3, 4, 5 periods and his target opinion is* $s = 0.5$*. The optimal state and control trajectories are presented in Table 1.6. The optimal cost for the player is 0.004486.*

Table 1.6: Optimal state trajectory and control

t	0	1	2	$t_1 = 3$	$t_2 = 4$	$t_3 = 5$
$x_1(t)$	0.9	0.6556	0.5061	0.4986	0.5003	0.5026
$x_2(t)$	0.1	0.42	0.5142	0.511	0.506	0.5037
$z(t)$	0.8	0.2356	-0.0082	-0.0124	-0.0057	-0.0012
$u(t)$	-0.1644	-0.126	-0.0083	0.0005	0.0017	-0.0005
t	6	7	8	9	10	
$x_1(t)$	0.5021	0.5028	0.5056	0.4987	0.4999	
$x_2(t)$	0.5033	0.5028	0.5028	0.5039	0.5018	
$z(t)$	-0.0011	-0.00002	0.0028	-0.0053	-0.0019	
$u(t)$	0.0006	0.0028	-0.0067	0.0007		

In this experiment, the player obtains the optimal state trajectory and optimal control by choosing periods 3, 4, 5 within the total time $T = 10$*, in Figures 1.12 and 1.13 respectively.*

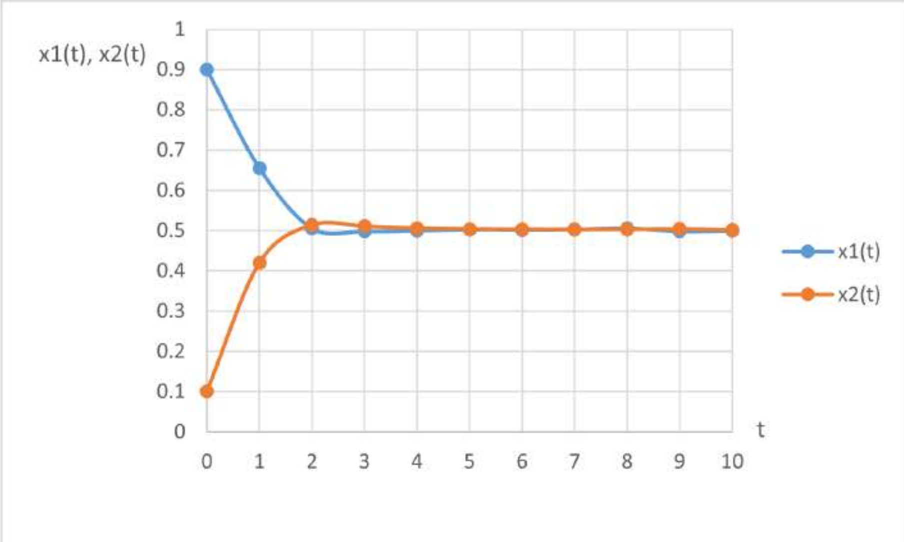

Figure 1.12: Optimal state trajectories, validation periods 3, 4, 5 (blue — $x_1(t)$, red — $x_2(t)$).

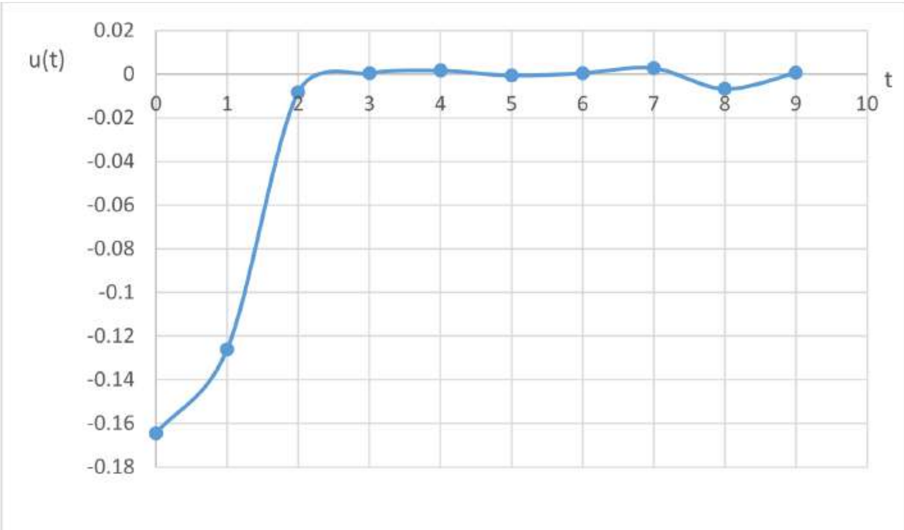

Figure 1.13: Optimal strategy trajectory $u(t)$.

Example 1.7. *Let the number of validation periods* $k = 4$*, the player monitors agents' opinions close to the target opinion* $s = 0.5$ *by choosing 3, 4, 5, 6 periods. The optimal state and control trajectories are presented in Table 1.7. The optimal cost for the player is 0.004472.*

Noticing that when we add one more period $t = 6$ *in the optimal solution of the problem. The optimal state trajectory and optimal control are obtained by choosing periods 3, 4, 5, 6 within the total time* $T = 10$*, in Figures 1.14 and 1.15*

Table 1.7: Optimal state trajectory and control

t	0	1	2	$t_1 = 3$	$t_2 = 4$	$t_3 = 5$
$x_1(t)$	0.9	0.6561	0.5057	0.4978	0.4987	0.4997
$x_2(t)$	0.1	0.42	0.5144	0.5109	0.5057	0.5029
$z(t)$	0.8	0.2361	-0.0088	-0.0132	-0.007	-0.0032
$u(t)$	-0.1639	-0.1268	-0.0088	-0.0004	0.0003	0.0005
t	$t_4 = 6$	7	8	9	10	
$x_1(t)$	0.5005	0.501	0.5021	0.4995	0.5	
$x_2(t)$	0.5016	0.5011	0.5011	0.5015	0.5007	
$z(t)$	-0.0011	-0.0002	0.001	-0.002	-0.0007	
$u(t)$	0.0004	0.0011	-0.0025	0.0003		

respectively.

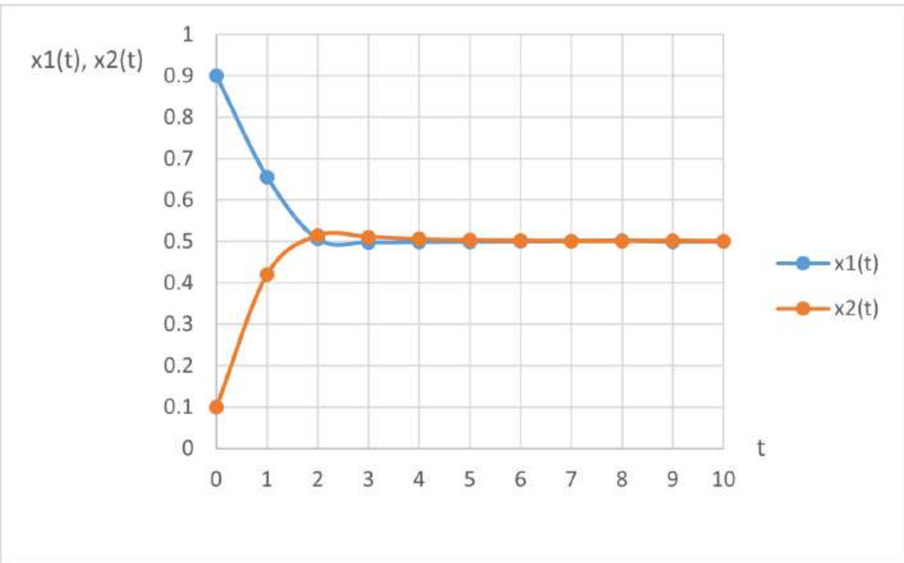

Figure 1.14: Optimal state trajectories, validation periods 3, 4, 5, 6 (blue — $x_1(t)$, red — $x_2(t)$).

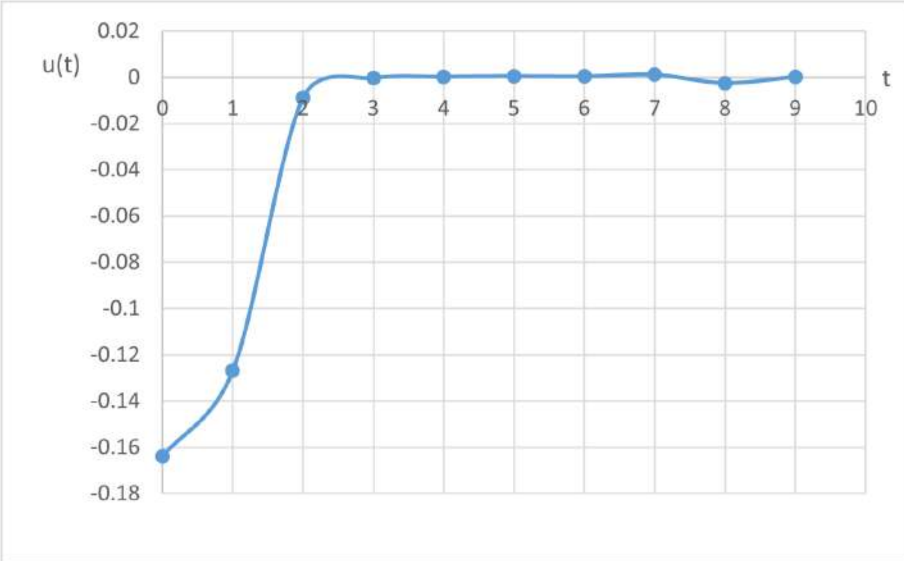

Figure 1.15: Optimal strategy trajectory $u(t)$.

Example 1.8. *Let $k = 5$, the player chooses 3, 4, 5, 6, 7 periods and his target opinion is $s = 0.5$. The optimal states and control trajectories are presented in Table 1.8. The optimal cost for the player is 0.004474.*

Table 1.8: Optimal state trajectory and control

t	0	1	2	$t_1 = 3$	$t_2 = 4$	$t_3 = 5$
$x_1(t)$	0.9	0.6561	0.5056	0.4977	0.4986	0.4994
$x_2(t)$	0.1	0.42	0.5145	0.5109	0.5056	0.5028
$z(t)$	0.8	0.2361	-0.0088	-0.0132	-0.0071	-0.0034
$u(t)$	-0.1639	-0.1269	-0.0088	-0.0005	0.0001	0.0002

t	$t_4 = 6$	$t_5 = 7$	8	9	10
$x_1(t)$	0.5	0.5006	0.5014	0.4996	0.5
$x_2(t)$	0.5015	0.5009	0.5008	0.501	0.5005
$z(t)$	-0.0015	-0.0003	0.0006	-0.0014	-0.0005
$u(t)$	0.0004	0.0008	-0.0017	0.0002	

In this experiment, by choosing periods 3, 4, 5, 6, 7 we obtain the optimal state trajectory and optimal control, given in Figures 1.16 and 1.17 respectively.

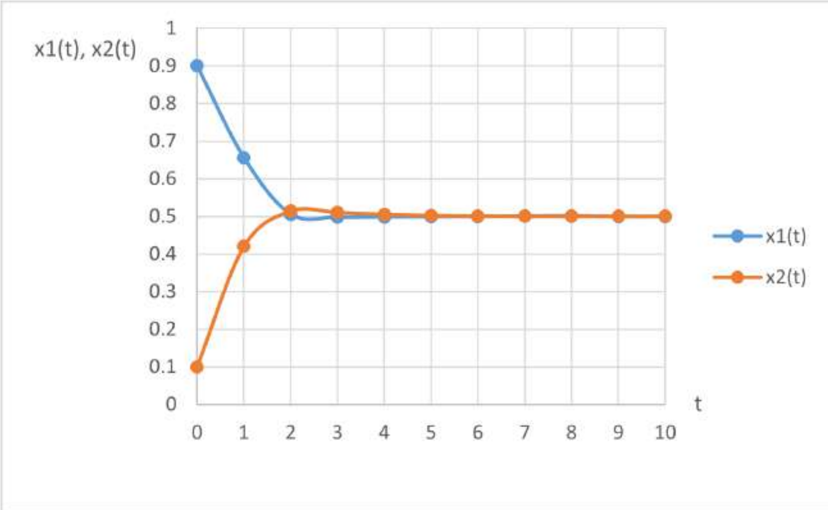

Figure 1.16: Optimal state trajectories, validation periods 3, 4, 5, 6, 7 (blue — $x_1(t)$, red — $x_2(t)$).

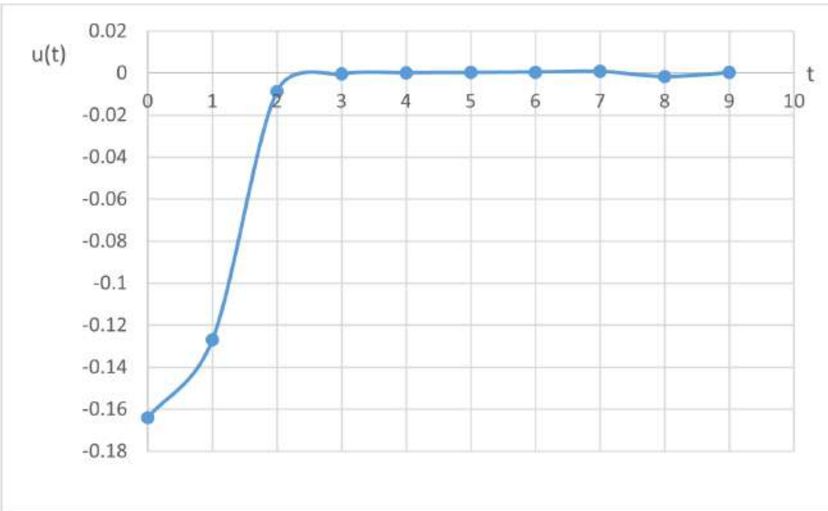

Figure 1.17: Optimal strategy trajectory $u(t)$.

From examples 1.6-1.8, with the remaining parameters unchanged, we only change the target opinion $s = 0.5$ and the 2 agents reach a consensus. Player's control is relatively stable after period 3. Although the latter three sets of experiments did not have much larger target opinions ($s = 0, 5$) than the first three sets of experiments ($s = 0.4$), we found that the change in the target opinion led agents to reach consensus more quickly. The optimal cost was smaller for the last three sets of experiments.

1.5 Conclusion to Chapter 1

We propose two models of opinion dynamics that facilitate the player's ability to monitor the state trajectory of the agent's opinion. The goal of the player in the first model is to bring the social opinion closer to the target opinion at the terminal time and to minimize its influence costs. When there is a unique center of influence in the social network, we can find the optimal control for this player. We use the Euler equation method to find the optimal strategy. In the numerical simulation, when the player validates the agent's opinion only at the terminal period and does not care about other periods, he can minimize the costs and make the agent's opinion close to the goal opinion.

For the case when the player validates the agent's opinion at multiple periods, we set up a comparison experiment with different target opinions, thus finding the optimal setting of the periods to validate opinions. We find that the optimal costs that decreases when the number of validation periods increases. We also note that if we increase the number of validation periods while other parameter values remain constant, the optimal set of validation periods under a smaller number of validation periods is included in the optimal set of validation periods under a larger number of them. We can easily extend our model to make it applicable to more agents and participants. When there is more than one participant, competition can arise if the participants have different goal perspectives and long-term goals.

Chapter 2

Average-oriented opinion dynamics in small groups when periods to control opinions are player's decision variable

2.1 Model of opinion dynamics in small social groups when player chooses to control opinions in the limited number of periods

We consider a multiagent system representing a small social network with two agents. Let $x_1(t) \in \mathbb{R}$ ($x_2(t) \in \mathbb{R}$) be the opinion of agent 1 (agent 2) at time t, $t = 0, \dots, T$. We assume that the player, who is not an agent in the system, can control agent 1 at several (limited number of) periods but he finds opinions of the agents significant at any time t. We denote the player's influence on agent 1 at time t by $u(t) \in \mathbb{R}$, $t = 0, \dots, T-1$. The set of periods at which player controls agent 1 is denoted by G and called the *set of control periods*. The number of elements in the set G is given, and it is equal to $k < T$. Therefore, we consider the problem, when k is known to the player, but the set of control periods G is not fixed. Let this set be represented as $G = \{t_1, t_2, \dots, t_k\}$.

When time t belongs to the set of control periods G, agent 1's future opinion depends on his own present opinion, the present average opinion of the society, and the player's present control. When time t does not belong to the set of control periods G, agent 1's future opinion depends on his own present opinion and the present average opinion of the society. Agent 2 is not influenced by the player, and his future opinion depends on his own present opinion and the present average opinion of the society. The dynamics of the agents' opinions are defined by the following equations:

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right) + u(t), \quad t \in G, \tag{2.1}$$

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right), \quad t \notin G, \tag{2.2}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right), \quad t = 0, \ldots, T-1, \tag{2.3}$$

with the initial condition

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0. \tag{2.4}$$

In the equations of dynamics (2.1)–(2.3), the constants $a_1 > 0$, $a_2 > 0$ denote the agent 1 and agent 2's beliefs about the average social opinion, respectively.

The player needs to define a set of control periods $G = \{t_1, t_2, \ldots, t_k\}$ for a given k. Let $0 \leq t_1 < t_2 < \ldots < t_k \leq T-1$, where $k < T$. The player's target opinion is $s \in \mathbb{R}$. The player aims to minimize his costs by choosing the set of control periods and choosing the values of controls for the periods from the set of control periods. We first solve the optimization problem over the set of controls for a given set of control periods G. The player's functional is

$$\min_u J(u) = \sum_{j=1}^{k} \delta^{t_j}\left(cu^2(t_j)\right) + \sum_{t=0}^{T} \delta^t\left((x_1(t)-s)^2 + (x_2(t)-s)^2\right), \tag{2.5}$$

where $\delta \in (0,1]$ is the discount factor and $c > 0$ is the player's cost per unit level of influence.

Second, we choose the set of control periods over all possible ones with the player's minimal costs.

The necessary conditions of the optimal control problem (2.5) s.t. (2.1)–(2.3) with initial condition (2.4) are given in the following theorem.

Theorem 2.1. *Let* $\{u^*(t) : t = t_1, t_2, \ldots, t_k\}$ *be the optimal strategy minimizing functional (2.5) subject to initial conditions (2.4) and state dynamics equations (2.1), (2.2) and (2.3), and* $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ *be the corresponding state trajectory. The periods* $0 \le t_1 < t_2 < \ldots < t_k \le T - 1$ *are given. Then the optimal strategy* $u^*(t), t = t_1, t_2, \ldots, t_k$ *is defined as*

$$u^*(t) = z^*(t+1) - Az^*(t)$$

and corresponding optimal state trajectory $(x_1^*(t), x_2^*(t)), t = 1, \ldots, T$ *satisfy the* *system of equations:*

$$\begin{cases} \left(\frac{a_2\delta}{2}-\delta\right)z(t)+z(t-1)=(\delta-a_2\delta)(x_2(t)-s)-x_2(t-1)+s, \\ \quad t=1,\ldots,T-1, \quad t,t-1\notin\{t_1,t_2,\ldots t_k,\}, \\ Bz(t)+Cz(t-1)-Acz(t-2)=\left(\delta^2-a_2\delta^2\right)(x_2(t)-s) \\ \quad -\delta(x_2(t-1)-s), \quad t=1,\ldots,T-1, \\ \quad t\notin\{t_1,t_2,\ldots,t_k\}, t-1\in\{t_2,\ldots,t_k\}, \\ Dz(t)+Ez(t-1)-Acz(t-2)+Ac\delta^2 z(t+1) \\ \quad =\left(\delta^2-a_2\delta^2\right)(x_2(t)-s)-\delta(x_2(t-1)-s), \quad t=1,\ldots,T-1, \\ \quad t,t-1\in\{t_2,\ldots,t_k\}, \\ Fz(t)+\left(A^2c+1\right)z(t-1)+Ac\delta z(t+1)=(\delta-a_2\delta)(x_2(t)-s) \\ \quad -x_2(t-1)+s, \quad t=1,\ldots,T-1, \\ \quad t\in\{t_1,t_2,\ldots,t_k\}, t-1\notin\{t_1,t_2,\ldots,t_k\}, \\ \frac{a_2\delta}{2}z(t)+z(t-1)=-a_2\delta(x_2(t)-s)-x_2(t-1)+s, \\ \quad t=T,\ T\notin\{t_1,t_2,\ldots,t_k\}, T-1\notin\{t_1,t_2,\ldots,t_k\}, \\ \frac{a_2\delta^2}{2}z(t)+(c+\delta)z(t-1)-Acz(t-2)=-a_2\delta^2(x_2(t)-s) \\ \quad -x_2(t-1)+s, \quad t=T,\ T\notin\{t_1,t_2,\ldots,t_k\}, T-1\in\{t_2,\ldots,t_k\}, \\ z(t)+x_2(t)-s=0, \quad t=T,\ T-1\notin\{t_1,t_2,\ldots,t_k\}, \\ c(z(t)-Az(t-1))+\delta(z(t)+x_2(t)-s)=0, \\ \quad t=T,\ T-1\in\{t_1,t_2,\ldots,t_k\}, \\ x_2(t+1)=x_2(t)+\frac{a_2}{2}z(t), t=0,\ldots,T-1, \end{cases} \tag{2.6}$$

where $z^*(t) = x_1^*(t) - x_2^*(t)$, *and*

$$A = 1 - \frac{a_1 + a_2}{2},$$
$$B = \frac{a_2\delta^2}{2} - c\delta - \delta^2,$$
$$C = Ac\delta - c - \delta,$$
$$D = \frac{a_2\delta^2}{2} - Ac\delta - c\delta - A^2c\delta^2 - \delta^2,$$
$$E = Ac\delta + c + A^2c\delta + \delta,$$
$$F = \frac{a_2\delta}{2} - Ac - \delta - A^2c\delta.$$

Proof. We represent a new variable $z\,(t)$ as

$$z\,(t) = x_1\,(t) - x_2\,(t)\,,\ t = 0, \ldots, T.$$

From state equations (2.1), (2.2), and (2.3) taking into account expression of $z\,(t)$, we obtain the new state equations:

$$z\,(t+1) = Az\,(t) + u\,(t)\,, \quad t \in \{t_1, t_2, \ldots, t_k\}\,, \tag{2.7}$$
$$z\,(t+1) = Az\,(t)\,, \quad t \notin \{t_1, t_2, \ldots, t_k\}\,,$$
$$x_2\,(t+1) = x_2\,(t) + \frac{a_2}{2}z\,(t)\,,\ t = 0, \ldots, T-1, \tag{2.8}$$

with initial condition

$$z\,(0) = x_1^0 - x_2^0,\ x_2\,(0) = x_2^0,$$

where $A = 1 - \frac{a_1+a_2}{2}$.

