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На правах рукописи

**Су Шимай**

# ТЕОРЕТИКО-ИГРОВЫЕ МЕТОДЫ АНАЛИЗА УСТОЙЧИВОСТИ В ЗАДАЧАХ УПРАВЛЕНИЯ ЗАГРЯЗНЕНИЕМ ОКРУЖАЮЩЕЙ СРЕДЫ

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# GAME-THEORETIC METHODS FOR STABILITY ANALYSIS IN ENVIRONMENTAL POLLUTION CONTROL PROBLEMS

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## DISSERTATION

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# Contents

# Introduction

## Relevance of thesis topic

In general, stability analysis widely applied in mathematics, engineering, economics, environmental science, medicine, etc., is a comprehensive procedure to assess the stability of a complex system or process. The reason why stability analysis is so important is that the stability provides observable and dependable evidences for researchers to track down the possible trajectories of complex systems. Or conversely, the researchers are capable to detect the threshold such that crossing it means that the system enters into instability region or even fail. In this way, the equilibrium in game theory can be interpreted as a stable situation, in which the development of the game can be predicted accordingly. Furthermore, the ultimate purpose of stability analysis is to determine the conditions of system stability or design a well-suited mechanism under which the system remains stable. Similarly, turning to game theory, the players' strategies can be finally obtained in correspondence with the equilibrium conditions and this finishes the analysis.

Whereas it comes to the implementation of stability analysis, the story is different. The internal and external factors and elements are significant aspects that we need to take into consideration. On the one hand, the objective function and structure symbolizing the internal characteristics of the system have a salient impact on the way how the analysis should be conducted, whether from the prospective of linear stability analysis or nonlinear stability analysis. For instance, if the stable coalition structure in multi-agent system or more specifically, international environment agreements (IEAs) is our target, the analysis is nonlinear. Meanwhile, for the simulation of a system through linear regression model, it turns out to be linear. On the other hand, the external elements contain uncertain information or any other factors which could indirectly affect the system externally. In differential games, the players take their actions based on the current information pattern and once the information has been changed, players need to alter their strategies correspondingly.

Therefore, this thesis intends to concentrate on the investigation of two aspects in stability analysis. The focus of this thesis is on stability analysis in environmental problems, caused, for example, by greenhouse gas (GHG) emissions, which are very important and challenging in examination. It is much more meaningful to apply the stability analysis into pollution control games when players are willing to deviate from tough restrictions prescribed to them by agreements.

We observe how the climate change substantially caused by greenhouse gas (GHG) emissions is threatening a lot of creatures in a much more complex way than ever before. Natural disasters such as drought, flood, forest fire, glacial melting and something similar consistently remind us of the incredible challenges we are facing. At present, the European Union (EU) is taking more active steps by using emission trading system (ETS) launched by 2005 to continuously reduce emissions and achieve climate neutrality in 2050. The "net-zero emissions" goal by 2050 is written in the road map of the United States, Canada, Australia, South Korea, and Japan.

While the developed world is making its contribution to solving this problem, targeted solutions in the developing world are still in progress. The developing countries such as India and Morocco are undoubtedly leading the way in improving renewable energy among all the nations [112]. China, one of the most representative developing countries, has made a great achievement in green energy transition, and its own ETS has been operating to systematically limit and reduce $CO_2$ since last year [59, 109]. However, developing countries are not generally capable of rapidly implementing some efficient measures to cope with the climate change without sacrificing their economic growth in comparison with developed countries. Moreover, for the next decade, with further elimination of poverty and improvement of living standards in developing countries, they will make up the bulk of increase in energy consumption, i.e., higher level of emissions. The sustainable way to solve this problem is based on stable cooperation.

The thesis is devoted to studying the stability in pollution control problems involving one or many players on the basis of internal and external factors in terms of their objective functions, structure, mechanism implementation, and uncertain information. In addition, it is expected that the stability mechanism concluded in this thesis can also be extended and applied in other fields to significantly improve the performance and robustness of the original system.