We find an expression of $u\,(t)$ from (2.7) and obtain

$$u\,(t) = z\,(t+1) - Az\,(t)\,, \quad t \in \{t_1, t_2, \ldots, t_k\}\,. \tag{2.9}$$

Substituting these expressions into $\sum_{t=0}^{T} \delta^t g_t(x(t), x(t+1))$, we can rewrite the functional in the following form:

$$J(z,x_2) = (x_1(0)-s)^2 + (x_2(0)-s)^2 + \sum_{j=1}^{k} \delta^{t_j}\left[c\left(z\left(t_j+1\right) - Az\left(t_j\right)\right)^2\right]$$
$$+\sum_{t=1}^{T}\delta^t\left[\left(z(t)+x_2(t)-s\right)^2 + \left(x_2(t)-s\right)^2\right].$$

To minimize $J(z,x_2)$ under condition given by equations (2.8) and (2.9), we form the Lagrange function

$$L(z,x_2,k) = J(z,x_2) + \sum_{t=0}^{T-1} k_t\left(x_2(t+1) - x_2(t) - \frac{a_2}{2}z(t)\right).$$

The first-order conditions are $\frac{\partial L(z,x_2,k)}{\partial z(t)} = 0$, $t = 1,\dots,T$, and $\frac{\partial L(z,x_2,k)}{\partial x_2(t)} = 0$, $t = 1,\dots,T$.

First, we find the derivatives and get

$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^t 2\left(z(t)+x_2(t)-s\right), t=1,\dots,T-1, t, t-1 \notin \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^{t-1}2c\left(z(t)-Az(t-1)\right) + \delta^t 2\left(z(t)+x_2(t)-s\right),$$
$$t \notin \{t_1,t_2,\cdots t_k\}, t-1 \in \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^{t-1}2c\left(z(t)-Az(t-1)\right) - \delta^t 2Ac\left(z(t+1)-Az(t)\right)$$
$$+\delta^t 2\left(z(t)+x_2(t)-s\right), \quad t, t-1 \in \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial z(t)} = -\delta^t 2Ac\left(z(t+1)-Az(t)\right) + \delta^t 2\left(z(t)+x_2(t)-s\right),$$
$$t \in \{t_1,t_2,\cdots t_k\}, t-1 \notin \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^t 2\left(z(t)+x_2(t)-s\right), \quad t=T,\ t-1 \notin \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial z(t)} = \delta^{t-1}2c\left(z(t)-Az(t-1)\right) + \delta^t 2\left(z(t)+x_2(t)-s\right),$$
$$t=T, t-1 \in \{t_1,t_2,\dots,t_k\},$$
$$\frac{\partial J(z,x_2)}{\partial x_2(t)} = \delta^t\left[2\left(z(t)+x_2(t)-s\right) + 2\left(x_2(t)-s\right)\right], \quad t=1,\dots,T.$$

Second, we rewrite the system of the first-order conditions in the following form:

$$\begin{cases} z(t) + x_2(t) - s = \frac{a_2}{4} k_t \delta^{-t}, \quad t = 1, \ldots, T-1, \\ \quad t, t-1 \notin \{t_1, t_2, \ldots, t_k\}, \\ c(z(t) - Az(t-1)) + \delta(z(t) + x_2(t) - s) = \frac{a_2}{4} k_t \delta^{-t+1}, \\ \quad t \notin \{t_1, t_2, \ldots, t_k\}, t-1 \in \{t_1, t_2, \ldots, t_k\}, \\ \frac{c}{\delta}(z(t) - Az(t-1)) - Ac(z(t+1) - Az(t)) + z(t) + x_2(t) - s \\ \quad = \frac{a_2}{4} k_t \delta^{-t}, \quad t, t-1 \in \{t_1, t_2, \ldots, t_k\}, \\ -Ac(z(t+1) - Az(t)) + z(t) + x_2(t) - s = \frac{a_2}{4} k_t \delta^{-t}, \\ \quad t \in \{t_1, t_2, \cdots t_k\}, t-1 \notin \{t_1, t_2, \ldots, t_k\}, \\ z(t) + x_2(t) - s = 0, \quad t = T, t-1 \notin \{t_1, t_2, \ldots, t_k\}, \\ c(z(t) - Az(t-1)) + \delta(z(t) + x_2(t) - s) = 0, \\ \quad t = T, t-1 \in \{t_1, t_2, \ldots, t_k\}, \\ \delta^t [2z(t) + 4(x_2(t) - s)] + k_{t-1} - k_t = 0, \quad t = 1, \ldots, T-1, \\ \delta^t [2z(t) + 4(x_2(t) - s)] + k_{t-1} = 0, \quad t = T, \end{cases} \tag{2.10}$$

with initial conditions $z(0) = x_1^0 - x_2^0$, $x_2(0) = x_2^0$.

Excluding k_t from system (2.10), finally we obtain the system of equations

$$\begin{cases} \left(\frac{a_2\delta}{2}-\delta\right) z(t)+z(t-1)=(\delta-a_2\delta)(x_2(t)-s)-x_2(t-1)+s, \\ \quad t=1,\dots,T-1, \quad t,t-1\notin\{t_1,t_2,\dots,t_k\}, \\ Bz(t)+Cz(t-1)-Acz(t-2)=\left(\delta^2-a_2\delta^2\right)(x_2(t)-s)-\delta(x_2(t-1)-s), \\ \quad t=1,\dots,T-1, \quad t\notin\{t_1,t_2,\dots,t_k\}, t-1\in\{t_2,\dots,t_k\}, \\ Dz(t)+Ez(t-1)-Acz(t-2)+Ac\delta^2 z(t+1) \\ \quad =\left(\delta^2-a_2\delta^2\right)(x_2(t)-s)-\delta(x_2(t-1)-s), \quad t=1,\dots,T-1, \\ \quad t,t-1\in\{t_2,\dots,t_k\}, \\ Fz(t)+\left(A^2c+1\right)z(t-1)+Ac\delta z(t+1)=(\delta-a_2\delta)(x_2(t)-s) \\ \quad -x_2(t-1)+s, \quad t=1,\dots,T-1, t\in\{t_1,t_2,\dots,t_k\}, t-1\notin\{t_1,t_2,\dots,t_k\}, \\ \frac{a_2\delta}{2}z(t)+z(t-1)=-a_2\delta(x_2(t)-s)-x_2(t-1)+s, \\ \quad t=T, T\notin\{t_1,t_2,\dots,t_k\}, T-1\notin\{t_1,t_2,\dots,t_k\}, \\ \frac{a_2\delta^2}{2}z(t)+(c+\delta)z(t-1)-Acz(t-2)=-a_2\delta^2(x_2(t)-s)-x_2(t-1)+s, \\ \quad t=T, T\notin\{t_1,t_2,\cdots t_k\}, T-1\in\{t_2,\dots,t_k\}, \\ z(t)+x_2(t)-s=0, \quad t=T, T-1\notin\{t_1,t_2,\dots,t_k\}, \\ c(z(t)-Az(t-1))+\delta(z(t)+x_2(t)-s)=0, t=T, T-1\in\{t_1,t_2,\dots,t_k\}, \end{cases}$$

where $B=\frac{a_2\delta^2}{2}-c\delta-\delta^2$, $C=Ac\delta-c-\delta$, $D=\frac{a_2\delta^2}{2}-Ac\delta-c\delta-A^2c\delta^2-\delta^2$, $E=Ac\delta+c+A^2c\delta+\delta$, $F=\frac{a_2\delta}{2}-Ac-\delta-A^2c\delta$.

The theorem is proved. □

2.2 The case when the player validates agents' opinions and controls them in the same periods

We consider a simple social network, i.e., a two-agent opinion dynamic system. The characteristic of the network is that there exists a player, different from the agents, who can choose to validate the opinion of agents 1 and 2, and use a strategy to influence opinion of agent 1 at that time. The state variables are the agents' opinions $x_i(t) \in R$ at time $t \in [0,T]$, $i = 1,2$. The control variable, that is, the influence strategy of the player, is intensity $u(t) \in R$. First, we assume that the validation and influence set of periods $U \subset \{1,2,\dots,T-1\}$ is given to the player. We provide the solution of the optimal control problem for set U, then allow to choose set U optimally. Let the number of elements k in set U be given, and $k < T$. In a nutshell, we have

- $x_i(t), i = 1,2$: the opinion of agent i at time $t \in [0,T]$;
- $u(t)$: the player's influence on agent 1's opinion at time $t \in [0,T-1]$;
- $U = \{t_1,\dots,t_k \mid 0 \le t_1 < t_2 < \cdots < t_k \le T-1\}$: the set of periods, in which the player adds a control and validates agents' opinions.

The dynamics of agents' opinions are defined by the following equations:

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right) + u(t),\ t \in U, \quad (2.11)$$

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right),\ t \notin U, \quad (2.12)$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right),\ t = 0,\dots,T-1 \quad (2.13)$$

with initial condition

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0. \quad (2.14)$$

In the above equation, $a_1 > 0$, $a_2 > 0$ denote the agents' beliefs about the average social opinion, respectively.

The dynamics of agents' opinion are describing the following idea:

1. When t belongs to set U, agent 1 updates his opinion according to equation (2.11). The agent 1's opinion, the social average opinion, and the player's strategy at the previous stage will influence the formation of the opinion at the next stage.

2. When t does not belong to set U, agent 1 updates his opinion according to equation (2.12). The agent 1's opinion and the social average opinion but not player's strategy at the previous stage will influence the formation of the opinion in the next stage.

3. When $t \in \{0, \ldots, T-1\}$, agent 2 updates his opinion according to equation (2.13) at any stage. Agent 2's opinion and the social average opinion at the previous stage will influence the formation of the opinion in the next stage, as agent 2 is not influenced by the player.

The player's target opinion is $s \in R$. The player aims to minimize his costs, which are as follows:

$$\begin{aligned} J(u) &= \sum_{j=1}^{k} \delta^{t_j} \left((x_1(t_j) - s)^2 + (x_2(t_j) - s)^2 + cu^2(t_j) \right) \\ &\quad + \delta^T \left((x_1(T) - s)^2 + (x_2(T) - s)^2 \right), \end{aligned} \tag{2.15}$$

where $\delta \in (0, 1]$ is the discount factor and $c > 0$ is the player's cost per unit level of influence.

Therefore, the LQ optimization problem with a given set U may be rewritten in the following way:

$$\begin{cases} \text{minimize (2.15)} \\ \text{subject to } u \text{ satisfying (2.11), (2.12), and (2.13).} \end{cases} \tag{2.16}$$

The Euler-equation method gives the necessary conditions (see e.g. [17, 37, 38]). We can notice that the problem considered in the paper belongs to the class of linear-quadratic optimization problems. We apply the Euler-equation method to find the player's optimal strategy in the dynamic problem with average-oriented opinion dynamics (see [63]). The same method is used in [30, 31] to find the optimal control in opinion dynamics problem.

Theorem 2.2. *Let $\{u^*(t) : t = t_1, t_2, \ldots, t_k\}$ be the optimal strategy minimizing functional (2.15) subject to state dynamics equations (2.11), (2.12), and (2.13) with initial conditions (2.14), and $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ be the corresponding state trajectory. The periods $0 \le t_1 < t_2 < \cdots < t_k \le T-1$, are given. Then the optimal strategy $u^*(t)$, $t = t_1, t_2, \ldots, t_k$ is defined as*

$$u^*(t) = z^*(t+1) - Az^*(t),$$

and corresponding optimal state trajectory $(x_1^*(t), x_2^*(t))$, $t = 1, \ldots, T$ satisfy the system of equations:

$$\begin{cases} z(t+1) = Az(t), \quad t = 1, \ldots, T-1, t, t-1 \notin U, \\ \delta c z(t) - (c + Ac\delta) z(t-1) + Acz(t-2) = 0, \quad t \notin U, t-1 \in U, \\ Bz(t) + Cz(t-1) + Ac\delta z(t+1) - \frac{Ac}{\delta} z(t-2) \\ \quad = (\delta - a_2\delta)(x_2(t) - s) - x_2(t-1) + s, \quad t, t-1 \in \{t_2, \ldots, t_k\}, \\ Dz(t) + \left(1 + A^2 c\right) z(t-1) + Ac\delta z(t+1) = (\delta - a_2\delta)(x_2(t) - s) \\ \quad -x_2(t-1) + s, \quad t \in U, t-1 \notin U, \\ (c+\delta) z(t) - Acz(t-1) + \delta(x_2(t) - s) = 0, \quad t = T, t-1 \in U, \\ z(t) + x_2(t) - s = 0, \quad t = T, t-1 \notin U, \\ \frac{a_2\delta^2}{2} z(t) + cz(t-1) - Acz(t-2) = a_2\delta^2 (x_2(t) - s), \\ \quad t = T, t-1 \in \{t_2, \ldots, t_k\}, \\ z(t) + 2(x_2(t) - s) = 0, \quad t = T, t-1 \notin U, \\ x_2(t+1) = x_2(t) + \frac{a_2}{2} z(t), \quad t = 1, \ldots, T-1, \end{cases} \tag{2.17}$$

where $z^*(t) = x_1^*(t) - x_2^*(t)$, $A = 1 - \frac{a_1 + a_2}{2}$, $B = \frac{a_2\delta}{2} - c - A^2 c\delta - \delta - Ac$, $C = Ac - \frac{c}{\delta} - A^2 c - 1$, $D = \frac{a_2\delta}{2} - A^2 c\delta - \delta - Ac$.

Proof. We represent a new variable $z(t)$ as

$$z(t) = x_1(t) - x_2(t), \; t = 0, \ldots, T.$$

From state equations (2.11), (2.12) taking into account expression of $z(t)$, we obtain the new state equations:

$$\begin{aligned} z(t+1) &= Az(t) + u(t), \quad t \in U, && (2.18) \\ z(t+1) &= Az(t), \quad t \notin U, \\ x_2(t+1) &= x_2(t) + \frac{a_2}{2} z(t), \quad t = 0, \dots, T-1, && (2.19) \end{aligned}$$

with initial condition

$$z(0) = x_1^0 - x_2^0, x_2(0) = x_2^0,$$

where $A = 1 - \frac{a_1 + a_2}{2}$.

We find an expression of $u(t)$ from (2.18) and obtain

$$u(t) = z(t+1) - Az(t), \quad t \in \{t_1, t_2, \dots, t_k\}, \tag{2.20}$$

Substitute these expressions into the functional, we can rewrite it in the following form:

$$\begin{aligned} J(z, x_2) = &\sum_{j=1}^{k} \delta^{t_j} \left[(z(t_j) + x_2(t_j) - s)^2 + (x_2(t_j) - s)^2 + c(z(t_j+1) - Az(t_j))^2 \right] \\ &+ \delta^T \left[(z(T) + x_2(T) - s)^2 + (x_2(T) - s)^2 \right]. \end{aligned}$$

To minimize $J(z, x_2)$ under condition given by equations (2.18) and (2.20), we form the Lagrange function

$$L(z, x_2, l) = J(z, x_2) + \sum_{t=1}^{T-1} l_t \left(x_2(t+1) - x_2(t) - \frac{a_2}{2} z(t) \right),$$

where $l = (l_1, \dots, l_{T-1})$.

The first-order conditions are $\frac{\partial L(z, x_2, l)}{\partial z(t)} = 0, t = 1, \dots, T$ and $\frac{\partial L(z, x_2, l)}{\partial x_2(t)} = 0$, $t = 1, \dots, T$.

First, we find the derivatives and get

$$\begin{aligned}
\frac{\partial J(z,x_2)}{\partial z(t)} &= 0, \quad t=1,\ldots,T-1, t, t-1 \notin U, \\
\frac{\partial J(z,x_2)}{\partial z(t)} &= \delta^{t-1}2c\left(z(t)-Az(t-1)\right), \quad t, \notin U, t-1 \in U, \\
\frac{\partial J(z,x_2)}{\partial z(t)} &= \delta^{t-1}2c\left(z(t)-Az(t-1)\right)-\delta^{t}2Ac\left(z(t+1)-Az(t)\right) \\
&\quad +\delta^{t}2\left(z(t)+x_2(t)-s\right), \quad t, t-1 \in U, \\
\frac{\partial J(z,x_2)}{\partial z(t)} &= -\delta^{t}2Ac\left(z(t+1)-Az(t)\right)+\delta^{t}2\left(z(t)+x_2(t)-s\right), \\
&\quad t \in U, t-1 \notin U \\
\frac{\partial J(z,x_2)}{\partial z(t)} &= \delta^{t-1}2c\left(z(t)-Az(t-1)\right)+\delta^{t}2\left(z(t)+x_2(t)-s\right), \\
&\quad t=T, t-1 \in U,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J(z,x_2)}{\partial z(t)} &= \delta^{t}2\left(z(t)+x_2(t)-s\right), \quad t=T, t-1 \notin U, \\
\frac{\partial J(z,x_2)}{\partial x_2(t)} &= 0, \quad t=1,\ldots,T-1, t \notin U, \\
\frac{\partial J(z,x_2)}{\partial x_2(t)} &= \delta^{t}\left[2\left(z(t)+x_2(t)-s\right)+2\left(x_2(t)-s\right)\right], \quad t \in U, \\
\frac{\partial J(z,x_2)}{\partial x_2(t)} &= \delta^{t}\left[2\left(z(t)+x_2(t)-s\right)+2\left(x_2(t)-s\right)\right], \quad t=T,
\end{aligned}$$

Second, we write the system of the first-order conditions, that is,

$$\begin{cases} l_t = 0, \quad t = 1, \dots, T-1, t, t-1 \notin U, \\ c\left(z\left(t\right) - A * z\left(t-1\right)\right) = \frac{a_2}{4} l_t \delta^{-t+1}, \quad t \notin U, t-1 \in U, \\ \left(\frac{c}{\delta} + A^2 c + 1\right) z\left(t\right) - \frac{Ac}{\delta} z\left(t-1\right) - Acz\left(t+1\right) + x_2\left(t\right) - s \\ \quad = \frac{a_2}{4} l_t \delta^{-t}, \quad t, t-1 \in U, \\ \left(A^2 c + 1\right) z\left(t\right) - Acz\left(t+1\right) + x_2\left(t\right) - s = \frac{a_2}{4} l_t \delta^{-t}, \\ \quad t \in U, t-1 \notin U, \\ \left(c + \delta\right) z\left(t\right) - Acz\left(t-1\right) + \delta\left(x_2\left(t\right) - s\right) = 0, \quad t = T, t-1 \in U, \\ z\left(t\right) + x_2\left(t\right) - s = 0, \quad t = T, t-1 \notin U, \\ l_{t-1} - l_t = 0, \quad t = 1, \dots, T-1, t \notin U, \\ \delta^t \left[2z\left(t\right) + 4\left(x_2\left(t\right) - s\right)\right] + l_{t-1} - l_t = 0, \quad t \in U, \\ \delta^t \left[2z\left(t\right) + 4\left(x_2\left(t\right) - s\right)\right] + l_{t-1} = 0, \quad t = T, \end{cases} \tag{2.21}$$

with initial conditions $z\left(0\right) = x_1^0 - x_2^0$, $x_2\left(0\right) = x_2^0$.

Excluding l_t from system (2.21), finally we obtain the system of equations

$$\begin{cases} z\left(t+1\right) = Az\left(t\right), \quad t = 1, \dots, T-1, t, t-1 \notin U, \\ \delta cz\left(t\right) - \left(c + Ac\delta\right) z\left(t-1\right) + Acz\left(t-2\right) = 0, \quad t \notin U, t-1 \in U \\ Bz\left(t\right) + Cz\left(t-1\right) + Ac\delta z\left(t+1\right) - \frac{Ac}{\delta} z\left(t-2\right) = \left(\delta - a_2\delta\right)\left(x_2\left(t\right) - s\right) \\ \quad - x_2\left(t-1\right) + s, \quad t, t-1 \in \left\{t_2, \dots, t_k\right\}, \\ Dz\left(t\right) + \left(1 + A^2 c\right) z\left(t-1\right) + Ac\delta z\left(t+1\right) = \left(\delta - a_2\delta\right)\left(x_2\left(t\right) - s\right) \\ \quad - x_2\left(t-1\right) + s, \quad t \in U, t-1 \notin U, \\ \left(c + \delta\right) z\left(t\right) - Acz\left(t-1\right) + \delta\left(x_2\left(t\right) - s\right) = 0, \quad t = T, t-1 \in U, \\ z\left(t\right) + x_2\left(t\right) - s = 0, \quad t = T, t-1 \notin U, \\ \frac{a_2\delta^2}{2} z\left(t\right) + cz\left(t-1\right) - Acz\left(t-2\right) = a_2\delta^2\left(x_2\left(t\right) - s\right), \\ \quad t = T, t-1 \in \left\{t_2, \dots, t_k\right\}, \\ z\left(t\right) + 2\left(x_2\left(t\right) - s\right) = 0, \quad t = T, t-1 \notin U, \end{cases}$$

where $B = \frac{a_2\delta}{2} - c - A^2 c\delta - \delta - Ac$, $C = Ac - \frac{c}{\delta} - A^2 c - 1$, $D = \frac{a_2\delta}{2} - A^2 c\delta - \delta - Ac$.

The proof is finished. □

The system given in Theorem 2.2 represents the necessary conditions for the optimal control problem.

Remark 2.1. *In Theorem 2.2, given set U, the solution of system (2.17) gives the optimal state trajectory and the unique corresponding optimal control. When the set U is not given to the player, he may consider an optimization problem and consider all possible sets U and then compare the costs corresponding to the possible sets to get the minimal costs. Therefore, the set U corresponding to the minimal costs is optimal.*

2.3 The case when validation and control time sets are different

We consider also a two-agent opinion dynamic system, in which the player can choose different sets of periods to validate and control opinions. Assume that for the player the two time sets are given, i.e., the control time set M and the validation time set N. We assume that a player divide the set of periods $\{0, 1, \ldots, T\}$ into two disjoint subsets: (i) when he influences an agent, set M, and (ii) when he validates the agents' opinions, set N. Therefore, $t \in \{0, 1, ..., T-1, T\} = M \cup N$, where $M = \{m_1, m_2, \ldots, m_p\}$, and $0 \le m_1 < m_2 < \cdots < m_p \le T-1$ with $p < T$. Let N be the set $\{n_1, n_2, \ldots, n_k\}$, $1 \le n_1 < n_2 < \cdots < n_k \le T$ with $k < T$, and $M \cap N = \varnothing$. The notations are as follows:

- $x_i(t)$: the opinion of agent i at time $t \in [0, T]$, $i = 1, 2$;
- $u(t)$: the player's influence level on agent 1's opinion at time $t \in [0, T-1]$;
- $M = \{m_1, m_2, \ldots, m_p\}$: the control time set of the player, $p < T$;

- $N = \{n_1, n_2, \ldots, n_k\}$: the validation time set, $k < T$.

The dynamics of agents' opinions are defined by the following equations:

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right) + u(t), \quad t \in M, \tag{2.22}$$

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t)+x_2(t)}{2} - x_1(t)\right), \quad t \notin M, \tag{2.23}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right), \quad t = 0, \ldots, T-1 \tag{2.24}$$

with initial condition

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0. \tag{2.25}$$

In the above equations, $a_1 > 0, a_2 > 0$ denote the agent's beliefs about the average social opinion, respectively.

The dynamics of agents' opinion satisfy the following idea:

1. When t belongs to the set M, agent 1 updates his opinion according to equation (2.22). Agent 1's opinion, the social average opinion and player's strategy at the previous stage will influence the formation of the opinion at the next stage.

2. When t belongs to the set N, agent 1 updates his opinion according to equation (2.23). Agent 1's opinion and the social average opinion at the previous stage will influence the formation of the opinion at the next stage.

3. When $t \in \{0, \ldots, T-1\}$, agent 2 updates his opinion according to equation (2.24) at any stage. Agent 2's opinion and the social average opinion at the previous stage will influence the formation of the opinion at the next stage as agent 2 is not influenced by the player.

The player's target opinion is $s \in R$. The player aims to minimize his costs, which are as follows:

$$J(u) = \sum_{i=1}^{p} \delta^{m_i} \left(cu^2(m_i) \right) + \sum_{j=1}^{k} \delta^{n_j} \left((x_1(n_j) - s)^2 + (x_2(n_j) - s)^2 \right) \\ + \delta^T \left((x_1(T) - s)^2 + (x_2(T) - s)^2 \right), \tag{2.26}$$

where $\delta \in (0, 1]$ is the discount factor and $c > 0$ is the player's cost per unit level of influence.

Therefore, LQ optimization problem may be rewritten in the following way:

$$\begin{cases} \text{minimize } (2.26) \\ \text{subject to } u, \text{ satisfying } (2.22), (2.23), \text{and } (2.24). \end{cases} \tag{2.27}$$

Theorem 2.3. *Let $\{u^*(t) : t = m_1, m_2, \ldots, m_p\}$ be the optimal strategy minimizing functional (2.26) subject to initial conditions (2.25), state dynamics equations (2.22), (2.23), and (2.24), and $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ be the corresponding state trajectory. The set of periods M and N are given. Then the optimal strategy $u^*(t)$, $t \in M$ is defined as*

$$u^*(t) = z^*(t+1) - Az^*(t),$$

and corresponding optimal state trajectory $(x_1^*(t), x_2^*(t))$, $t = 1, \ldots, T$, satisfy the *system of equations:*

$$\begin{cases} Bz\left(t\right)+Ac\delta z\left(t+1\right)+Cz\left(t-1\right)-\frac{Ac}{\delta}z\left(t-2\right)=0, \\ \quad t\in M, \quad t=m_i, i=2,\ldots,p, \\ \left(\frac{a_2\delta}{2}-\delta\right)z\left(t\right)+z\left(t-1\right)=\left(\delta-a_2\delta\right)\left(x_2\left(t\right)-s\right)-x_2\left(t\right)+s, \\ \quad t\in N, \\ \left(Ac-\frac{a_2\delta}{2}\right)z\left(t\right)-\left(\frac{c}{\delta}+A^2c\right)z\left(t-1\right)+\frac{Ac}{\delta}z\left(t-2\right) \\ \quad =a_2\delta\left(x_2\left(t\right)-s\right), \quad t=T, t-1\in M, \quad t=m_i, i=2,\ldots,p, \\ -\frac{a_2\delta}{2}z\left(t\right)-z\left(t-1\right)=a_2\delta\left(x_2\left(t\right)-s\right)+x_2\left(t\right)-s, \\ \quad t=T, t-1\in N, \\ \left(c+\delta\right)z\left(t\right)-Acz\left(t-1\right)+\delta\left(x_2\left(t\right)-s\right)=0, \quad t=T, t-1\in M, \\ z\left(t\right)+x_2\left(t\right)-s=0, \quad t=T, t-1\in N, \\ x_2\left(t+1\right)=x_2\left(t\right)+\frac{a_2}{2}z\left(t\right), t=1,\ldots,T-1, \end{cases} \tag{2.28}$$

where $z^*(t) = x_1^*(t) - x_2^*(t)$, $A = 1 - \frac{a_1+a_2}{2}$, $B = -\left(Ac - c - A^2c\delta\right)$, $C = \frac{c}{\delta} + A^2c + Ac$.

Proof. We represent a new variable $z\left(t\right)$ as

$$z\left(t\right)=x_1\left(t\right)-x_2\left(t\right),\ t=0,\ldots,T.$$

From state equations (2.22), (2.23) and (2.24)taking into account expression of $z\left(t\right)$, we obtain the new state equations:

$$z\left(t+1\right) = Az\left(t\right)+u\left(t\right), t\in M, \tag{2.29}$$

$$z\left(t+1\right) = Az\left(t\right), t\notin M,$$

$$x_2\left(t+1\right) = x_2\left(t\right)+\frac{a_2}{2}z\left(t\right), \tag{2.30}$$

with initial condition

$$z\left(0\right)=x_1^0-x_2^0,\ x_2\left(0\right)=x_2^0,$$

where $A=1-\frac{a_1+a_2}{2}$.

We find an expression of $u(t)$ from (2.22) and obtain

$$u(t) = z(t+1) - Az(t), t \in M. \tag{2.31}$$

Substituting these expressions into functional, we can rewrite it in the following form:

$$\begin{aligned} J(z, x_2) =& \sum_{i=1}^{p} \delta^{m_i} \left[c\left(z(m_i+1) - Az(m_i)\right)^2 \right] \\ &+ \sum_{j=1}^{k} \delta^{n_j} \left[\left(z(n_j) + x_2(n_j) - s\right)^2 + \left(x_2(n_j) - s\right)^2 \right] \\ &+ \delta^{T} \left[\left(z(T) + x_2(T) - s\right)^2 + \left(x_2(T) - s\right)^2 \right]. \end{aligned}$$

To minimize $J(z, x_2)$ under condition given by equations (2.24) and (2.31), we form the Lagrange function

$$L(z, x_2, k) = J(z, x_2) + \sum_{t=1}^{T-1} k_t \left(x_2(t+1) - x_2(t) - \frac{a_2}{2} z(t) \right).$$

where $k = (k_1, \ldots, k_{T-1})$. The first-order conditions are $\frac{\partial L(z,x_2,k)}{\partial z(t)} = 0, t = 1, \ldots, T$ and $\frac{\partial L(z,x_2,k)}{\partial x_2(t)} = 0$, $t = 1, \ldots, T$.

First, we find the derivatives and get

$$\begin{aligned} \frac{\partial J(z, x_2)}{\partial z(t)} &= \delta^{t-1} 2c\left(z(t) - Az(t-1)\right) - \delta^{t} 2Ac\left(z(t+1) - Az(t)\right), \quad t \in M, \\ \frac{\partial J(z, x_2)}{\partial z(t)} &= \delta^{t} 2\left(z(t) + x_2(t) - s\right), \quad t \in N, \\ \frac{\partial J(z, x_2)}{\partial z(t)} &= \delta^{t-1} 2c\left(z(t) - Az(t-1)\right) + \delta^{t} 2\left(z(t) + x_2(t) - s\right), \\ & \quad t = T, t-1 \in M, \\ \frac{\partial J(z, x_2)}{\partial z(t)} &= \delta^{t} 2\left(z(t) + x_2(t) - s\right), \quad t = T, t-1 \in N, \\ \frac{\partial J(z, x_2)}{\partial x_2(t)} &= 0, \quad t \in M, \\ \frac{\partial J(z, x_2)}{\partial x_2(t)} &= \delta^{t} \left[2\left(z(t) + x_2(t) - s\right) + 2\left(x_2(t) - s\right)\right], \quad t \in N, \\ \frac{\partial J(z, x_2)}{\partial x_2(t)} &= \delta^{t} \left[2\left(z(t) + x_2(t) - s\right) + 2\left(x_2(t) - s\right)\right], \quad t = T. \end{aligned}$$

Second, we write the system of the first-order conditions, that is

$$\begin{cases} \left(\frac{c}{\delta}+A^2c\right)z(t)-\frac{Ac}{\delta}z(t-1)-Acz(t+1)=\frac{a_2}{4}k_t\delta^{-t}, \quad t\in M,\\ z(t)+x_2(t)-s=\frac{a_2}{4}k_t\delta^{-t}, \quad t\in N,\\ (c+\delta)z(t)-Acz(t-1)+\delta(x_2(t)-s)=0, \quad t=T, t-1\in M,\\ z(t)+x_2(t)-s=0, \quad t=T, t-1\in N,\\ k_{t-1}-k_t=0, \quad t\in M,\\ \delta^t\left[2z(t)+4(x_2(t)-s)\right]+k_{t-1}-k_t=0, \quad t\in N,\\ \delta^t\left[2z(t)+4(x_2(t)-s)\right]+k_{t-1}=0, \quad t=T, \end{cases} \tag{2.32}$$

with initial conditions $z(0)=x_1^0-x_2^0$, $x_2(0)=x_2^0$.

Excluding k_t from system (2.32), finally we obtain the system of equations

$$\begin{cases} Bz(t)+Ac\delta z(t+1)+Cz(t-1)-\frac{Ac}{\delta}z(t-2)=0,\\ \quad t\in M, \quad t=m_i, i=2,\dots,p,\\ \left(\frac{a_2\delta}{2}-\delta\right)z(t)+z(t-1)=(\delta-a_2\delta)(x_2(t)-s)-x_2(t)+s, \quad t\in N,\\ \left(Ac-\frac{a_2\delta}{2}\right)z(t)-\left(\frac{c}{\delta}+A^2c\right)z(t-1)+\frac{Ac}{\delta}z(t-2)=a_2\delta(x_2(t)-s),\\ \quad t=T, t-1\in M, \quad t=m_i, i=2,\dots,p,\\ -\frac{a_2\delta}{2}z(t)-z(t-1)=a_2\delta(x_2(t)-s)+x_2(t)-s, \quad t=T, t-1\in N,\\ (c+\delta)z(t)-Acz(t-1)+\delta(x_2(t)-s)=0, \quad t=T, t-1\in M,\\ z(t)+x_2(t)-s=0, \quad t=T, t-1\in N, \end{cases}$$

where $B=-\left(Ac-c-A^2c\delta\right)$, $C=\frac{c}{\delta}+A^2c+Ac$.

The theorem is proved. □

Remark 2.2. *In Theorem 2.3, given the validation and control sets M and N, the solution of system (2.28) gives the optimal state trajectory and the unique corresponding optimal control trajectory. If sets M and N are not given, the player can find them in an optimal way by considering all possible sets N (after that, M is uniquely defined) and comparing the costs corresponding to all these sets to get* *the minimum costs. Therefore, the set N corresponding to the minimum costs is optimal.*

2.4 Numerical simulations

2.4.1 Numerical example for Section 2.1

Let $a_1 = 0.2$, $a_2 = 0.9$, $\delta = 1$, $c = 0.8$ and initial agents' opinions be $x_1(0) = 0.7$, $x_2(0) = 0.2$. The player's target opinion is $s = 0.5$. We also assume that k is equal to three. For the time horizon $T = 10$, we realize the algorithm and obtain that the player's minimal costs are obtained when the set of control periods is $\{0, 8, 9\}$. The values of the optimal agents' opinion trajectories and the optimal control trajectory are given in Table 2.1. The optimal value of functional (2.5) is 0.1511.

Table 2.1: Optimal control trajectories and state.

t	$t_1 = 0$	1	2	3	4	5
$x_1(t)$	0.7000	0.5193	0.5084	0.5036	0.5016	0.5007
$x_2(t)$	0.2000	0.4250	0.4674	0.4858	0.4939	0.4973
$z(t)$	0.5000	0.0943	0.0409	0.0178	0.0077	0.0034
$u(t)$	-0.1307					
t	6	7	$t_2 = 8$	$t_3 = 9$	10	
$x_1(t)$	0.5003	0.5001	0.5001	0.5000	0.5000	
$x_2(t)$	0.4988	0.4995	0.4998	0.4999	0.5000	
$z(t)$	0.0015	0.0006	0.0003	0.0001	0.00005	
$u(t)$			-0.000007	-0.000006		

We introduce the optimal opinion trajectory (for both agents 1 and 2) and player's strategy trajectory on Figures 2.1 and 2.2 respectively.

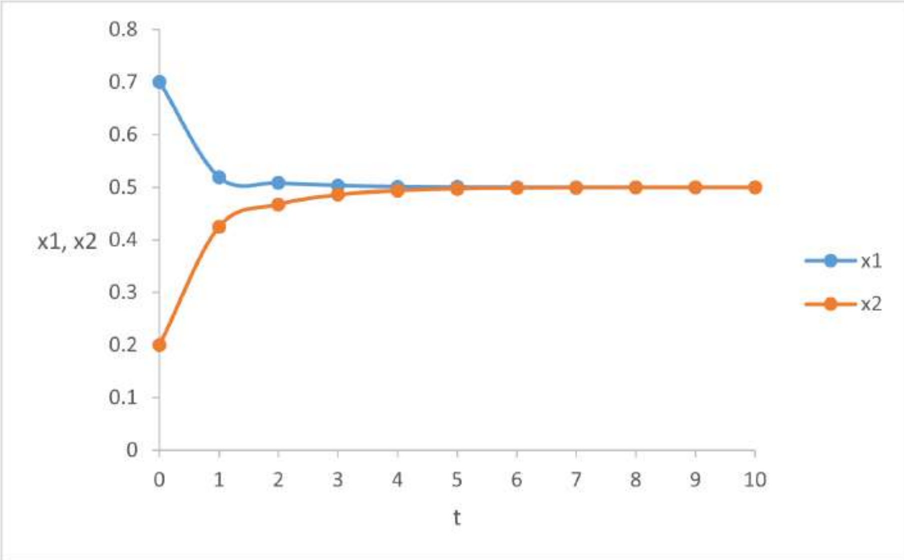

Figure 2.1: Optimal state trajectories (blue — $x_1(t)$, red — $x_2(t)$).

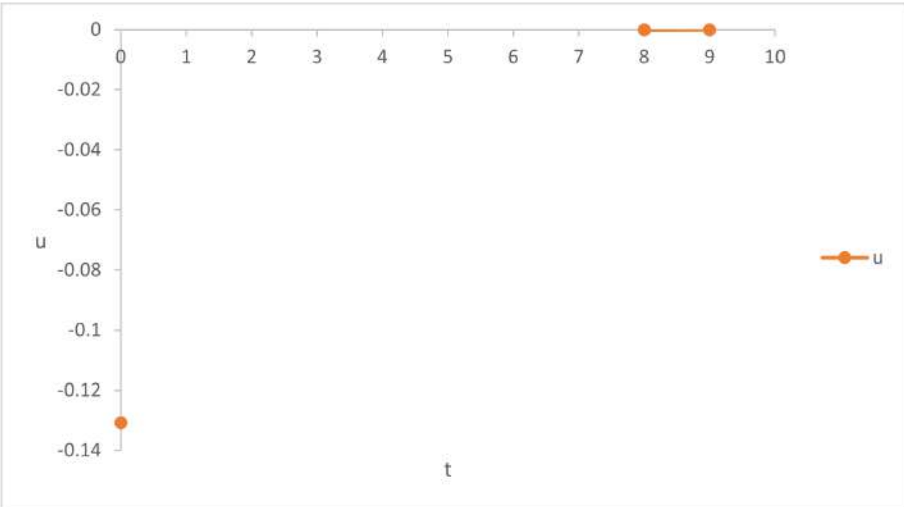

Figure 2.2: Optimal strategy trajectory $u(t)$.

Behaving optimally, the player chooses to control agent 1 at periods 0, 8, and 9 to influence his opinion. We should remind that the player validates the opinions of both agents at each period. Calculations show that the player finds this set of the control periods optimal, i.e. the set of control periods $\{0, 8, 9\}$ minimizes his total costs which are 0.1511 over all possible set of periods. We can easily notice looking at Figure 2.1 that after period 4 the opinions of both agents almost reach the target opinion $s = 0.5$.

2.4.2 Numerical example for Section 2.2

Consider 11-stage problem with $T = 10$ (periods 0, ..., 10). Let $k = 3$. The parameters are as follows:

$$a_1 = 0.4, a_2 = 0.5, \delta = 1, c = 0.7, s = 0.5,$$

$$x_1(0) = 0.2, x_2(0) = 0.8.$$

There are 2 agents in a network. The initial opinion state is $x(0) = (0.2, 0.8)$, i.e. $x_1(0) = 0.2$ and $x_2(0) = 0.8$ for any $i = 1, 2$. The agent 1 and agent 2' beliefs about the average social opinion are $a_1 = 0.4$ and $a_2 = 0.5$, respectively. The discount factor is $\delta = 1$. The unit costs for level of influence is $c = 0.7$ for player 1. His target opinion is $s = 0.5$.

Table 2.2: Optimal control and state trajectories.

t	0	1	2	3	4	5
$x_1(t)$	0.2000	0.3930	0.4444	0.4727	0.4882	0.4968
$x_2(t)$	0.8000	0.6500	0.5858	0.5504	0.5310	0.5203
$z(t)$	-0.6000	-0.2570	-0.1413	-0.0777	-0.0428	-0.0235
$u(t)$						

t	$t_1 = 6^a$	$t_2 = 7^a$	$t_3 = 8^a$	9	10
$x_1(t)$	0.4987	0.4955	0.4929	0.4901	0.5000
$x_2(t)$	0.5144	0.5105	0.5067	0.5033	0.5000
$z(t)$	-0.0157	-0.0150	-0.0138	-0.0132	0
$u(t)$	-0.0063	-0.0056	-0.0056		

[a] The player validates and influences agent 1's opinion in these three periods to get the minimum cost.

Based on Theorem 2.2, we find the necessary conditions, and solve system (2.17). The player's minimal costs are obtained when the set of control and validation periods is $\{6, 7, 8\}$. This set is optimally chosen among 120 sets. The values of the optimal agents' opinion trajectories and optimal control trajectory are given in Table 2.2. The optimal value of functional (2.15) is 0.000507.