**Overview of the results in this area**

Starting with the internal factors, the fact is that the United Nations (UN) working as a significant coordinator is urging all the nations to take concrete actions to reduce gas emissions. The Kyoto Protocol and Paris Agreement are the international commitments of the specific actions for the countries in the way to solve the problem. But an announcement of the US in 2017 to withdraw from the Paris Agreement of 2015 motivates us to carefully think about maintaining the stability of international environmental agreements (IEAs) and to make an effort to prevent other failures [110]. The idea of taking into account countries' reputations in IEA modeling, when participants suffer from some members' deviations from agreement conditions, is the way to predict players' behavior [13].

Actually, the analysis of stable IEAs from a game-theoretic perspective can be traced back to the late 1900s, since then the research in this area has started in view of different aspects. The necessity of cooperation between nations or players in pollution control problems is highlighted in [20, 26]. In many publications on IEAs from a game-theoretical perspective, the comparison of cooperative and noncooperative countries' behavior when there are only two participants is investigated in [33, 49, 60, 64, 95]. The interaction between two countries different in terms of their vulnerabilities to emissions, i.e., vulnerable (developed) and invulnerable (developing) countries, is modeled as an asymmetric differential game [33, 64, 105]. In [103] a differential game between developed and developing countries is examined when the players interact in stochastic environment. A different optimization problem relying on the same system dynamics is raised in [64]. In [66], a two-player asymmetric game symbolizing the factor of economic efficiency and environmental vulnerability in the North-South competitive system is studied. The case of participating more than two countries in an agreement is considered in [28, 52, 77, 93, 99], where the authors compare two cases: (i) countries behave individually rational, or (ii) they cooperate forming the grand coalition. The model of environmental agreements when participants include adaptive measures in them is proposed in [12], where the authors reveal that when adaptation is regulated by an agreement, a stability of a one-coalition structure is reached with a particular coalition size.

Meanwhile, partial cooperation or the coalition structure formed by coalitions of different sizes is another option to model IEAs according to [35]. In this research, the partial cooperation structure is the main feature to investigate IEAs. First of all, the variety of different coalitions that can be formed implies more scenarios and provides more solutions to the problem of emission reduction different from "one coalition plus many singletons" scenario. As the profit of a coalition depends on other players' partitioning, it makes the game different from hedonic games, in which the payoff of any coalition is independent of the players' behavior outside the coalition [9, 46, 47]. Besides, based on the empirical evidence on IEAs, the grand coalition usually can not be created because of a possible conflict of interests [111], and such a scenario may be not desirable under the lack of multilateral coordination [95]. Therefore, consideration of nontrivial or multiple coalition scenarios can be motivated by exogenous factors, e.g., countries' locations, political situation, communication possibilities, etc., not allowing to form the grand coalition.

As mentioned, the typical situation examined in the literature is when the authority proposes signing an environmental agreement to a set of countries. They may join the agreement, and form a coalition, while those who disagree to join it become singletons [4, 20, 37, 63]. In contrast, there may be several agreements proposed to the players at the same time and they can choose the one to join. A question arises if a single-agreement coalition structure is better in some sense than a multiple-agreement coalition structure. It is worth mentioning that in [2, 16, 18, 34, 36] the authors conclude that a multiple coalition structure can outperform the former one, although in [11] it is formulated that no multiple coalition structure is profitable. In this research, we do not restrict the number of agreements in the society meaning that several coalitions of different sizes can be simultaneously formed.