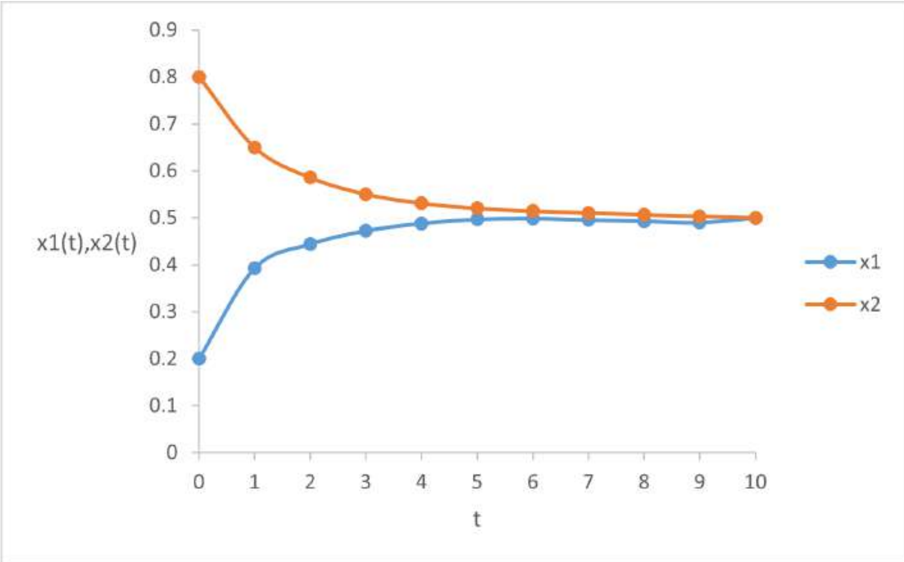

Figure 2.3: Optimal state trajectories (blue — $x_1(t)$, red — $x_2(t)$).

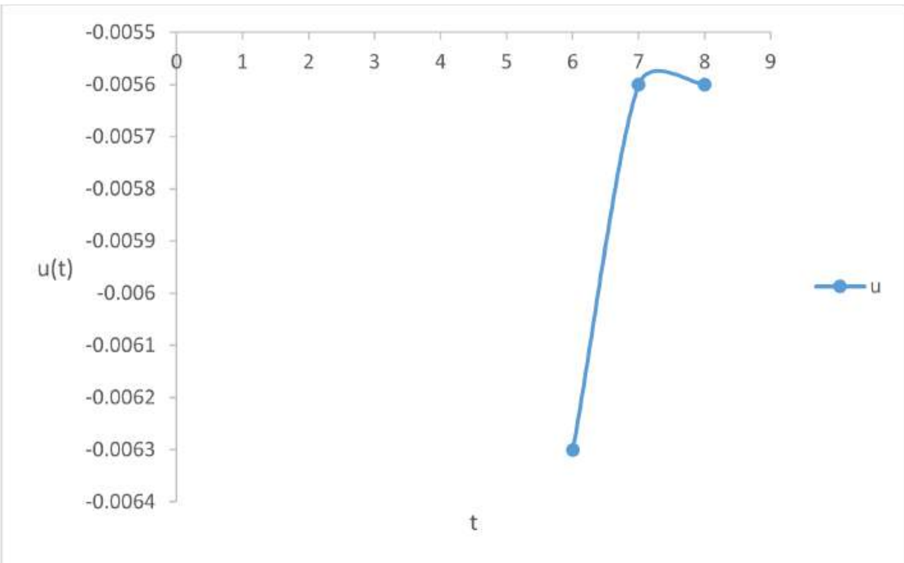

Figure 2.4: Optimal control trajectory $u(t)$.

Agents' opinions are "stabilized" over time. Player controls the opinion of agent 1. The optimal state trajectories of the agents are shown in Figure 2.3. The optimal control trajectory is represented in Figure 2.4.

We can see in Figure 2.3 that the player chooses to control an agent and validates the agents' opinions at periods 6, 7, and 8, which minimizes his cost, up to 0.000507. After period 5, the agents' opinions almost reach target opinion $s = 0.5$.

The minimum costs correspond to the "best" result (optimal value of the costs), and we can also examine if these costs are much less then the costs in the "worst" result (when the validation periods are chosen to have maximal costs, or this can be called as the worst-case scenario). The maximum cost can be calculated in the same way. The player's maximum costs are 2.07173297, and they are obtained when the set of control and validation periods is $\{1, 3, 5\}$. Comparing the gap between the "worst" and "best" results, i.e., $\frac{J_{max} - J_{min}}{J_{max}} \cdot 100\% = 99.98\%$, and we can notice that if the player optimizes following the procedure proposed in the paper, he can reduce almost all his costs by zero level.

Remark 2.3. *We calculate the set of all possible periods by C_T^k ways. When $T = 10$, the player chooses three periods to validate and influence the agent's opinion, i.e. $C_{10}^3 = 120$. To find the optimal solution of the optimization problem, we need to solve 120 systems given in Theorem 2.2.*

2.4.3 Numerical example for Section 2.3

We also assume that k is equal to 3 like in the previous section. Let the parameters be as follows:

$$fa_1 = 0.2, a_2 = 0.6, \delta = 1, c = 0.7, s = 0.5,$$

$$x_1(0) = 0.3, x_2(0) = 0.9, T = 10.$$

By Theorem 2.3, we need to solve system (2.28) and obtain player's minimal costs when the set of control periods is $\{0, 1, 9\}$. The values of optimal agents' opinion trajectories and the optimal control trajectory are given in Table 2.3. The optimal value of functional (2.26) is 0.034872.

Table 2.3: Optimal control and state trajectories.

t	$t_1 = 0^a$	$t_2 = 1^a$	2	3	4	5
$x_1(t)$	0.3000	0.4312	0.4588	0.4759	0.4868	0.4943
$x_2(t)$	0.9000	0.7200	0.6334	0.5810	0.5495	0.5307
$z(t)$	-0.6000	-0.2888	-0.1745	-0.1051	-0.0627	-0.0364
$u(t)$	0.0712	-0.0012				
t	6	7	8	$t_3 = 9^a$	10	
$x_1(t)$	0.5003	0.5064	0.5141	0.5059	0.5027	
$x_2(t)$	0.5198	0.5139	0.5117	0.5124	0.5104	
$z(t)$	-0.0195	-0.0075	0.0024	-0.0065	-0.0077	
$u(t)$				-0.0039		

[a] The player can get the minimum cost by influencing agent 1's opinion in these three periods.

Agents' opinions are "stabilized" over time. Player controls the opinion of agent 1. The optimal state trajectories are presented in Figure 2.5. The optimal control trajectory is shown in Figure 2.6.

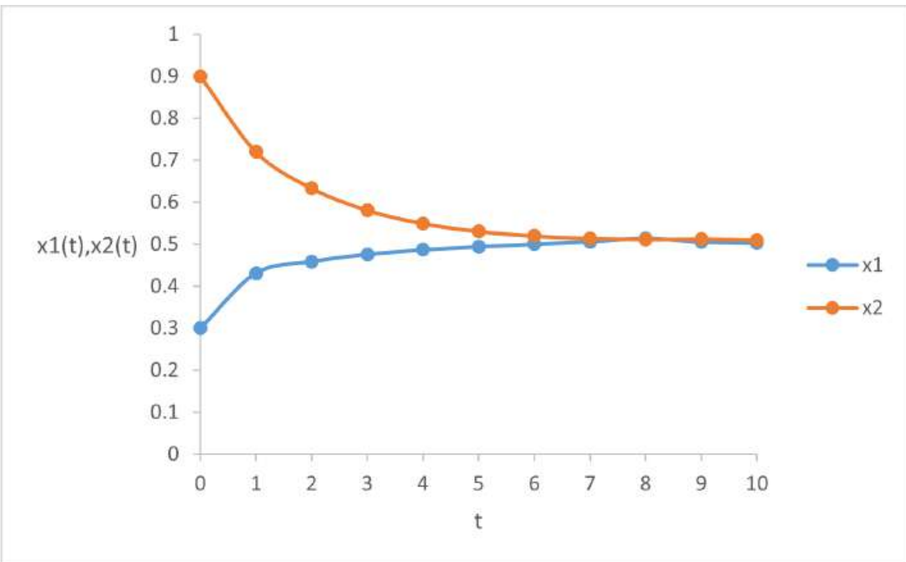

Figure 2.5: Optimal state trajectories (blue — $x_1(t)$, red — $x_2(t)$).

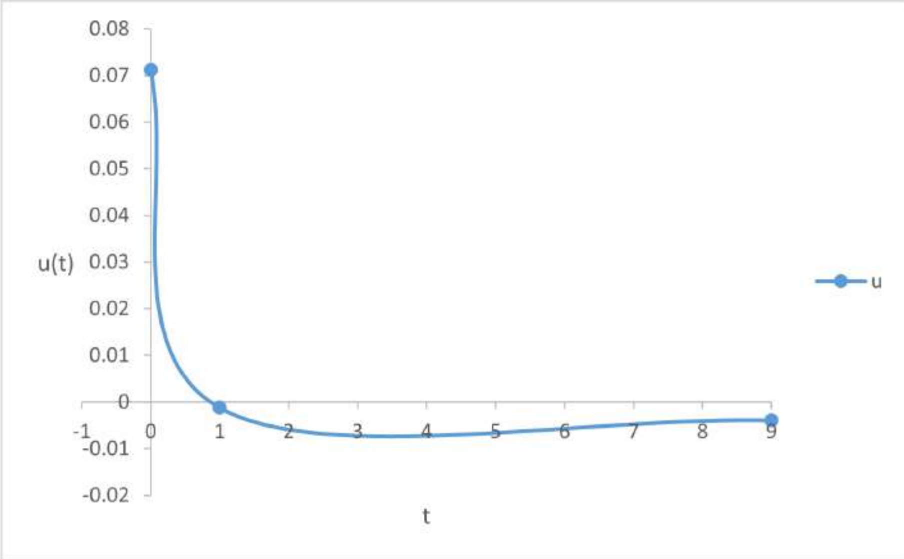

Figure 2.6: Optimal control trajectory $u(t)$.

From Figure 2.5, we can see that the player chooses to add controls at periods 0, 1, and 9, and monitor the agents' opinion at the remaining periods, making his costs minimal, that are 0.034872. After period 5, the agents' opinions almost reach the target opinion $s = 0.5$.

We use the same idea as in Example 2.4.2 to highlight that Theorem 2.3 is also very efficient to reduce the player's costs. The player's maximum costs (the worst-case scenario) are 44.0362762, and they are obtained when the set of control and validation periods is $\{3, 4, 6\}$. Comparing the gap between the "worst" and "best" scenarios, we get $\frac{J_{max} - J_{min}}{J_{max}} \cdot 100\% = 99.92\%$. Therefore, the player can reduce almost all his costs to zero by using optimal control obtained from Theorem 2.3.

One can mention that for the this case (this section) of the numerical simulations the player's costs are larger for the previous case (Section 2.2), and moreover, the control sets are different for these two cases. When the player has a possibility to monitor the opinion of the agent at the same time as he controls it, then he prefers to do it at the end of the time interval. In the second case, he first controls the agent's opinion at the beginning of the time interval, then only validates the opinion, and finally, controls the agents at the end of time interval at $t = 9$.

2.5 Conclusions to Chapter 2

We present three mathematical models of small social networks in multi-agent case with the presence of a player or a decision maker. In these models, the player's objective is to bring the social opinion closer to the target opinion minimizing costs, then he takes the approach of optimally choosing the "right periods" to influence an agent, thus minimizing the cost. The first model is characterized by the fact that the player can control only one agent in the system and can influence that agent for a limited time. We find necessary conditions for the optimal solution of the problem, i.e., the minimization of his costs. In numerical simulations, we find the set of optimal control periods chosen by the player to minimize the costs.

We next simulated two more cases when players have restrictions to create the set of periods: 1) player validates agents' opinions and controls one agent's opinion at the same time, and 2) player validates agents' opinions and controls them at different time, but without overlapping. These two ideas allow players to add control more efficiently and avoid wasting time and money. We obtain the optimal time-controlled set and optimal control for the player in numerical simulations. The models presented in this chapter can also be extended to larger number of agents and players.

Chapter 3

Opinion dynamics in small social groups with two influencers

3.1 The case when both influencers validate opinions in the last time

We assume that there are two influencers or players and they directly influence agent 1 and agent 2, respectively. The players have the same discount factors but they are different in their costs per unit of influence intense (c_1 and c_2) and target opinion levels (s_1 and s_2). The state equations for opinions of agents 1 and 2 are

$$x_1(t+1) = x_1(t) + a_1\left(\tfrac{x_1(t)+x_2(t)}{2} - x_1(t)\right) + u_1(t), \tag{3.1}$$

$$x_2(t+1) = x_2(t) + a_2\left(\tfrac{x_1(t)+x_2(t)}{2} - x_2(t)\right) + u_2(t), \tag{3.2}$$

with initial condition

$$x_1(0) = x_1^0, x_2(0) = x_2^0. \tag{3.3}$$

.

Players 1 and 2 are willing to minimize their functionals:

$$J_1(u_1, u_2) = \sum_{t=0}^{T-1} \delta^t \left(c_1 u_1^2(t)\right) + \delta^T \left((x_1(T) - s_1)^2 + (x_2(T) - s_1)^2\right), \tag{3.4}$$

$$J_2\left(u_1,u_2\right)=\sum_{t=0}^{T-1}\delta^t\left(c_2u_2^2(t)\right)+\delta^T\left(\left(x_1(T)-s_2\right)^2+\left(x_2(T)-s_2\right)^2\right),\qquad(3.5)$$

correspondingly, where $c_i>0$ is player i's costs per unit of influence intensity.

We define a two-player game in a normal-form representation with the set of players' strategies U_1,U_2, where $U_j=\left(u_j(t)\in\mathbb{R}:t=0,\dots,T-1\right),j=1,2$, players' cost functions J_1,J_2, defined by formulae (3.4) and (3.5) s.t. state equations (3.1) and (3.2) with initial state $x(0)=(x_1(0),x_2(0))=\left(x_1^0,x_2^0\right)$. The game belongs to the class of linear-quadratic games.

The following theorem gives the necessary conditions of the Nash equilibrium in the game described above.

Theorem 3.1. *Let* $\left\{\left(u_1,u_2\right),u_i=\left(u_i(t):t=0,\dots,T-1\right),i=1,2\right\}$ *be the Nash equilibrium in the game described above, then the Nash equilibrium is defined by*

$$u_i(t)=x_i(t+1)-\left(1-\frac{a_i}{2}\right)x_i(t)-\frac{a_i}{2}x_j(t),\qquad(3.6)$$

and $\{(x_1(t),x_2(t)):t=0,\dots,T\}$ *be the corresponding equilibrium state trajectory with initial condition* $x_1(0)=x_1^0,x_2(0)=x_2^0$*, they satisfy the system:*

$$\begin{cases}-a_ic_i\left(x_i\left(t+1\right)-\left(1-\frac{a_i}{2}\right)x_i\left(t\right)-\frac{a_i}{2}x_j\left(t\right)\right)\\ \quad=\left(1-\frac{a_j}{2}\right)\hat{k}_t^i-\frac{\hat{k}_{t-1}^i}{\delta},\quad t=2,\dots,T-1,\\ c_i\left(x_i\left(t\right)-\left(1-\frac{a_i}{2}\right)x_i\left(t-1\right)-\frac{a_i}{2}x_j\left(t-1\right)\right)\\ \quad+\delta\left(x_i\left(t\right)-s_i\right)=0,t=T,\\ x_j(t)=s_i-\frac{1}{2\delta}\hat{k}_{t-1}^i,\quad t=T,\end{cases}\qquad(3.7)$$

where $i,j=1,2,i\neq j$ *and*

$$\hat{k}_t^i=\frac{4}{a_j\delta}\left[\left(c_i+\delta c_i\left(1-\frac{a_i}{2}\right)^2\right)x_i\left(t\right)-c_i\left(1-\frac{a_i}{2}\right)x_i\left(t-1\right)\right.$$
$$\left.-\delta c_i\left(1-\frac{a_i}{2}\right)x_i\left(t+1\right)-\frac{c_ia_i}{2}x_j\left(t-1\right)+\frac{\delta c_ia_i}{2}\left(1-\frac{a_i}{2}\right)x_j\left(t\right)\right],$$

$t=1,\dots,T-1$ *taking into account the state equations (3.1) and (3.2) and initial state* $(x_1(0),x_2(0))=\left(x_1^0,x_2^0\right)$.

Proof. Let us fix $u_2(t), t=0,\ldots,T-1$ and find the best response of player 1. We use the Euler equation approach to find the Nash equilibrium in the game. First, we find expressions of $u_1(t)$ from state equation (3.1) as a function of x_1 and x_2, and $u_2(t)$ from state equation (3.2) as a function of x_1 and x_2:

$$u_1(t)=x_1(t+1)-\left(1-\frac{a_1}{2}\right)x_1(t)-\frac{a_1}{2}x_2(t).$$

The goal of the first player is to minimize

$$J_1\left(x_1,x_2,u_2\right)=\sum_{t=0}^{T-1}\delta^t\left(c_1\left(x_1(t+1)-\left(1-\frac{a_1}{2}\right)x_1(t)-\frac{a_1}{2}x_2(t)\right)^2\right)$$
$$+\,\delta^T\left(\left(x_1(T)-s_1\right)^2+\left(x_2(T)-s_1\right)^2\right),$$

subject to condition (3.2). We form the Lagrange function

$$L_1\left(x_1,x_2,k^1\right)=J_1\left(x_1,x_2,u_2\right)+\sum_{t=0}^{T-1}k_t^1\left(x_2(t+1)-\left(1-\frac{a_2}{2}\right)x_2(t)\right.$$
$$\left.-\frac{a_2}{2}x_1(t)-u_2(t)\right),$$

where $k^1=\left\{k_t^1, t=0,\ldots,T-1\right\}$. Finding the derivatives and solving equations: $\frac{\partial L_1\left(x_1,x_2,k^1\right)}{\partial x_1(t)}=0$ and $\frac{\partial L_1\left(x_1,x_2,k^1\right)}{\partial x_2(t)}=0, t=1,\ldots,T$, we obtain the corresponding systems:

$$\begin{cases}\left(c_1+\delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1\left(t\right)-c_1\left(1-\frac{a_1}{2}\right)x_1\left(t-1\right)\\ \quad-\delta c_1\left(1-\frac{a_1}{2}\right)x_1\left(t+1\right)-\frac{c_1a_1}{2}x_2\left(t-1\right)+\frac{\delta c_1a_1}{2}\left(1-\frac{a_1}{2}\right)x_2\left(t\right)\\ \quad=\frac{a_2}{4}k_t^1\delta^{-(t-1)},\quad t=1,\ldots,T-1,\\ c_1\left(x_1\left(t\right)-\left(1-\frac{a_1}{2}\right)x_1\left(t-1\right)-\frac{a_1}{2}x_2\left(t-1\right)\right)\\ \quad+\delta\left(x_1\left(t\right)-s_1\right)=0, t=T,\end{cases}\tag{3.8}$$

$$\begin{cases}-a_1c_1\left(x_1\left(t+1\right)-\left(1-\frac{a_1}{2}\right)x_1\left(t\right)-\frac{a_1}{2}x_2\left(t\right)\right)\\ \quad=\left(\left(1-\frac{a_2}{2}\right)k_t^1-k_{t-1}^1\right)\delta^{-t}, t=1,\ldots,T-1,\\ x_2(t)-s_1=-\frac{1}{2}k_{t-1}^1\delta^{-t},\quad t=T.\end{cases}\tag{3.9}$$

From the first equation of system (3.8) we obtain

$$k_t^1 = \frac{4}{a_2}\delta^{t-1}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$\left.\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1a_1}{2}x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t)\right],$$
$$t = 1,\ldots,T-1.$$

Denoting $\hat{k}_t^1 = k_t^1\delta^{-t}$ and substituting it to system (3.9), and taking into account the second equation in system (3.8), we obtain the system of equations to define the best response strategy of player 1:

$$\begin{cases} -a_1c_1\left(x_1(t+1) - \left(1-\frac{a_1}{2}\right)x_1(t) - \frac{a_1}{2}x_2(t)\right) \\ \quad = \left(1-\frac{a_2}{2}\right)\hat{k}_t^1 - \frac{\hat{k}_{t-1}^1}{\delta}, \quad t = 2,\ldots,T-1, \\ c_1\left(x_1(t) - \left(1-\frac{a_1}{2}\right)x_1(t-1) - \frac{a_1}{2}x_2(t-1)\right) \\ \quad +\delta\left(x_i(t) - s_i\right) = 0, t = T, \\ x_2(t) = s_1 - \frac{1}{2\delta}\hat{k}_{t-1}^1, \quad t = T, \end{cases}$$

where

$$\hat{k}_t^1 = \frac{4}{a_2\delta}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$\left.-\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1a_1}{2}x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t)\right],$$
$$t = 1,\ldots,T-1.$$

We determine the best response strategy of player 2 in the same way.

Let us fix $u_1(t), t = 0,\ldots,T-1$ and find the best response of player 2. We use the Euler-equation approach to find the Nash equilibrium in the game described. First, we find expressions of $u_2(t)$ from state equation (3.2) as a function of x_1 and x_2 and $u_1(t)$ from state equation (3.1) as a function of x_1 and x_2:

$$u_2(t) = x_2(t+1) - \left(1-\frac{a_2}{2}\right)x_2(t) - \frac{a_2}{2}x_1(t).$$

The goal of the second player is to minimize

$$J_2\left(u_1,x_1,x_2\right)=\sum_{t=0}^{T-1}\delta^t\left(c_2\left(x_2(t+1)-\left(1-\frac{a_2}{2}\right)x_2(t)-\frac{a_2}{2}x_1(t)\right)^2\right)$$
$$+\,\delta^T\left(\left(x_1(T)-s_2\right)^2+\left(x_2(T)-s_2\right)^2\right),$$

subject to condition (3.12). We form the Lagrange function

$$L_2\left(x_1,x_2,k^2\right)=J_2\left(u_1,x_1,x_2\right)+\sum_{t=0}^{T-1}k_t^2\left(x_1(t+1)-\left(1-\frac{a_1}{2}\right)x_1(t)\right.$$
$$\left.-\frac{a_1}{2}x_2(t)-u_1(t)\right),$$

where $k^2=\left\{k_t^2,t=0,\dots,T-1\right\}$. Finding the derivatives and solving equations: $\frac{\partial L_2\left(x_1,x_2,k^2\right)}{\partial x_1(t)}=0$ and $\frac{\partial L_2\left(x_1,x_2,k^2\right)}{\partial x_2(t)}=0,t=1,\dots,T$, we obtain the corresponding systems:

$$\begin{cases}-a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right)\\ \quad=\left(\left(1-\frac{a_1}{2}\right)k_t^2-k_{t-1}^2\right)\delta^{-t},t=1,\dots,T-1,\\ x_1(t)-s_2=-\frac{1}{2}k_{t-1}^2\delta^{-t},\quad t=T,\end{cases}\tag{3.10}$$

$$\begin{cases}\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\\ \quad-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)\\ \quad=\frac{a_1}{4}k_t^2\delta^{-(t-1)},\quad t=1,\dots,T-1,\\ c_2\left(x_2\left(t\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)-\frac{a_2}{2}x_2\left(t-1\right)\right)\\ \quad+\delta\left(x_2\left(t\right)-s_2\right)=0,t=T.\end{cases}\tag{3.11}$$

From the first equation of system (3.11) we obtain

$$k_t^2=\frac{4}{a_1}\delta^{t-1}\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right.$$
$$\left.-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)\right],$$
$$t=1,\dots,T-1.$$

Denoting $\hat{k}_t^1=k_t^1\delta^{-t}$ and substituting it to system (3.10) and taking into account the second equation in system (3.11), we obtain the system of equations to define the best response strategy of player 2:

$$\begin{cases} -a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right) \\ =\left(1-\frac{a_1}{2}\right)\hat{k}_t^2-\frac{\hat{k}_{t-1}^2}{\delta}, \quad t=2,\ldots,T-1, \\ c_2\left(x_2\left(t\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)-\frac{a_2}{2}x_1\left(t-1\right)\right)+\delta\left(x_2\left(t\right)-s_2\right)=0, t=T \\ x_1(t)=s_2-\frac{1}{2\delta}\hat{k}_{t-1}^2, \quad t=T, \end{cases}$$

where

$$\hat{k}_t^2=\frac{4}{a_1\delta}\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right.$$
$$\left.-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)\right],$$
$$t=1,\ldots,T-1.$$

The theorem is proved. □

3.2 The case when players have the same set of validation periods

Based on the above model, we can consider the model with 2 players involved in the same situation. We assume that there are two players and they directly influences agent 1 and agent 2, respectively. The players have the same discount factors but they are different in their costs per unit of influence intense (c_1 and c_2) and target opinion levels (s_1 and s_2). The state equations for opinions of agents 1 and 2 are

$$x_1(t+1)=x_1(t)+a_1\left(\frac{x_1(t)+x_2(t)}{2}-x_1(t)\right)+u_1(t), \tag{3.12}$$

$$x_2(t+1)=x_2(t)+a_2\left(\frac{x_1(t)+x_2(t)}{2}-x_2(t)\right)+u_2(t) \tag{3.13}$$

with initial condition

$$x_1\left(0\right)=x_1^0, x_2\left(0\right)=x_2^0. \tag{3.14}$$

In comparison with the previous section, players 1 and 2 can choose the validation periods of agents' opinions and these sets are the same for both players. The players are willing to minimize the functionals:

$$J_1(u_1,u_2)=\sum_{t=0}^{T-1}\delta^t\left(c_1u_1^2(t)\right)+\sum_{j=1}^{k}\delta^{t_j}\Big(\left(x_1(t_j)-s_1\right)^2 + \left(x_2(t_j)-s_1\right)^2\Big)+\delta^T\Big(\left(x_1(T)-s_1\right)^2+\left(x_2(T)-s_1\right)^2\Big), \tag{3.15}$$

$$J_2(u_1,u_2)=\sum_{t=0}^{T-1}\delta^t\left(c_2u_2^2(t)\right)+\sum_{j=1}^{k}\delta^{t_j}\Big(\left(x_1(t_j)-s_2\right)^2 + \left(x_2(t_j)-s_2\right)^2\Big)+\delta^T\Big(\left(x_1(T)-s_2\right)^2+\left(x_2(T)-s_2\right)^2\Big) \tag{3.16}$$

correspondingly, where $c_i > 0$ is player i's costs per unit of influence intensity, $s_i \in \mathbb{R}, i = 1, 2$.

We can define a two-player game in a normal-form representation with the set of players' strategies U_1, U_2, where $U_j = (u_j(t) \in \mathbb{R} : t = 0, \ldots, T-1)$, $j = 1, 2$, players' cost functions J_1, J_2, defined by formulae (3.15) and (3.16) s.t. state equations (3.12) and (3.13) with initial state $x(0) = (x_1(0), x_2(0)) = \left(x_1^0, x_2^0\right)$. The game belongs to the class of linear-quadratic games. We provide the necessary conditions for the Nash equilibrium for the case when the set of validation periods is given by $\{1 \le t_1 < \ldots < t_k \le T-1\}$.

Theorem 3.2. *Let* $\{(u_1, u_2), u_i = (u_i(t) : t = 0, \ldots, T-1), i = 1, 2\}$ *be the Nash equilibrium in the game described above in this section, then the Nash equilibrium is defined as*

$$u_i(t) = x_i(t+1) - \left(1 - \frac{a_i}{2}\right) x_i(t) - \frac{a_i}{2} x_j(t), \tag{3.17}$$

and $\{(x_1(t), x_2(t)) : t = 0, \ldots, T\}$ *be the state trajectory corresponding to this equilibrium with initial condition* $x_1(0) = x_1^0, x_2(0) = x_2^0$*, then they satisfy the* *system:*

$$\begin{cases} -a_i c_i \left(x_i\left(t+1\right)-\left(1-\frac{a_i}{2}\right)x_i\left(t\right)-\frac{a_i}{2}x_j\left(t\right)\right)=\left(1-\frac{a_j}{2}\right)\hat{k}_t^i-\frac{\hat{k}_{t-1}^i}{\delta}, \\ \quad t=2,\ldots,T-1, \\ -a_i c_i \left(x_i\left(t+1\right)-\left(1-\frac{a_i}{2}\right)x_i\left(t\right)-\frac{a_i}{2}x_j\left(t\right)\right)-2\left(x_j\left(t\right)-s_i\right) \\ \quad =\left(1-\frac{a_j}{2}\right)\hat{k}_t^i-\frac{\hat{k}_{t-1}^i}{\delta}, \quad t=t_j, j=2,\ldots,k, \\ c_i\left(x_i\left(t\right)-\left(1-\frac{a_i}{2}\right)x_i\left(t-1\right)-\frac{a_i}{2}x_j\left(t-1\right)\right) \\ \quad +\delta\left(x_i\left(t\right)-s_i\right)=0,\ t=T, \\ x_j(t)=s_i-\frac{1}{2\delta}\hat{k}_{t-1}^i, \quad t=T, \end{cases} \tag{3.18}$$

where $i,j=1,2,i\neq j$ and

$$\begin{aligned} \hat{k}_t^i = {} & \frac{4}{a_j\delta}\left[\left(c_i+\delta c_i\left(1-\frac{a_i}{2}\right)^2\right)x_i\left(t\right)-c_i\left(1-\frac{a_i}{2}\right)x_i\left(t-1\right)\right. \\ & \left.-\delta c_i\left(1-\frac{a_i}{2}\right)x_i\left(t+1\right)-\frac{c_i a_i}{2}x_j\left(t-1\right)+\frac{\delta c_i a_i}{2}\left(1-\frac{a_i}{2}\right)x_j\left(t\right)\right], \\ & \quad t=1,\ldots,T-1, \\ \hat{k}_t^i = {} & \frac{4}{a_j\delta}\left[\left(c_i+\delta c_i\left(1-\frac{a_i}{2}\right)^2\right)x_i\left(t\right)-c_i\left(1-\frac{a_i}{2}\right)x_i\left(t-1\right)\right. \\ & -\delta c_i\left(1-\frac{a_i}{2}\right)x_i\left(t+1\right)-\frac{c_i a_i}{2}x_j\left(t-1\right)+\frac{\delta c_i a_i}{2}\left(1-\frac{a_i}{2}\right)x_j\left(t\right) \\ & \left.+\delta\left(x_i\left(t\right)-s_i\right)\right], \quad t=t_j, j=1,\ldots,k, \end{aligned}$$

taking into account the state equations (3.12) and (3.13) and initial state $(x_1(0),x_2(0))=\left(x_1^0,x_2^0\right)$.

Proof. Let us fix $u_2(t), t=0,\ldots,T-1$ and find the best response of player 1. We use the Euler-equation approach to find the Nash equilibrium in the game described. First, we find expressions of $u_1(t)$ from state equation (3.12) as a function of x_1 and x_2 and $u_2(t)$ from state equation (3.13) as a function of x_1 and x_2:

$$u_1(t)=x_1(t+1)-\left(1-\frac{a_1}{2}\right)x_1(t)-\frac{a_1}{2}x_2(t).$$

The goal of the first player is to minimize

$$J_1(x_1, x_2, u_2) = \sum_{t=0}^{T-1} \delta^t \left(c_1 \left(x_1(t+1) - \left(1 - \frac{a_1}{2}\right) x_1(t) - \frac{a_1}{2} x_2(t) \right)^2 \right)$$
$$+ \sum_{j=1}^{k} \delta^{t_j} \left((x_1(t_j) - s_1)^2 + (x_2(t_j) - s_1)^2 \right)$$
$$+ \delta^T \left((x_1(T) - s_1)^2 + (x_2(T) - s_1)^2 \right),$$

subject to condition (3.13). We form the Lagrange function

$$L_1\left(x_1, x_2, k^1\right) = J_1(x_1, x_2, u_2) + \sum_{t=0}^{T-1} k_t^1 \left(x_2(t+1) - \left(1 - \frac{a_2}{2}\right) x_2(t) \right.$$
$$\left. - \frac{a_2}{2} x_1(t) - u_2(t) \right),$$

where $k^1 = \left\{k_t^1, t = 0, \ldots, T-1\right\}$. Finding the derivatives and solving the equations: $\frac{\partial L_1\left(x_1, x_2, k^1\right)}{\partial x_1(t)} = 0$ and $\frac{\partial L_1\left(x_1, x_2, k^1\right)}{\partial x_2(t)} = 0, t = 1, \ldots, T$, we obtain the corresponding systems:

$$\begin{cases} \left(c_1 + \delta c_1 \left(1 - \frac{a_1}{2}\right)^2\right) x_1(t) - c_1 \left(1 - \frac{a_1}{2}\right) x_1(t-1) \\ \quad -\delta c_1 \left(1 - \frac{a_1}{2}\right) x_1(t+1) - \frac{c_1 a_1}{2} x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1 - \frac{a_1}{2}\right) x_2(t) \\ \quad = \frac{a_2}{4} k_t^1 \delta^{-(t-1)}, \quad t = 1, \ldots, T-1, t \neq t_j, \\ \left(c_1 + \delta c_1 \left(1 - \frac{a_1}{2}\right)^2\right) x_1(t) - c_1 \left(1 - \frac{a_1}{2}\right) x_1(t-1) \\ \quad -\delta c_1 \left(1 - \frac{a_1}{2}\right) x_1(t+1) - \frac{c_1 a_1}{2} x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1 - \frac{a_1}{2}\right) x_2(t) \\ \quad +\delta (x_1(t) - s_1) = \frac{a_2}{4} k_t^1 \delta^{-(t-1)}, t = t_j, j = 1, \ldots, k, \\ c_1 \left(x_1(t) - \left(1 - \frac{a_1}{2}\right) x_1(t-1) - \frac{a_1}{2} x_2(t-1)\right) \\ \quad +\delta (x_1(t) - s_1) = 0, t = T, \end{cases} \tag{3.19}$$

$$\begin{cases} -a_1 c_1 \left(x_1(t+1) - \left(1 - \frac{a_1}{2}\right) x_1(t) - \frac{a_1}{2} x_2(t)\right) \\ \quad = \left(\left(1 - \frac{a_2}{2}\right) k_t^1 - k_{t-1}^1\right) \delta^{-t}, \quad t = 1, \ldots, T-1, \\ -a_1 c_1 \left(x_1(t+1) - \left(1 - \frac{a_1}{2}\right) x_1(t) - \frac{a_1}{2} x_2(t)\right) + 2(x_2(t) - s_1) \\ \quad = \left(\left(1 - \frac{a_2}{2}\right) k_t^1 - k_{t-1}^1\right) \delta^{-t}, \quad t = t_j, j = 1, \ldots, k, \\ x_2(t) - s_1 = -\frac{1}{2} k_{t-1}^1 \delta^{-t}, \quad t = T. \end{cases} \tag{3.20}$$

From the first equation of system (3.19) we obtain

$$k_t^1 = \frac{4}{a_2}\delta^{t-1}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$\left. -\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1 a_1}{2}x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t)\right],$$
$$t = 1,\dots,T-1, t \neq t_j,$$
$$k_t^1 = \frac{4}{a_2}\delta^{t-1}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$-\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1 a_1}{2}x_2(t-1)$$
$$\left. +\frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t) + \delta\left(x_1(t) - s_1\right)\right], \quad t = t_j, j = 1,\dots,k.$$

Denoting $\hat{k}_t^1 = k_t^1\delta^{-t}$ and substituting it to system (3.20) and taking into account the second equation in system (3.19), we obtain the system of equations to define the best response strategy of player 1:

$$\begin{cases} -a_1 c_1\left(x_1(t+1) - \left(1-\frac{a_1}{2}\right)x_1(t) - \frac{a_1}{2}x_2(t)\right) = \left(1-\frac{a_2}{2}\right)\hat{k}_t^1 - \frac{\hat{k}_{t-1}^1}{\delta}, \\ \quad t = 1,\dots,T-1, t \neq t_j, \\ -a_1 c_1\left(x_1(t+1) - \left(1-\frac{a_1}{2}\right)x_1(t) - \frac{a_1}{2}x_2(t)\right) + 2\left(x_2(t) - s_1\right) \\ \quad = \left(1-\frac{a_2}{2}\right)\hat{k}_t^1 - \frac{\hat{k}_{t-1}^1}{\delta}, \quad t = t_j, j = 1,\dots,k, \\ x_2(t) - s_1 = -\frac{1}{2\delta}\hat{k}_{t-1}^1, \quad t = T, \\ c_1\left(x_1(t) - \left(1-\frac{a_1}{2}\right)x_1(t-1) - \frac{a_1}{2}x_2(t-1)\right) + \delta\left(x_1(t) - s_1\right) = 0, t = T, \end{cases}$$

where

$$\hat{k}_t^1 = \frac{4}{a_2\delta}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$\left. -\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1 a_1}{2}x_2(t-1) + \frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t)\right],$$
$$t = 1,\dots,T-1, t \neq t_j,$$

$$\hat{k}_t^1 = \frac{4}{a_2\delta}\left[\left(c_1 + \delta c_1\left(1-\frac{a_1}{2}\right)^2\right)x_1(t) - c_1\left(1-\frac{a_1}{2}\right)x_1(t-1)\right.$$
$$-\delta c_1\left(1-\frac{a_1}{2}\right)x_1(t+1) - \frac{c_1 a_1}{2}x_2(t-1)$$
$$\left.+\frac{\delta c_1 a_1}{2}\left(1-\frac{a_1}{2}\right)x_2(t) + \delta\left(x_1(t)-s_1\right)\right], \quad t = t_j, j = 1,\ldots,k.$$

We determine the best response strategy of player 2 in the same way.

Let us fix $u_1(t), t = 0,\ldots,T-1$ and find the best response of player 2. We use the Euler-equation approach to find the Nash equilibrium in the game described. First, we find expressions of $u_2(t)$ from state equation (3.13) as a function of x_1 and x_2 and $u_1(t)$ from state equation (3.12) as a function of x_1 and x_2:

$$u_2(t) = x_2(t+1) - \left(1-\frac{a_2}{2}\right)x_2(t) - \frac{a_2}{2}x_1(t).$$

The goal of the second player is to minimize

$$J_2(u_1, x_1, x_2) = \sum_{t=0}^{T-1}\delta^t\left(c_2\left(x_2(t+1) - \left(1-\frac{a_2}{2}\right)x_2(t) - \frac{a_2}{2}x_1(t)\right)^2\right)$$
$$+\sum_{j=1}^{k}\delta^{t_j}\left(\left(x_1(t_j)-s_2\right)^2 + \left(x_2(t_j)-s_2\right)^2\right)$$
$$+\delta^T\left(\left(x_1(T)-s_2\right)^2 + \left(x_2(T)-s_2\right)^2\right),$$

subject to condition (3.12). We form the Lagrange function

$$L_2\left(x_1, x_2, k^2\right) = J_2(u_1, x_1, x_2) + \sum_{t=0}^{T-1}k_t^2\left(x_1(t+1) - \left(1-\frac{a_1}{2}\right)x_1(t)\right.$$
$$\left.-\frac{a_1}{2}x_2(t) - u_1(t)\right),$$

where $k^2 = \left\{k_t^2, t = 0,\ldots,T-1\right\}$. Finding the derivatives and solving the equations: $\frac{\partial L_2\left(x_1,x_2,k^2\right)}{\partial x_1(t)} = 0$ and $\frac{\partial L_2\left(x_1,x_2,k^2\right)}{\partial x_2(t)} = 0, t = 1,\ldots,T$, we obtain the corresponding systems:

$$\begin{cases} -a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right) \\ \quad =\left(\left(1-\frac{a_1}{2}\right)k_t^2-k_{t-1}^2\right)\delta^{-t}, t=1,\dots,T-1, t\neq t_j, \\ -a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right)+2\left(x_1\left(t\right)-s_2\right) \\ \quad =\left(\left(1-\frac{a_1}{2}\right)k_t^2-k_{t-1}^2\right)\delta^{-t}, \quad t=t_j, j=1,\dots,k, \\ x_2(t)-s_2=-\frac{1}{2}k_{t-1}^2\delta^{-t}, \quad t=T, \end{cases} \tag{3.21}$$

$$\begin{cases} \left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right) \\ \quad -\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right) \\ \quad =\frac{a_1}{4}k_t^2\delta^{-(t-1)}, \quad t=1,\dots,T-1, t\neq t_j, \\ \left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right) \\ \quad -\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right) \\ \quad +\delta\left(x_2\left(t\right)-s_2\right)=\frac{a_1}{4}k_t^2\delta^{-(t-1)}, \quad t=t_j, j=1,\dots,k, \\ c_2\left(x_2\left(t\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)-\frac{a_2}{2}x_2\left(t-1\right)\right) \\ \quad +\delta\left(x_1\left(t\right)-s_2\right)=0, \quad t=T. \end{cases} \tag{3.22}$$

From the first equation of system (3.22) we obtain

$$\begin{aligned} k_t^2=\frac{4}{a_1}\delta^{t-1}&\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right. \\ &\left.-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)\right], \\ &t=1,\dots,T-1, t\neq t_j, \end{aligned}$$

$$\begin{aligned} k_t^2=\frac{4}{a_1}\delta^{t-1}&\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right. \\ &-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right) \\ &\left.+\delta\left(x_2\left(t\right)-s_2\right)\right], \quad t=t_j, j=1,\dots,k. \end{aligned}$$

Denoting $\hat{k}_t^1 = k_t^1\delta^{-t}$ and substituting it to system (3.21) and taking into account the second equation in system (3.22), we obtain the system of equations to define the best response strategy of player 2:

$$\begin{cases} -a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right)=\left(1-\frac{a_1}{2}\right)\hat{k}_t^2-\frac{\hat{k}_{t-1}^2}{\delta}, \\ \quad t=2,\ldots,T-1, t\neq t_j, \\ -a_2c_2\left(x_2\left(t+1\right)-\left(1-\frac{a_2}{2}\right)x_2\left(t\right)-\frac{a_2}{2}x_1\left(t\right)\right)+2\left(x_1\left(t\right)-s_2\right) \\ \quad =\left(1-\frac{a_1}{2}\right)\hat{k}_t^2-\frac{\hat{k}_{t-1}^2}{\delta}, \quad t=t_j, j=2,\ldots,k, \\ x_2(t)-s_2=-\frac{1}{2\delta}\hat{k}_{t-1}^2, \quad t=T, \end{cases}$$

where

$$\hat{k}_t^2=\frac{4}{a_1\delta}\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right.$$
$$\left.-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)\right],$$
$$t=1,\ldots,T-1, t\neq t_j,$$

$$\hat{k}_t^2=\frac{4}{a_1\delta}\left[\left(c_2+\delta c_2\left(1-\frac{a_2}{2}\right)^2\right)x_2\left(t\right)-c_2\left(1-\frac{a_2}{2}\right)x_2\left(t-1\right)\right.$$
$$-\delta c_2\left(1-\frac{a_2}{2}\right)x_2\left(t+1\right)-\frac{c_2a_2}{2}x_1\left(t-1\right)+\frac{\delta c_2a_2}{2}\left(1-\frac{a_2}{2}\right)x_1\left(t\right)$$
$$\left.+\delta\left(x_2\left(t\right)-s_2\right)\right], \quad t=t_j, j=1,\ldots,k.$$

The theorem is proved. □

3.3 The case when two players choose different sets of the periods to influence agents' opinions

In this section, we propose the following model: in a small social network, the opinions of agents are represented by $x_i(t)$ at time t, where i is the number of an agent. Suppose there are two players who directly influence opinions of agents 1 and 2, respectively, and the level of influence is denoted by $u_j(t)$, j is the number of a player. The sets V_1, V_2, where $V_j=\left\{t_1^j,\ldots,t_k^j\right\}$, $j=1,2$, are the sets of periods, in which players control the opinions of agents, and the number of elements k in set V_j is given. We assume that k is the same for both players, but sets V_1 and

V_2 may be different. Define a two-player game of competition for agents' opinions with the set of players' strategies U_1, U_2, where $U_j = (u_j(t) \in \mathbb{R} \mid t \in V_j), j = 1, 2$. The players have the same discount factor, but their levels of influence per unit cost and target opinions are different. Summarize the notations:

- $x_i(t), i = 1, 2$: the opinion of agent i at time $t \in \{0, 1, \ldots, T\}$;
- $u_j\left(t\right),\ j = 1, 2$: player 1 influences agent 1's opinion with $u_1(t),\ t \in V_1$, player 2 influences agent 2's opinion with $u_2(t)$ at time $t \in V_2$;
- $V_j = \left\{t_1^j, \ldots, t_k^j \mid 0 \le t_1^j < t_2^j < \cdots < t_k^j \le T - 1\right\},\ j = 1, 2$: the set of periods when player j controls the corresponding agent's opinion;
- $U_j = (u_j(t) \in R \mid t \in V_j), j = 1, 2$: players' strategy sets of control variables.