More importantly, if any coalition and any coalition structure can be formed, we need to be sure that they are stable in some sense to be practically realized. In existing literature on dynamic games, the study of stability is focused on the structure represented by a unique grand coalition [71, 75, 77, 101, 104]. In international pollution control problems, it is usually assumed that one "big" coalition may be formed and all other players who have not joined it behave as singletons [4, 20, 63]. The concept of a coalition stability determined by international environmental agreements (e.g., see [53]) is defined by external and internal stability conditions [19, 82]. These conditions assume that no player has an incentive to deviate from a coalition and no singleton would benefit from joining this coalition. But under an assumption on multiple coalition formation, the agreement stability concept including internal and external stability conditions cannot capture all possible players' deviations [16, 18, 23]. One can allow to change one agreement to another which may be profitable in a multiple coalition case. In this notion, we can use stability concepts proposed for the games with coalition structures. In the research, we verify Nash stability [9, 46, 47, 70] and individual stability [9, 91] conditions for any possible coalition structure. In a nontransferable case, when players do not redistribute payoffs inside the coalitions, it is difficult to expect existence of stable coalition structures. Therefore, we aim at proposing the mechanisms to design stable scenarios of cooperation or stable coalition structures. We propose three such mechanisms. The first one is based on defining the transfers between coalition members to increase the payoffs to the players who may have profitable deviations. The transfer scheme can be defined basing on cooperative solutions adopted for the games with coalition structures [3, 55, 68]. One of such schemes is the Nash bargaining solution [65] which is mostly used for two-player games [14, 15, 31, 33, 92, 103] modeling cooperative environmental problems. For more than two players, the Shapley value [84] has become a powerful tool to maintain stable cooperation in differential games [85, 43, 76]. The transfer scheme is also applied in [88] for designing environmental agreements for different cooperative scenarios. The transferable utilities are considered in [11, 17, 20, 32] in solving environmental problems. The paper [96] examines a repeated game with transfer at each stage between countries polluting the atmosphere. Under given conditions, the profile of constructed strategies is a subgame perfect equilibrium realizing Pareto-optimal payoffs in each period of the game. The second mechanism is based on defining a taxation scheme for players' profitable deviations. We define the value of a uniform tax for all players for any particular scenario when the players pay the same tax in the case of deviation. We provide the definition of a stable scenario when the taxation scheme is adopted. The third mechanism proposed to make a desired scenario stable is to define the set of restricted coalitions or the set of feasible coalition structures. This approach can be determined basing on the theory of cooperative games with restricted cooperation and the solutions defined for this class of games [1, 8, 69]. Restricting the formation of some coalitions we can prevent undesirable scenario to be realized. There are different approaches to define the set of feasible coalitions, e.g., permission structures, matroids, antimatroids etc., and for a selected class, one can use modified cooperative solution concepts based on, for example, the Shapley value [1, 8]. Moreover, even with the mechanisms described above, the existence of stable structure is still an open question. In this research, we can only prove it theoretically under static model, because the complexity is much higher in a dynamic setting.

Apart from the investigation on coalition structure, a trade-off mechanism [24] which differs in its objective functions working in a two-player differential game is proposed in this research. The idea of a trade-off mechanism comes from a supply chain theory. Generally, there are two prominent supply chain models in use: forward supply chain and closed-loop supply chain. The forward supply chain, namely, the flow of the product is one-directional through the chain. Meanwhile, the closed-loop supply chain (CLSC), in which the used product can be recycled and sold again after remanufacturing, is another popular model. Moreover, contrary to forward supply chains, CLSC has its inherent characteristics: closed-loop, which naturally make it implement in an environment-friendly and profitable way [24]. However, no matter which model is used, various pollution control policies or constraints have been leveraged concerning the exacerbating environment problems, such as carbon tax [97, 98], cap-and-trade [40, 58, 106], green supply chain management [50, 107], consumers' low-carbon preference [40, 51, 97, 100], low-carbon subsidy [98, 108], contract design [24], etc. The trade-off mechanism is similar to the contract design, where players sign a contract and behave following the contract rules over time, and it is different from a fully cooperative scenario, in which players completely coordinate their actions to maximize the total profit. The latter scenario requires a total control of players' actions along the cooperative trajectory while in a trade-off mechanism, once the contract is signed, the players act individually and adopt the Nash equilibrium in a redefined differential game. Therefore, there is no need to adopt any allocation mechanism [42, 73, 74, 84] along the state trajectory. What is more, the power structure [58] in the supply chain model indicates if the manufacture is acting as the bellwether or the retailer is dominating. Noticeably, this distinguishes the trade-off mechanism from the cost-revenue sharing contract [24] because the former does not require coordination of players' order in decision making.