The small social network we examine is represented in Figure 3.1.

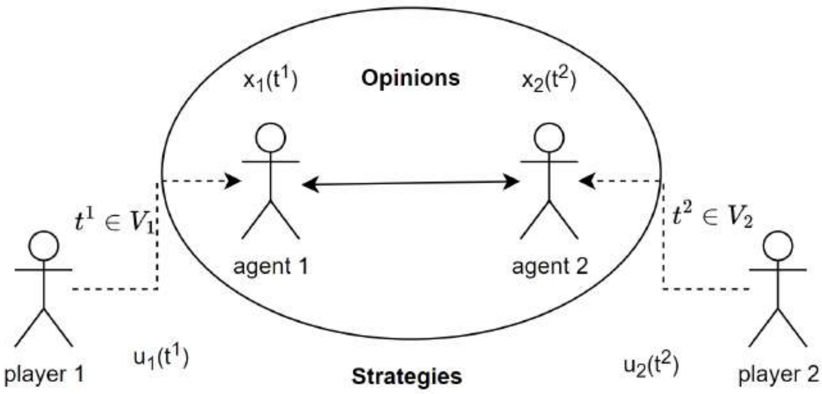

Figure 3.1: Small social network.

The dynamics of agents' opinions are defined by the following equations:

$$x_1\left(t+1\right) = x_1\left(t\right) + a_1\left(\frac{x_1\left(t\right) + x_2\left(t\right)}{2} - x_1\left(t\right)\right) + u_1\left(t\right),\ \ t \in V_1, \tag{3.23}$$

$$x_1\left(t+1\right) = x_1\left(t\right) + a_1\left(\frac{x_1\left(t\right) + x_2\left(t\right)}{2} - x_1\left(t\right)\right),\ \ t \notin V_1, \tag{3.24}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right) + u_2(t)\,,\ t \in V_2, \qquad (3.25)$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t)+x_2(t)}{2} - x_2(t)\right),\ t \notin V_2, \qquad (3.26)$$

with initial condition

$$x_1(0) = x_1^0,\ x_2(0) = x_2^0.$$

In equations (3.23)–(3.26), $a_1 > 0$, $a_2 > 0$ denote agent 1 and 2's beliefs about the average social opinion, respectively.

The players' target opinions are s_1 and $s_2 \in \mathbb{R}$. Players 1 and 2 are willing to minimize functionals:

$$J_1(u_1, u_2) = \sum_{t_i \in V_1} \delta^{t_i}\left(c_1 u_1^2(t_i)\right) + \sum_{t=0}^{T} \delta^t\left((x_1(t) - s_1)^2 + (x_2(t) - s_1)^2\right),$$

$$J_2(u_1, u_2) = \sum_{t_i \in V_2} \delta^{t_i}\left(c_2 u_2^2(t_i)\right) + \sum_{t=0}^{T} \delta^t\left((x_1(t) - s_2)^2 + (x_2(t) - s_2)^2\right),$$

where $\delta \in (0,1]$ is a discount factor and $c_j > 0$ is player j's cost per unit level of influence.

Theorem 3.3. *Let* $\{(u_1^*, u_2^*), u_j = (u_j(t) : t \in V_j), j = 1, 2\}$ *be the Nash equilibrium in the game described above in this section and* $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ *be a state trajectory corresponding to this equilibrium with initial condition* $x_1(0) =$

$x_1^0, x_2(0) = x_2^0$, *then they satisfy the system:*

$$\begin{cases} u_1(t) = \frac{\delta}{2c_1}\lambda_1^1\left(t+1\right), \ t \in V_1, \\ \lambda_1^1\left(t+1\right) = \frac{2}{\delta(2-a_1)}\left[2\left(x_1\left(t\right)-s_1\right) - \lambda_1^1\left(t\right) - \lambda_1^2\left(t+1\right)\frac{a_2\delta}{2}\right], \ t = 0, \ldots, T-1, \\ \lambda_1^2\left(t+1\right) = \frac{2}{\delta(2-a_2)}\left[2\left(x_2\left(t\right)-s_1\right) - \lambda_1^1\left(t+1\right)\frac{a_1\delta}{2} - \lambda_1^2\left(t\right)\right], \ t = 0, \ldots, T-1, \\ \lambda_1^1\left(T\right) = 2\left(x_1\left(T\right)-s_1\right), \\ \lambda_1^2\left(T\right) = 2\left(x_2\left(T\right)-s_1\right), \\ u_2(t) = \frac{\delta}{2c_2}\lambda_2^2\left(t+1\right), \ t \in V_2, \\ \lambda_2^1\left(t+1\right) = \frac{2}{\delta(2-a_1)}\left[2\left(x_1\left(t\right)-s_2\right) - \lambda_2^1\left(t\right) - \lambda_2^2\left(t+1\right)\frac{a_2\delta}{2}\right], \ t = 0, \ldots, T-1, \\ \lambda_2^2\left(t+1\right) = \frac{2}{\delta(2-a_2)}\left[2\left(x_2\left(t\right)-s_2\right) - \lambda_2^1\left(t+1\right)\frac{a_1\delta}{2} - \lambda_2^2\left(t\right)\right], \ t = 0, \ldots, T-1, \\ \lambda_2^1\left(T\right) = 2\left(x_1\left(T\right)-s_2\right), \\ \lambda_2^2\left(T\right) = 2\left(x_2\left(T\right)-s_2\right), \end{cases}$$

taking into account state equations (3.23)–(3.26) *and initial state* $(x_1(0), \ x_2(0)) = \left(x_1^0, \ x_2^0\right)$.

Proof. We find the strategy profile $(u_1^*, \ u_2^*)$, which is the Nash equilibrium in the game described above. We find the equilibrium in open-loop strategies using the Pontryagin maximum principle. The Hamiltonian of player 1 is

$$\begin{aligned} &H_1^1\left(x_1\left(t\right), x_2\left(t\right), \lambda_1^1\left(t+1\right), \lambda_1^2\left(t+1\right), u_1\left(t\right), u_2\left(t\right), t\right) \\ &\quad = c_1 u_1^2(t) + \left(x_1\left(t\right)-s_1\right)^2 + \left(x_2\left(t\right)-s_1\right)^2 \\ &\quad + \delta\lambda_1^1\left(t+1\right)\left(x_1\left(t+1\right) - x_1\left(t\right) - a_1\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_1\left(t\right)\right) - u_1\left(t\right)\right) \\ &\quad + \delta\lambda_1^2\left(t+1\right)\left(x_2\left(t+1\right) - x_2\left(t\right) - a_2\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_2\left(t\right)\right) - u_2\left(t\right)\right), \end{aligned}$$

for any $t \in V_1$, and it takes the form

$$H_1^2 \left(x_1 (t), x_2 (t), \lambda_1^1 (t+1), \lambda_1^2 (t+1), t\right)$$
$$= (x_1 (t) - s_1)^2 + (x_2 (t) - s_1)^2$$
$$+ \delta \lambda_1^1 (t+1) \left(x_1 (t+1) - x_1 (t) - a_1 \left(\frac{x_1 (t) + x_2 (t)}{2} - x_1 (t) \right) \right)$$
$$+ \delta \lambda_1^2 (t+1) \left(x_2 (t+1) - x_2 (t) - a_2 \left(\frac{x_1 (t) + x_2 (t)}{2} - x_2 (t) \right) \right),$$

for any $t \notin V_1$.

Finding the derivatives $\frac{\partial H_1^1(t)}{\partial u_1(t)} = 0,\ t \in V_1$, $\lambda_1^1 (t) = \frac{\partial H_1^1(t)}{\partial x_1(t)} = \frac{\partial H_1^2(t)}{\partial x_1(t)}$ and $\lambda_1^2 (t) = \frac{\partial H_1^1(t)}{\partial x_2(t)} = \frac{\partial H_1^2(t)}{\partial x_2(t)},\ t = 1, \ldots, T-1$, we obtain the system of equations:

$$\frac{\partial H_1^1 (t)}{\partial u_1 (t)} = 2 c_1 u_1(t) - \delta \lambda_1^1 (t+1) = 0,\ t \in V_1,$$
$$\lambda_1^1 (t) = \frac{\partial H_1^1 (t)}{\partial x_1 (t)} = \frac{\partial H_1^2 (t)}{\partial x_1 (t)}$$
$$= 2 (x_1 (t) - s_1) - \delta \lambda_1^1 (t+1) \left(1 - \frac{a_1}{2}\right) - \delta \lambda_1^2 (t+1) \frac{a_2}{2},$$
$$t = 1, \ldots, T-1,$$
$$\lambda_1^2 (t) = \frac{\partial H_1^1 (t)}{\partial x_2 (t)} = \frac{\partial H_1^2 (t)}{\partial x_2 (t)}$$
$$= 2 (x_2 (t) - s_1) - \delta \lambda_1^1 (t+1) \frac{a_1}{2} - \delta \lambda_1^2 (t+1) \left(1 - \frac{a_2}{2}\right),$$
$$t = 1, \ldots, T-1,$$

$$\lambda_1^1 (T) = \frac{\partial \left((x_1 (T) - s_1)^2 + (x_2 (T) - s_1)^2 \right)}{\partial x_1 (T)} = 2 (x_1 (T) - s_1),$$
$$\lambda_1^2 (T) = \frac{\partial \left((x_1 (T) - s_1)^2 + (x_2 (T) - s_1)^2 \right)}{\partial x_2 (T)} = 2 (x_2 (T) - s_1).$$

It can be rewritten as:

$$\begin{cases} u_1(t) = \frac{\delta}{2c_1}\lambda_1^1\left(t+1\right), \ t \in V_1, \\ \lambda_1^1\left(t+1\right) = \frac{2}{\delta(2-a_1)}\left[2\left(x_1\left(t\right)-s_1\right) - \lambda_1^1\left(t\right) - \lambda_1^2\left(t+1\right)\frac{a_2\delta}{2}\right], \\ \quad t = 0, \ldots, T-1, \\ \lambda_1^2\left(t+1\right) = \frac{2}{\delta(2-a_2)}\left[2\left(x_2\left(t\right)-s_1\right) - \lambda_1^1\left(t+1\right)\frac{a_1\delta}{2} - \lambda_1^2\left(t\right)\right], \\ \quad t = 0, \ldots, T-1, \\ \lambda_1^1\left(T\right) = 2\left(x_1\left(T\right)-s_1\right), \\ \lambda_1^2\left(T\right) = 2\left(x_2\left(T\right)-s_1\right). \end{cases} \tag{3.27}$$

From the last four equations of system (1.5) we obtain expressions of $\lambda_1^1(t)$ and $\lambda_1^2(t)$ as functions of x_1 and x_2, $t = 0, \ldots, T$. We substitute these expressions of $\lambda_1^1(t)$ and $\lambda_1^2(t)$ into the first equation of system (1.5) if t belongs to V_1. We get an expression of u_1 as a function of x_1 and x_2. Substituting the new expression of u_1 into equation (3.23), we get new state equation $x_1(t+1)$ as a function of x_1 and x_2.

Then, we write the Hamiltonian of player 2 as

$$\begin{aligned} &H_2^1\left(x_1\left(t\right), x_2\left(t\right), \lambda_2^1\left(t+1\right), \lambda_2^2\left(t+1\right), u_1\left(t\right), u_2\left(t\right), t\right) \\ &\quad = c_2u_2^2(t) + \left(x_1\left(t\right)-s_2\right)^2 + \left(x_2\left(t\right)-s_2\right)^2 \\ &\quad + \delta\lambda_2^1\left(t+1\right)\left(x_1\left(t+1\right) - x_1\left(t\right) - a_1\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_1\left(t\right)\right) - u_1\left(t\right)\right) \\ &\quad + \delta\lambda_2^2\left(t+1\right)\left(x_2\left(t+1\right) - x_2\left(t\right) - a_2\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_2\left(t\right)\right) - u_2\left(t\right)\right), \end{aligned}$$

for any $t \in V_2$, and

$$\begin{aligned} &H_2^2\left(x_1\left(t\right), x_2\left(t\right), \lambda_2^1\left(t+1\right), \lambda_2^2\left(t+1\right), t\right) \\ &\quad = \left(x_1\left(t\right)-s_2\right)^2 + \left(x_2\left(t\right)-s_2\right)^2 \\ &\quad + \delta\lambda_2^1\left(t+1\right)\left(x_1\left(t+1\right) - x_1\left(t\right) - a_1\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_1\left(t\right)\right)\right) \\ &\quad + \delta\lambda_2^2\left(t+1\right)\left(x_2\left(t+1\right) - x_2\left(t\right) - a_2\left(\frac{x_1\left(t\right)+x_2\left(t\right)}{2} - x_2\left(t\right)\right)\right), \end{aligned}$$

for any $t \notin V_2$.

Finding the derivatives $\frac{\partial H_2^1(t)}{\partial u_2(t)} = 0,\ t \in V_2$, $\lambda_2^1(t) = \frac{\partial H_2^1(t)}{\partial x_1(t)} = \frac{\partial H_2^2(t)}{\partial x_1(t)}$ and $\lambda_2^2(t) = \frac{\partial H_2^1(t)}{\partial x_2(t)} = \frac{\partial H_2^2(t)}{\partial x_2(t)},\ t = 1, \ldots, T-1$, we obtain the system:

$$\frac{\partial H_2^1(t)}{\partial u_2(t)} = 2c_2 u_2(t) - \delta\lambda_2^2(t+1) = 0,\ t \in V_2,$$

$$\lambda_2^1(t) = \frac{\partial H_2^1(t)}{\partial x_1(t)} = \frac{\partial H_2^2(t)}{\partial x_1(t)}$$

$$= 2(x_1(t) - s_2) - \delta\lambda_2^1(t+1)\left(1 - \frac{a_1}{2}\right) - \delta\lambda_2^2(t+1)\frac{a_2}{2},$$

$$t = 1, \ldots, T-1,$$

$$\lambda_2^2(t) = \frac{\partial H_2^1(t)}{\partial x_2(t)} = \frac{\partial H_2^2(t)}{\partial x_2(t)}$$

$$= 2(x_2(t) - s_2) - \delta\lambda_2^1(t+1)\frac{a_1}{2} - \delta\lambda_2^2(t+1)\left(1 - \frac{a_2}{2}\right),$$

$$t = 1, \ldots, T-1,$$

$$\lambda_2^1(T) = \frac{\partial\left((x_1(T) - s_2)^2 + (x_2(T) - s_2)^2\right)}{\partial x_1(T)} = 2(x_1(T) - s_2),$$

$$\lambda_2^2(T) = \frac{\partial\left((x_1(T) - s_2)^2 + (x_2(T) - s_2)^2\right)}{\partial x_2(T)} = 2(x_2(T) - s_2).$$

Finally, we rewrite the system as follows:

$$\begin{cases} u_2(t) = \frac{\delta}{2c_2}\lambda_2^2(t+1),\ t \in V_2, \\ \lambda_2^1(t+1) = \frac{2}{\delta(2-a_1)}\left[2(x_1(t) - s_2) - \lambda_2^1(t) - \lambda_2^2(t+1)\frac{a_2\delta}{2}\right], \\ \quad t = 0, \ldots, T-1, \\ \lambda_2^2(t+1) = \frac{2}{\delta(2-a_2)}\left[2(x_2(t) - s_2) - \lambda_2^1(t+1)\frac{a_1\delta}{2} - \lambda_2^2(t)\right], \\ \quad t = 0, \ldots, T-1, \\ \lambda_2^1(T) = 2(x_1(T) - s_2), \\ \lambda_2^2(T) = 2(x_2(T) - s_2). \end{cases} \tag{3.28}$$

We use the same idea as above to find new state equation $x_2(t+1)$ as a function of x_1 and x_2. Taking into account the state equation (3.24) and (3.26), we can find the equilibrium state trajectories of agent 1 and agent 2 according to the initial condition $x_1(0) = x_1^0,\ x_2(0) = x_2^0$. The equilibrium strategy trajectories of players 1 and 2 are also found. Joining two systems (3.27) and (3.28) we finish the proof. □

Remark 3.1. *In Theorem 3.3, the Nash equilibrium is found under an assumption that the sets of periods V_1 and V_2, when players 1 and 2 choose their controls, are given. These sets may be different for the players. If we consider the problem of choosing these sets from the optimization perspective, then we need to find all possible combinations of periods for a given number k, and find the Nash equilibrium for any such a pair of sets V_1 and V_2. Moreover, some pair of sets may be preferable (in terms of minimizing the costs) for one player, and another pair may be preferable for another player. Therefore, we could find Pareto optimal sets V_1 and V_2 such that no other pair of sets can give at least the same costs and strictly smaller costs for at least one player. We demonstrate how we find such Pareto-optimal sets V_1 and V_2 for numerical examples provided in Section 3.4.2.*

3.4 Numerical simulations

3.4.1 Numerical examples for Section 3.1

Example 3.1. *Let $a_1 = 0.3, a_2 = 0.6, \delta = 1, c_1 = 8, c_2 = 7$ and initial opinions be $x_1(0) = 0.7, x_2(0) = 0.6$. For time horizon $T = 10$ and target opinion $s_1 = 0.3, s_2 = 0.1$, the equilibrium state and control trajectories are presented in Table 3.1. The equilibrium costs to players 1 and 2 are 0.125 and 1.305, respectively.*

Table 3.1: Equilibrium state and control trajectories

t	0	1	2	3	4	5
$x_1(t)$	0.7	0.7129	0.6337	0.5663	0.5087	0.4591
$x_2(t)$	0.6	0.6372	0.4532	0.3245	0.2343	0.1712
$u_1(t)$	0.0279	-0.0678	-0.0403	-0.0213	-0.0084	0.0002
$u_2(t)$	0.0072	-0.2067	-0.1829	-0.1627	-0.1454	-0.1305
t	6	7	8	9	10	
$x_1(t)$	0.4161	0.3777	0.3419	0.3057	0.2643	
$x_2(t)$	0.1271	0.0962	0.0745	0.0594	0.0489	
$u_1(t)$	0.0050	0.0064	0.0040	-0.0045		
$u_2(t)$	-0.1176	-0.1062	-0.0953	-0.0844		

We also introduce equilibrium state and strategy trajectories on Figures 3.2 and 3.3.