Coming to the external factors, we can say that the economic activity including pollution reduction is a mixture of various joint ingredients, and corresponding decisions are constructed based on its statistical estimation but not on its real values. It is quite obvious that information plays an important role here. Since Shannon-Weaver's model of communication [83] came to the public in 1948, the various concepts and details for information communication such as information source, transmitter, channel, noise, message, receiver, information destination, encoder, and decoder, have been widely accepted. Because of the intrinsic character of information, a value of information (VI) can explain information in connection with its actual description. In a case of ecology, Vl analysis identifies the best information collection strategy that leads to the largest net benefit [38]. In case of medicine, the disclosure and ability to react to the diagnosis test could comprehensively depict the value of it [67]. In commerce, VI is embedded into the product flow among different roles [81]. To put it briefly, VI has a broad application in various subjects in which uncertainty plays an essential part in the decision process. As indicated in [54], the use of VI analysis at least has started since the 1990s and the rea of application is widespread over economics, infrastructure, environment, energy, medical systems, and other fields partially shown in [7, 30, 57, 61, 86, 102]. Among them, VI hidden in the estimated value of one specific parameter in a model, e.g., the estimation of potential amount of oil underground, has a vital impact on the stability of players' decisions in economic activities. Thinking of the role of the estimated value in VI, usually when we are trying to calculate the velocity of an object moving with a constant acceleration, the initial speed $v_0$ has to be observed. It is also necessary to determine the initial condition when we choose to utilize a gradient descent method to solve the optimization problem. The list of such similar cases can be enormous, but the unanimous feature of these cases is that the estimated value remains a required part in all the problems, some of which may even dramatically affect the interests of a decision maker. In differential games, the accuracy or sufficiency of information in model formulation is not always guaranteed. Thus, VI is presented to measure the lack or misjudgment of information when players make their decisions, and to estimate the impact of their final profits. In [41], the authors elaborately describe the impact of information of the uncertain estimation of initial storage which could lead to the profit volatility. One can calculate VI in differential games or in optimization problems and may highlight how the information originated from parameter uncertainty actually affects the outcomes. There are several works on the uncertainty of the parameters in differential games [21, 94]. In particular, in [94], the pollution control problem with rehabilitation process is considered for the first time as far as I know. There is a research on the value of cooperation [22], where the information is represented by the comparison of benefits under cooperation contrary to noncooperation.

To conclude, we can say that the two aspects: internal and external elements — are of great importance for stability analysis. It is critical and helpful to figure out how the objective functions, structure, mechanism implementation and uncertain information could influence the stability of environment protection and economic activities.

**Goals of the thesis**

The thesis is devoted to contributions in solving the environmental problems through a deeper research of stability analysis in these problems. The subject of stability analysis is pollution control games with many players (in special case, optimization problems with one player). The stability analysis mainly focuses on the internal and external elements. In this thesis, the research of the impact of internal elements including objective functions, structure, and mechanism implementation is more oriented on the investigation of influence of inherent characteristics imposed on the stability. In terms of uncertain information caused by external elements, including changes of upper boundary for control and terminal costs, estimation of initial stock, one may expect to find out the possible changes or instability in players' decisions under the lack or absence of necessary information.

Broadly speaking, this thesis can be considered as a tutorial on the stability analysis in a range of problems and one can apply existing or new concepts and analytical techniques to the problems of job change, moving to another country, divorce or generally to any multiagent problems.

**Main tasks**

To fulfill the research plan, key tasks can be identified as follows:

1. Under the static model setting, a game between countries or companies polluting the common region, when countries are different in their attitudes to the pollution reduction policy should be considered. One of the tasks is to examine different scenarios: (i) when all countries behave individually rational, (ii) when they all form a unique coalition, and (iii) when the countries partially cooperate implying the formation of different coalition structures. The stability of all coalition structures is going to be examined. The research proposes three ways of designing a stable coalition structure in case it is not essentially stable. These ways are based on transfer payments, restrictions on coalition formation, and design of the system of transition costs defining a contract between cooperating countries. All these schemes support Nash stability and/or individual

stability properties of the preferred coalition structure.