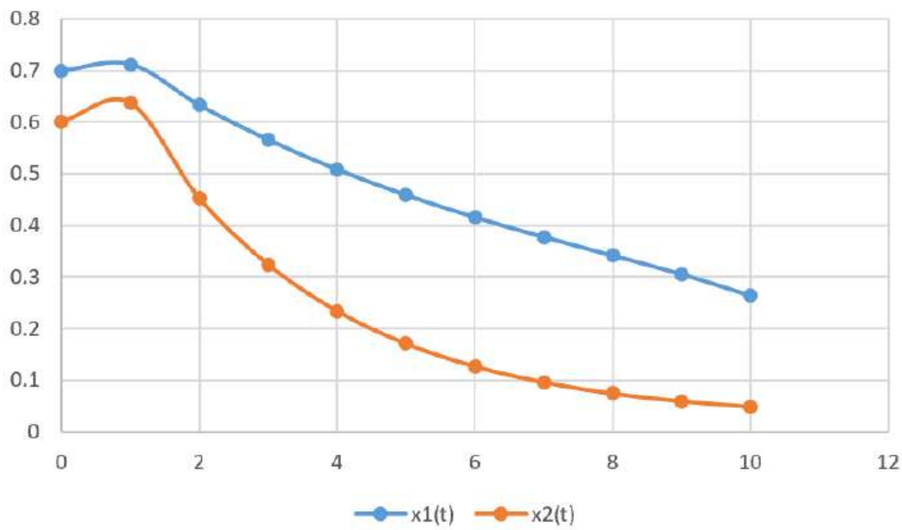

Figure 3.2: State trajectories (blue - $x_1(t)$, red - $x_2(t)$).

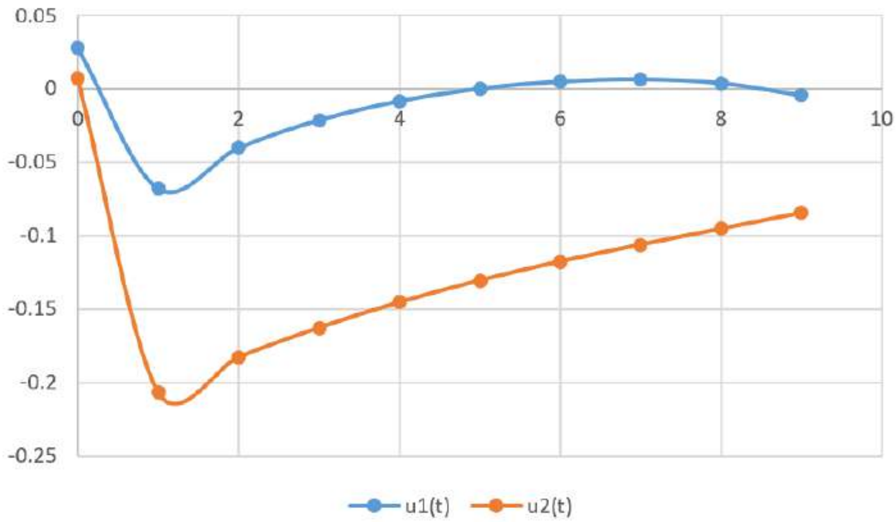

Figure 3.3: Strategy trajectory (blue - $u_1(t)$, red - $u_2(t)$).

Example 3.2. *Let* $a_1 = 0.1, a_2 = 0.2, \delta = 1, c_1 = 9, c_2 = 5$ *and initial opinions be* $x_1(0) = 0.4, x_2(0) = 0.5$. *For time horizon* $T = 10$ *and target opinion* $s_1 = 0.8, s_2 = 0.9$, *the equilibrium state and control trajectories are presented in Table 3.2. The equilibrium costs of players 1 and 2 are* 0.134 *and* 0.203, *respectively.*

Table 3.2: Equilibrium state and control trajectories

t	0	1	2	3	4	5
$x_1(t)$	0.4	0.4665	0.5025	0.5343	0.5615	0.5838
$x_2(t)$	0.5	0.5638	0.5803	0.5941	0.6053	0.6138
$u_1(t)$	0.0615	0.0311	0.0279	0.0242	0.0201	0.0157
$u_2(t)$	0.0738	0.0262	0.0216	0.0172	0.0129	0.0090
t	6	7	8	9	10	
$x_1(t)$	0.601	0.6125	0.6178	0.6162	0.6069	
$x_2(t)$	0.6198	0.6232	0.6238	0.6215	0.6161	
$u_1(t)$	0.0106	0.0048	-0.0019	-0.0096		
$u_2(t)$	0.0053	0.0017	-0.0017	-0.0049		

We also introduce the equilibrium state and control trajectories on Figures 3.4 *and 3.5.*

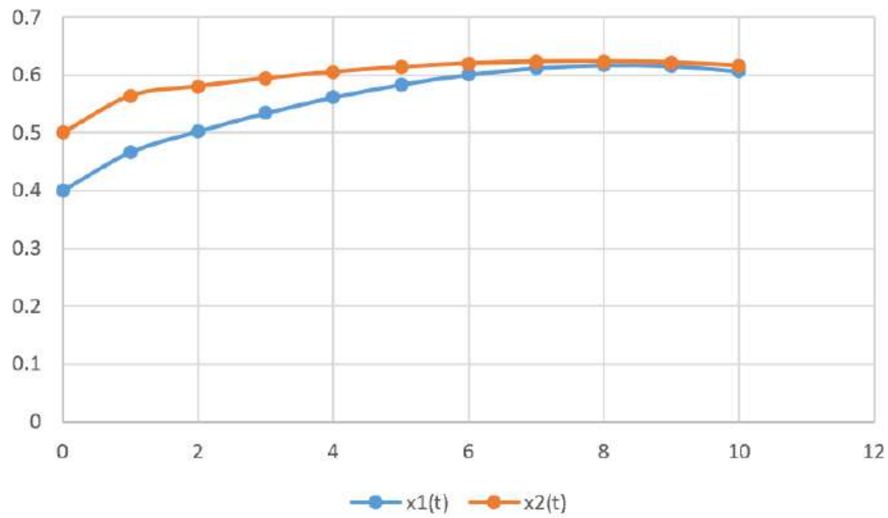

Figure 3.4: State trajectories (blue - $x_1(t)$, red - $x_2(t)$).

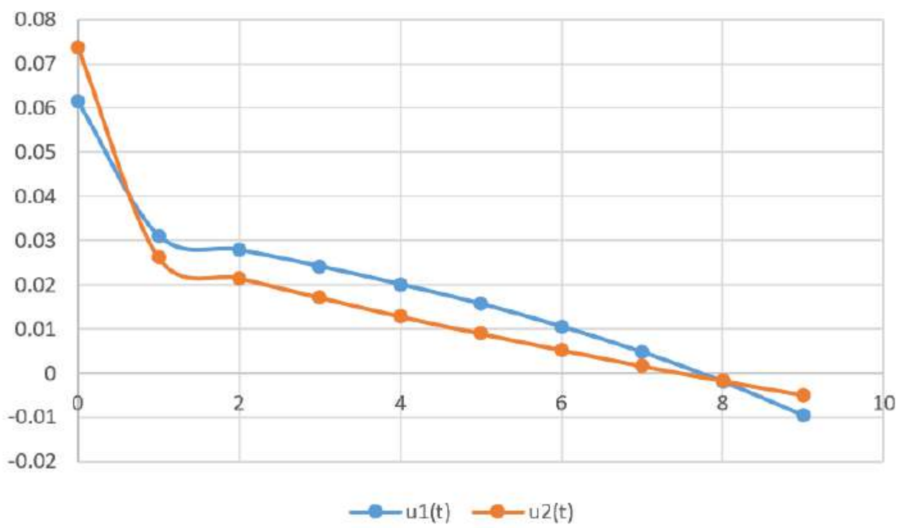

Figure 3.5: Strategy trajectory (blue - $u_1(t)$, red - $u_2(t)$).

Example 3.3. *Let* $a_1 = 0.2, a_2 = 0.1, \delta = 1, c_1 = 8, c_2 = 4$ *and initial opinions be* $x_1(0) = 0.8, x_2(0) = 0.2$. *For time horizon* $T = 10$ *and target opinions* $s_1 = 0.3, s_2 = 0.8$, *the equilibrium state and control trajectories are presented in Table 3.3. The equilibrium costs to players 1 and 2 are* 0.350 *and* 0.641, *respectively.*

Table 3.3: Equilibrium state and control trajectories

t	0	1	2	3	4	5
$x_1(t)$	0.8	0.7489	0.6824	0.6218	0.5666	0.516
$x_2(t)$	0.2	0.3459	0.4392	0.5218	0.593	0.6522
$u_1(t)$	0.0089	-0.0262	-0.03628	-0.0452	-0.05324	-0.06022
$u_2(t)$	0.1159	0.07315	0.07044	0.0662	0.06052	0.05311
t	6	7	8	9	10	
$x_1(t)$	0.4694	0.4262	0.3858	0.3478	0.3116	
$x_2(t)$	0.6985	0.7307	0.7475	0.7473	0.7279	
$u_1(t)$	-0.06611	-0.07085	-0.07417	-0.07615		
$u_2(t)$	0.043655	0.032025	0.017885	0.000575		

The equilibrium state and strategy trajectories are presented on Figures 3.6 and 3.7.

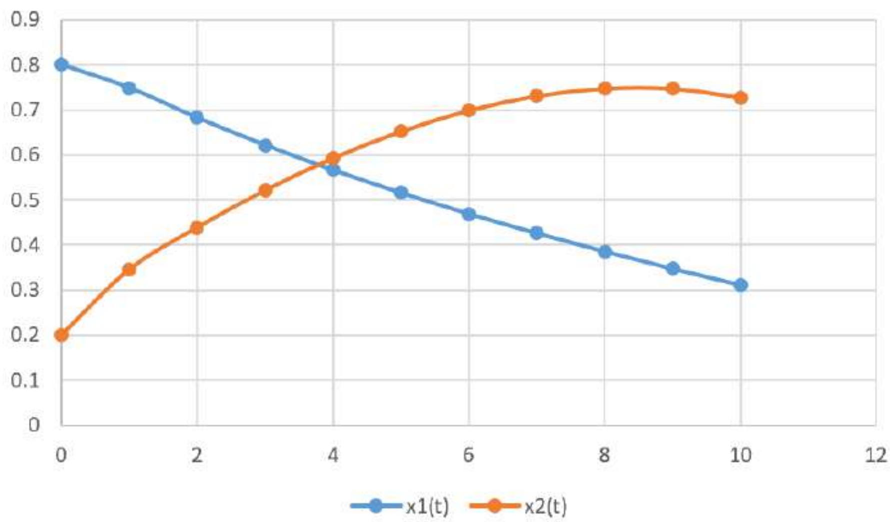

Figure 3.6: State trajectories ((blue - $x_1(t)$, red - $x_2(t)$).

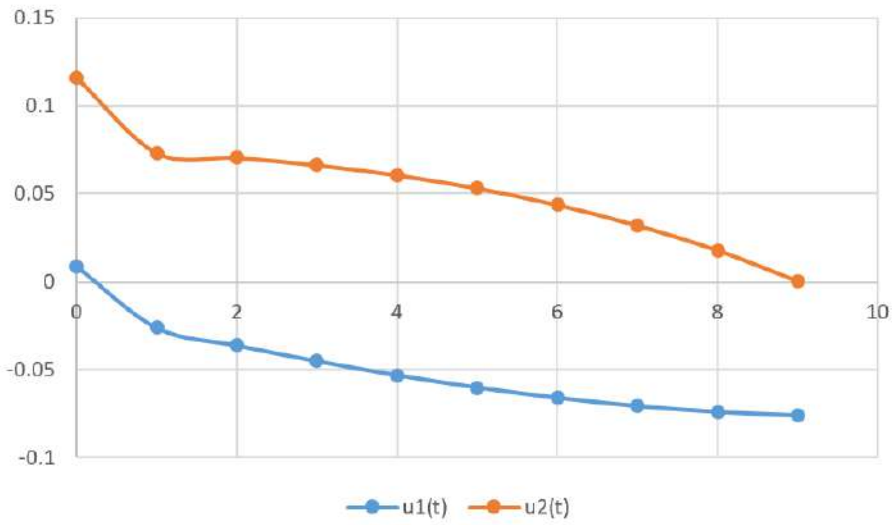

Figure 3.7: Strategy trajectory (blue - $u_1(t)$, red - $u_2(t)$).

3.4.2 Numerical examples for Section 3.2

Example 3.4. *We consider the following example. Let the parameters of the game be as follows:* $a_1 = 0.8, a_2 = 0.7, \delta = 1, c_1 = 0.7, c_2 = 0.9$, *and initial opinions be* $x_1(0) = 0.7, x_2(0) = 0.9$. *The time horizon is* $T = 10$ *and target opinions are* $s_1 = 0.3, s_2 = 0.2$. *The equilibrium costs of Players 1 and 2 are 1.3684 and 1.1657, respectively.*

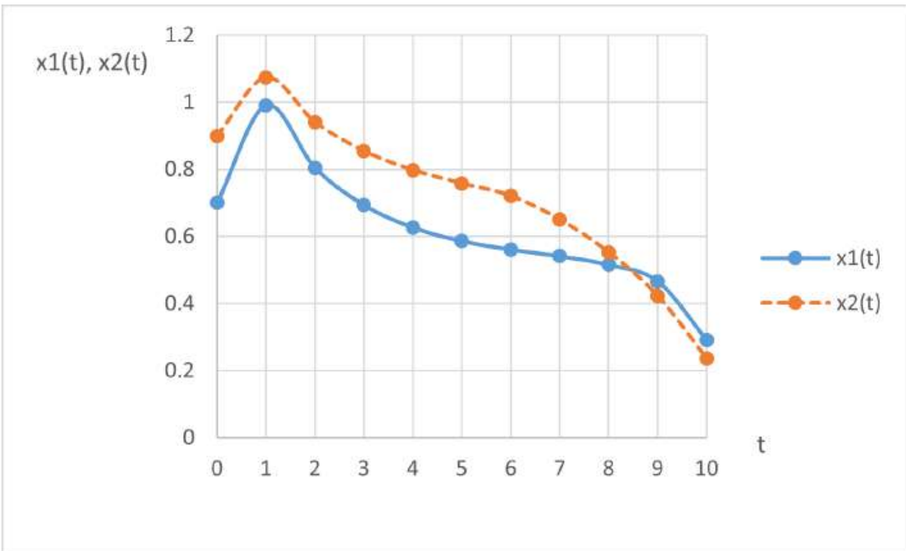

Figure 3.8: Equilibrium state trajectories, validation periods 7, 8, 9 (solid — $x_1(t)$, dashed— $x_2(t)$).

Table 3.4: Nash equilibrium state trajectories and controls

t	0	1	2	3	4	5
$x_1(t)$	0.7000	0.9906	0.805	0.6936	0.6266	0.5862
$x_2(t)$	0.9000	1.0733	0.9409	0.8547	0.7979	0.7581
$u_1(t)$	-0.7800	-0.4293	-0.3764	-0.3419	-0.3192	-0.3033
$u_2(t)$	0.2533	0.2969	0.2901	0.2851	0.2794	0.2658
t	6	7	8	9	10	
$x_1(t)$	0.5608	0.5409	0.5151	0.4662	0.2917	
$x_2(t)$	0.7207	0.6501	0.5520	0.4222	0.2353	
$u_1(t)$	-0.2883	-0.2600	-0.2208	-0.1689		
$u_2(t)$	0.2177	0.1619	0.0910	-0.0181		

The equilibrium state and strategy trajectories are presented in Fig. 3.8 and Fig. 3.9.

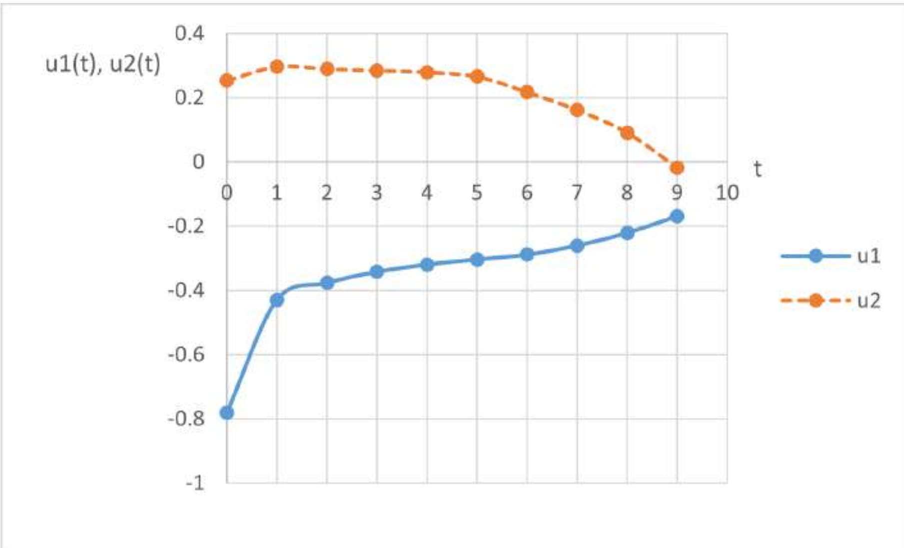

Figure 3.9: Equilibrium strategy trajectories (solid — $u_1(t)$, dashed — $u_2(t)$).

3.4.3 Numerical examples for Section 3.3

Example 3.5. *Let the time horizon be* $T = 9$ *(periods* $0, \ldots, 9$*), and* $k = 2$ *be the number of periods in which players influence agents. The parameters are as* *follows:*

$$a_1 = 0.7,\ a_2 = 0.5,\ \delta = 1,\ c_1 = 0.4,\ c_2 = 0.6,\ s_1 = 0.6,\ s_2 = 0.6,$$

$$x_1(0) = 0.9,\ x_2(0) = 0.1.$$

The initial opinion state of two agents is $x(0) = (0.9,\ 0.1)$, *i.e.* $x_1(0) = 0.9$, $x_2(0) = 0.1$. *The agent 1 and 2' beliefs about the average social opinion are* $a_1 = 0.7$, $a_2 = 0.5$, *respectively. The discount factor is* $\delta = 1$. *The unit costs of influence are* $c_1 = 0.4$, $c_2 = 0.6$ *for player 1 and player 2, respectively. Their target opinions are* $s_1 = 0.6$, $s_2 = 0.6$. *We find the Nash equilibrium for any possible sets* V_1 *and* V_2 *consisting of two periods of influence. We obtain that for the sets* $V_1 = \{2,\ 5\}$ *and* $V_2 = \{0,\ 2\}$, *both players have the lowest costs in the Nash equilibrium in comparison with all other Nash equilibria. So, this pair of sets* V_1 *and* V_2 *is Pareto optimal. We characterize this equilibrium describing equilibrium strategies and state trajectories (see Table 3.5). The equilibrium costs of players 1 and 2 are* 0.3845 *and* 0.3963, *respectively.*

Table 3.5: Nash equilibrium strategies and state trajectories, $V_1 = \{2,\ 5\}$ and $V_2 = \{0,\ 2\}$

t	$t_2^1 = 0$	1	$t_1^1 = t_2^2 = 2$	3	4
$x_1(t)$	0.9000	0.6200	0.5577	0.7001	0.6074
$x_2(t)$	0.1000	0.4420	0.3943	0.4352	0.5014
$u_1(t)$			0.1117		
$u_2(t)$	0.1515		0.0745		
t	$t_1^2 = 5$	6	7	8	9
$x_1(t)$	0.5703	0.5410	0.5401	0.5398	0.5396
$x_2(t)$	0.5279	0.5385	0.5391	0.5394	0.5395
$u_1(t)$	-0.0242				
$u_2(t)$					

Agents' opinions are becoming closer to target opinions over time. The equilibrium state and strategy trajectories are shown in Figures 3.10 and 3.11, respectively.

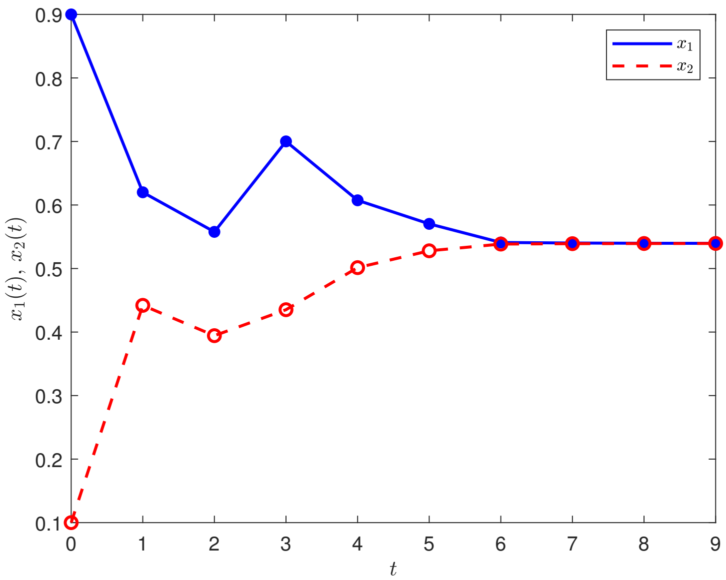

Figure 3.10: Equilibrium state trajectories (solid — $x_1(t)$, dotted — $x_2(t)$).

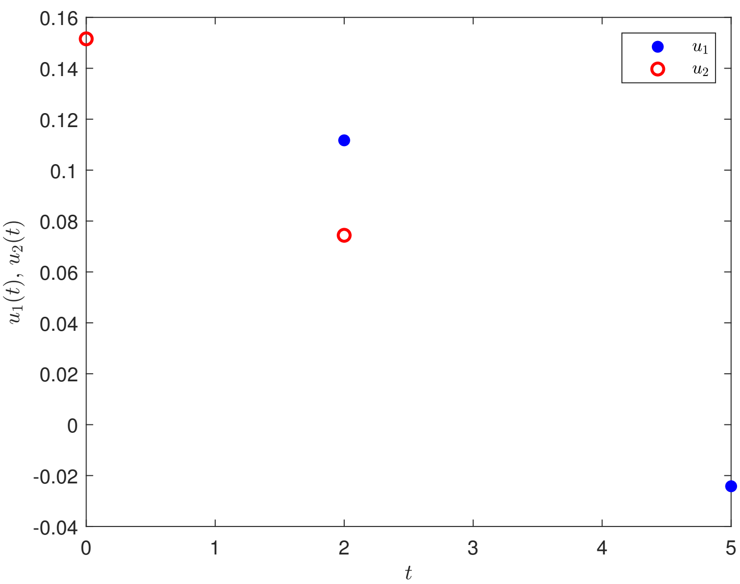

Figure 3.11: Equilibrium strategy trajectories (solid — $u_1(t)$, dotted — $u_2(t)$).

In this scenario, we see that among all Nash equilibria when we vary V_1 *and* V_2*, the minimal costs for both players emerge at the Nash equilibrium with* $V_1 =$

$\{2,\ 5\}$ *and* $V_2 = \{0,\ 2\}$.

Remark 3.2. *We calculate the set of all possible periods* V_1 *and* V_2 *by* C_T^k *combinations for each player. When* $T = 9$*, and players choose two periods to influence the agents' opinions, then* $C_9^2 = 36$*. Considering that the two players may have different choices, the number of all possible combinations is* $36 \cdot 36 = 1296$*. To find the Nash equilibrium, we solved* 1296 *systems given in Theorem 3.3 to find Pareto-optimal equilibrium costs.*

Example 3.6. *Let the time horizon be* $T = 8$ *(periods* $0, \ldots, 8$*), and* $k = 3$ *be the number of periods in which players influence agents. The parameters are as follows:*

$$a_1 = 0.1,\ a_2 = 0.7,\ \delta = 1,\ c_1 = 0.4,\ c_2 = 0.6,\ s_1 = 0.2,\ s_2 = 0.1,$$

$$x_1(0) = 0.7,\ x_2(0) = 0.9.$$

The initial opinion state of two agents is $x(0) = (0.7, 0.9)$*, i.e.* $x_1(0) = 0.7,\ x_2(0) = 0.9$*. The agent 1 and agent 2' beliefs about the average social opinion are* $a_1 = 0.1$*,* $a_2 = 0.7$*, respectively. The discount factor is* $\delta = 1$*. The unit costs of influence are* $c_1 = 0.4,\ c_2 = 0.6$ *for player 1 and player 2, respectively. Their target opinions are* $s_1 = 0.2,\ s_2 = 0.1$*. We consider all possible sets* V_1 *and* V_2 *consisting of three periods when players can influence agents' opinions. Remind that these sets may be different. In this example, we find that different Nash equilibria, i.e. different combinations of sets* V_1 *and* V_2 *give the lowest costs to different players. Therefore, the set of Pareto-optimal pairs of sets* V_1 *and* V_2 *consists of two elements. We describe both equilibria when (i)* $V_1 = \{0,\ 5,\ 7\}$ *and* $V_2 = \{0,\ 4,\ 5\}$ *and (ii)* $V_1 = \{0,\ 1,\ 2\}$ *and* $V_2 = \{0,\ 2,\ 7\}$*. Player 1 prefers case (i) and player 2 prefers case (ii). First, we characterize the Nash equilibrium for* $V_1 = \{0,\ 5,\ 7\}$ *and* $V_2 = \{0,\ 4,\ 5\}$ *(see Table 3.6). The equilibrium costs of players 1 and 2 are* 0.9491 *and* 1.4566*, respectively.*

Table 3.6: Nash equilibrium strategies and state trajectories, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$

t	$t_1^1 = t_2^1 = 0$	1	2	3	$t_2^2 = 4$
$x_1(t)$	0.7000	0.1584	0.1698	0.1766	0.1807
$x_2(t)$	0.9000	0.3855	0.3060	0.2583	0.2297
$u_1(t)$	-0.5516				
$u_2(t)$	-0.4445				-0.1421
t	$t_1^2 = t_2^3 = 5$	6	$t_1^3 = 7$	8	
$x_1(t)$	0.1831	0.1747	0.1812	0.2100	
$x_2(t)$	0.0704	0.3047	0.2592	0.2319	
$u_1(t)$	-0.0029		0.0249		
$u_2(t)$	0.1948				

Table 3.7: Nash equilibrium strategies and state trajectories, $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$

t	$t_1^1 = t_2^1 = 0$	$t_1^2 = 1$	$t_1^3 = t_2^2 = 2$	3	4
$x_1(t)$	0.7000	0.0130	0.2241	0.1555	0.1520
$x_2(t)$	0.9000	0.2885	0.1921	0.0842	0.1092
$u_1(t)$	-0.6970	0.1973	-0.0670		
$u_2(t)$	-0.5415		-0.1191		
t	5	6	$t_2^3 = 7$	8	
$x_1(t)$	0.1498	0.1486	0.1478	0.1473	
$x_2(t)$	0.1241	0.1331	0.1385	0.2206	
$u_1(t)$					
$u_2(t)$			0.0789		

Agents' opinions are becoming closer to target opinions over time (see Figure 3.12). The equilibrium state trajectories and strategy trajectories are shown in Figure 3.12 and Figure 3.14, respectively.

Second, we characterize the Nash equilibrium for the case when $V_1 = \{0,\ 1,\ 2\}$

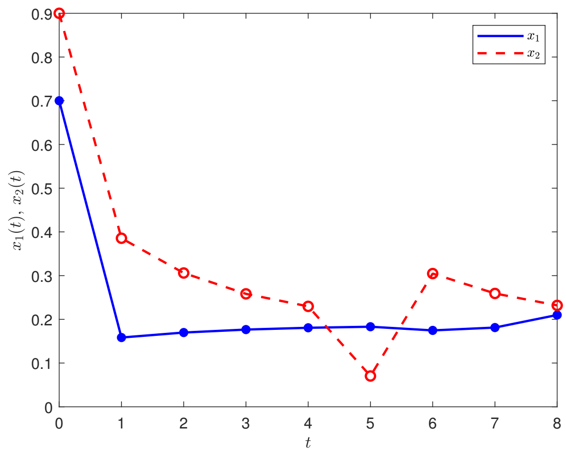

Figure 3.12: Equilibrium state trajectories, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ (solid — $x_1(t)$, dotted — $x_2(t)$)

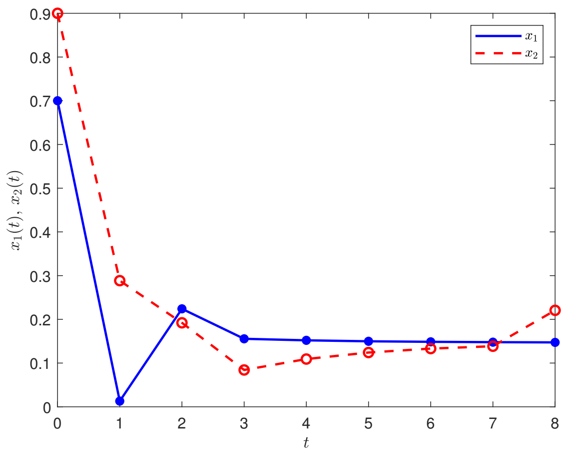

Figure 3.13: Equilibrium state trajectories, $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$ (solid — $x_1(t)$, dotted — $x_2(t)$)

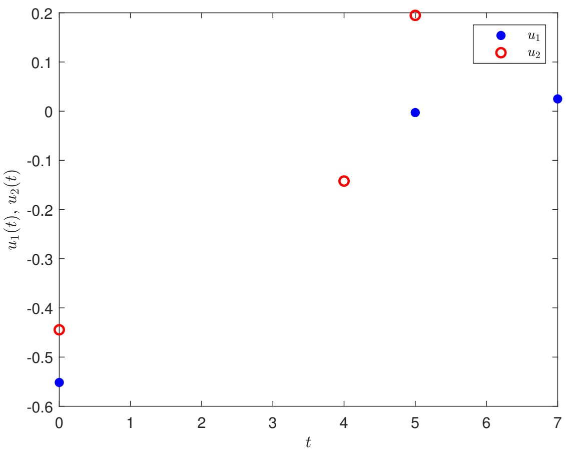

Figure 3.14: Equilibrium strategies, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ (solid — $u_1(t)$, dotted — $u_2(t)$)

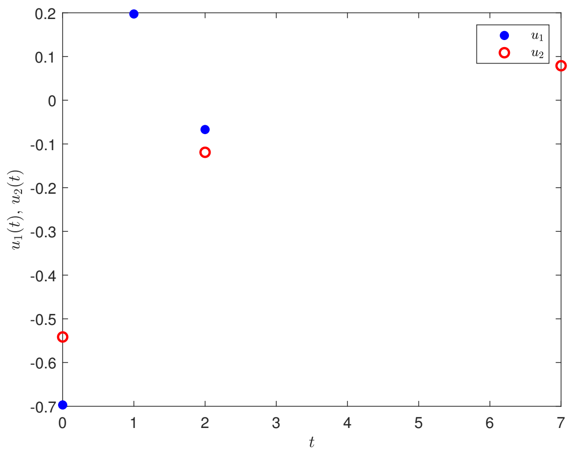

Figure 3.15: Equilibrium strategies, $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$ (solid — $u_1(t)$, dotted — $u_2(t)$)

and $V_2 = \{0,\ 2,\ 7\}$, *which is preferable for player 2. The equilibrium costs of player 1 and 2 are* 1.0462 *and* 1.2884, *respectively. The values of equilibrium state and strategy trajectories are given in Table 3.7, and they are represented in Figure 3.13 and Figure 3.15.*

In order to examine the two Nash equilibria and differences in players' costs, we conducted a comparative analysis in Table 3.8.

Table 3.8: Comparison of the two Nash equilibria

Players	Time set, equil. 1	Costs	Time set, equil. 2	Costs	Index
Player 1	$\{0,5,7\}$	0.9491*	$\{0,1,2\}$	1.0462	10.23%
Player 2	$\{0,4,5\}$	1.4566	$\{0,2,7\}$	1.2884*	13.05%

In Table 3.8, the costs with an asterisk are minimal for the corresponding player. To estimate the difference between the two Nash equilibria we use the following index for player 1:

$$\frac{J_1(equil_2) - J_1^*(equil_1)}{J_1^*\,(equil_1)} \cdot 100\% = 10.23\%,$$

where by $equil_1$ and $equil_2$ we mean the Nash equilibria with time sets $V_1 = \{0,\ 5,\ 7\}$, $V_2 = \{0,\ 4,\ 5\}$, and $V_1 = \{0,\ 1,\ 2\}$, $V_2 = \{0,\ 2,\ 7\}$, respectively. Although this value is not very large, it implies that in the second Nash equilibrium, the cost loss borne by player 1 is not too high compared to the first Nash equilibrium. Similarly, for player 2, the index comparing two Nash equilibria is

$$\frac{J_2(equil_1) - J_2^*(equil_2)}{J_2^*(equil_2)} \cdot 100\% = 13.05\%.$$

We could notice that player 2 does not bear much costs in the first Nash equilibrium relative to the second one.

To sum up, Theorem 3.3 provides the necessary conditions for the Nash equilibrium for a competition game when the periods of controls are fixed. As we showed above, some equilibrium may be preferable for one player while not for another player. In this case, if players have an option to choose sets V_1 and V_2 when they control the agents, there may be a conflict of interests between the players. We do not discuss how one of the Nash equilibria can be chosen, but it can be modeled as a bargaining process.

3.5 Conclusion to Chapter 3

We propose three models of opinion dynamics with two centers of influence (players). Two cases are considered: first, players validate agents' opinions only at the last time; second, players are willing to find the optimal validation periods by choosing a fixed number of periods k. In an arbitrary period, the player takes into account the difference between an agent's opinion and the socially desired opinion. We present the opinions of all members of the society as an average as a summary of the opinions of network members. We find the Nash equilibrium in the case of two centers, and the equilibrium set of periods that players choose. The Euler equation approach is used to find the Nash equilibrium.

The last model of opinion dynamics where agents' opinions are influenced by the players is such that the players are willing to minimize their costs represented by the sum of squared distances of the agents' opinions from the desired opinion and quadratic functions of controls. The main feature of the model is that the players can influence agents' opinions in a limited number of periods. We find the Nash equilibrium in the game where the number of such periods is given and it is the same for both players. But the sets of periods may be different. We also find the Pareto-optimal sets of periods in our numerical simulations.

Conclusions

The thesis focuses on modeling opinion dynamics in small social networks with different types of constraints. Combining opinion dynamics, noncooperative games, and optimization problems, it discusses how the opinions of social network members change over time under some outside influence which is also constrained. The study is divided into three main parts: studying effects of different modes of choosing significant periods to validate the agents' opinions in a network to reach the target opinion (Chapter 1); providing different modes to choose periods to control when players have a limit on the number of such periods to influence agents (Chapter 2); providing the necessary conditions for the Nash equilibrium in a competition on agents' opinions when players minimize their costs in a linear-quadratic form (Chapter 3).

The main results of the work are the following:

1. In Chapter 1, the models of opinion dynamics with different scenarios of restrictions on players' or influencers' behavior in small social networks are presented. The restrictions are as follows: (i) the player assumes the agent's opinion in the last time is significant and we include the term with the agents' opinions in the terminal time in her functional, no other opinions are considered in the functional, and (ii) the player can choose the set of periods to include into the set of significant ones and take them into her objective functional. For all models of Chapter 1, the optimization problems are formulated

for a one-player case. The necessary conditions for the optimal strategies are found. A series of numerical simulations is conducted to test the results and to make conclusions about parameter influence on optimal strategies. From our numerical experiments, we find that the increasing number of validation periods reduces the player's costs, and that if the validation period is optimal for the given number of such periods, then it will be also optimal if the number of periods is increased (see [30, 31, 33]).

2. In Chapter 2, the models of opinion dynamics with different scenarios of restrictions on players' or influencers' control strategy in small social networks are presented. The restrictions are as follows: (i) the player can control the agents or validate their opinions, but these two actions cannot be done at the same time, (ii) the player can choose the periods when she controls opinions and simultaneously validates the opinions, and the size of this set is limited, and (ii) the player can choose the set of periods to control the opinions but they should be different from the time when she is validating opinions of agents. For all these models, the optimization problems are formulated and necessary conditions for the optimal strategies are proved. It is worth mentioning that in the numerical simulations the control sets are different for the three cases. The numerical simulation shows that when the player has a possibility to validate the agent's opinion while controlling it, she prefers to do it at the end of the time horizon. In the first and third cases, she first controls the agent's opinions at the beginning of the planning time horizon, then only validates the opinions, and finally controls the agent at the end of time horizon (see [32, 34]).

3. Chapter 3 considers the models of opinion dynamics in small networks with two players or influencers and examines their competition on social opinion

under constraints on the number of validation periods or periods to control them. Three main scenarios are considered in this chapter: (i) players validate the agents' opinions only in the terminal time, and (ii) players can choose the set of validation periods with the limitation on the size of this set, and this set is the same for both players, or (iii) this set may be different for the players. I find the necessary conditions for the Nash equilibria for all scenarios when players' goals are to minimize the linear-quadratic costs. I determine the Pareto optimal sets of periods for validation and periods to control through numerical simulations (see [29, 35]).

I conclude that all the tasks formulated in this thesis are achieved, and the objectives are fully accomplished.

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Appendix

Research methods

The Euler-Equation Approach

Let $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be the state space and the control set, respectively. Given an initial state $x_0 \in X$, the state of a system evaluates with respect to dynamics:

$$x(t+1) = f_t(x(t), u(t)), t = 0, 1, \ldots, T-1. \tag{3.29}$$

The optimal control problem is to find a control $u(t) \in U$ maximizing functional:

$$\sum_{t=0}^{T} \delta^t r_t(x(t), u(t)) \tag{3.30}$$

with respect to the state dynamics equations (3.29) and a given initial condition $x(0) = x_0$, where $r_t(x(t), u(t))$ is a reward or cost function of a player.

We can reformulate this problem in terms of the state trajectory $x(t)$. Suppose that we can express $u(t)$ from equation (3.29) as a function of $x(t)$ and $x(t+1)$, say $u(t) = q(x(t), x(t+1))$. Therefore, we can rewrite functional (3.30) in the following form:

$$\sum_{t=0}^{T} \delta^t g_t(x(t), x(t+1)), \tag{3.31}$$

where $g_t(x(t), x(t+1)) = r_t(x(t), q(x(t), x(t+1)))$, $t = 0, 1, \ldots, T-1$. The Euler equation approach gives the necessary conditions (see [17, 37, 38]) for the optimal trajectory $x^*(t)$:

$$\frac{\partial g_{t-1}\left(x^*(t-1), x^*(t)\right)}{\partial y} + \delta \frac{\partial g_t\left(x^*(t), x^*(t+1)\right)}{\partial x} = 0, \quad t = 1, \ldots, T-1. \quad (3.32)$$

We can notice that games considered in the thesis belong to the class of linear-quadratic games. We will apply the Euler-equation method to find the optimal strategies in the dynamic games with average-oriented opinion dynamics (see [63]).

The Pontryagin maximum principle

This section is written following [43].

Definition 3.1. *The multistage two-player game is defined, for the finite-horizon case, by the following utility functions (or performance criterions) and state equations:*

$$J_j = \sum_{t=0}^{T-1} g_j\left(\mathbf{x}(t), \mathbf{u}_1(t), \mathbf{u}_2(t), t\right) + S_j(\mathbf{x}(T)), \ \textit{for } j = 1, 2, \qquad (3.33)$$

$$\mathbf{u}_j(t) \in U_j, \qquad (3.34)$$

$$\mathbf{x}(t+1) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}_1(t), \mathbf{u}_2(t), t\right), t = 0, 1, \ldots T-1, \qquad (3.35)$$

$$\mathbf{x}(0) = \mathbf{x}^0. \qquad (3.36)$$

If the game is played in open loop, each player, having observed the initial state x_0 chooses an admissible control sequence $\tilde{\mathbf{u}}_j^T = \left(\mathbf{u}_j(0), \ldots, \mathbf{u}_j(T-1)\right), j = 1, 2$. This generates, from the initial position $\left(0, \mathbf{x}^0\right)$, a state trajectory $\tilde{\mathbf{x}}^T$ solution.

We can utilize an optimal-control technique based on the maximum principle to characterize open-loop Nash-equilibrium solutions. For that purpose, we define the Hamiltonian for each Player j by

$$H_j\left(\mathbf{p}_j(t+1), \mathbf{x}(t), \mathbf{u}_1(t), \mathbf{u}_2(t), t\right) \equiv$$
$$g_j\left(\mathbf{x}(t+1), \mathbf{u}_1(t), \mathbf{u}_2(t), t\right) + \mathbf{p}_j(t+1)'\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}_1(t), \mathbf{u}_2(t), t\right),$$

where $\mathbf{p}_j(t)$ is a costate vector in $\mathbb{R}^n$ and indicates the transposition of the vector $p_j(t+1)$ in a scalar product.

Assumption 3.1. *Assume that* $\mathbf{f}(\mathbf{x}, \underline{\boldsymbol{u}}, t)$ *and* $g_j(\mathbf{x}, \underline{\mathbf{u}}, t)$ *are continuously differentiable in state* $\mathbf{x}$ *and continuous in controls* $\underline{\mathbf{u}}$ *for each* $t = 0, \ldots, T-1$ *and* $S_j(\mathbf{x})$ *is continuously differentiable in* $\mathbf{x}$*. Assume that, for each* j, U_j *is compact and convex. Assume also that, for each* $t, \mathbf{x}$*, the function* $H_j\left(\mathbf{p}, \mathbf{x}, \mathbf{u}_j, \mathbf{u}_{-j}, t\right)$ *is concave in* $\mathbf{u}_j$*.*

We can then formulate the following lemma that provides the necessary conditions that the open-loop equilibrium strategies need to satisfy:

Lemma 3.1. *Under assumption 3.1, if* $\underline{\tilde{\mathrm{u}}}^*$*is an open-loop Nash equilibrium pair of controls, generating the trajectory* $\tilde{\mathrm{x}}^*$ *from initial state* x^0 *for the game (3.33), (3.35), then there exist functions of time* $\mathbf{p}_j(\cdot)$*, with values in* $\mathbb{R}^n$*, such that the following relations hold*

$$\begin{aligned}
\mathbf{u}_j^*(t) &= \arg \max_{u_j(t) \in U_j} H_j\left(\mathbf{p}_j(t+1), \mathbf{x}^*(t), \mathbf{u}_j(t), \mathbf{u}_{-j}^*(t), t\right), \\
\mathbf{p}_j(t)' &= \frac{\partial}{\partial \mathbf{x}} H_j\left(\mathbf{p}_j(t+1), \mathbf{x}^*(t), \mathbf{u}_1^*(t), \mathbf{u}_2^*(t), t\right), \\
\mathbf{p}_j(T)' &= \frac{\partial}{\partial \mathbf{x}(T)} S_j\left(\mathbf{x}^*(T)\right), \; j = 1, 2.
\end{aligned}$$

Proof. See [43]. □