2. One of the tasks is to investigate an asymmetric differential game of pollution control with a developing country and two developed countries. The developing country is invulnerable to the pollution in contrast to the developed ones. Assuming partial cooperation, all coalition structures composed by three players should be examined and stability conditions for them should be provided based on two approaches: (i) Nash stability and (ii) individual stability. First, the case of nontransferable payoffs is examined. Second, a transfer payment scheme is proposed to make particular coalition structures stable. A trade-off mechanism is proposed by sharing partial workload of a pollution disposal between the developed and developing countries. In return, the developed country shares its profit with the developing country to make the trade work. The efficiency of the trade-off mechanism compared with the fully cooperative and noncooperative cases should be explored.

3. For differential games of pollution control, one of the tasks is to examine how the payoff can be affected when the required information is unknown. For this, in particular, two scenarios should be examined: (i) to study the role of knowledge about the terminal cost and (ii) to analyze the influence of knowledge about the exact value of the upper bound on the control set. In case of two-player games of pollution control with uncertain initial disturbance stock, we present a model of resource extraction with rehabilitation process in which the firms are required to compensate the local to rehabilitate the polluted and dilapidated areas. A simulation of initial stock estimation is alternatively investigated in cooperative and noncooperative cases. In both games, the task is to estimate VI or Normalized Value of Information (NVI) to quantify the influence of uncertainty on the final payoff.

**Scientific novelty**

In the thesis, with regard to the objective function as one of the internal elements of stability analysis, a trade-off mechanism merging two asymmetric players' objective functions in pollution control games is proposed for the first time. This trade-off mechanism is innovated from the supply chain theory in which the members' payoff functions are strongly correlated. The most distinguishable part of the trade-off mechanism is that it does not require complete coordination of players' action over time in maximizing the total profit. Moreover, a comparison between the trade-off mechanism and cooperative and noncooperative solutions is investigated. The numerical example demonstrates that the trade-off mechanism could do better than a cooperative scenario in pollution reduction task.

Referring to stability analysis of the cooperative structures or partitions formed by players, first, the variety of structures from grand coalition and the set of singletons to partial cooperative structures in which multiple coalitions can be constructed is examined. Partial cooperation is "grand coalition plus many singletons" scenario. Second, contrary to internal and external stability, two appropriate approaches to partition-form game: Nash stability and individual stability — are applied to verify the stability of coalition structures. It can be determined that individual stability is much more reasonable for environmental agreements because any player can reject the entry of another player if the acceptance of a new participant implies the losses in payoffs.

Concerning mechanism design of stable scenarios, a toolkit for making coalition structures stable is proposed. In this toolkit, three mechanisms are primarily presented: (i) transfer scheme when the utilities can be transferred among the players to maintain the stability of cooperation, (ii) taxation scheme when the value of a uniform tax is defined for all players to prevent possible deviations, and (iii) modeling the set of feasible coalitions which can be determined to restrict cooperation. Within the third approach, some of the coalition structures become prohibited so that more scenarios can be stable.

In the part of the thesis concerning uncertain information, three detailed cases are examined: (i) VI for uncertainty about terminal costs, (ii) VI for uncertainty about possible adjustment of a control upper boundary, and (iii) VI for initial pollution stock. The first two cases are caused by the government or regulator who implement new policies based on the current production. The third case concerning estimation of the initial pollution stock arises under the influence of a decision maker and affects the stability of a production plan. As far as I know, the pollution control problem with rehabilitation process is examined from VI prospective for the first time in this thesis. In this part of the thesis, the new characteristic of NVI is proposed for measuring the benefit of uncertain information in the three cases mentioned above.

**Research methods**

This thesis uses the methods of static game theory (stability conditions and mechanism design of stable coalition structures, Nash equilibrium, Pareto optimality), solutions of differential games (stability conditions, Pontryagin's maximum principle, Hamiltonian-Jacobi-Bellman equation, subgame consistency), cooperative game theory (with nontransferable and transferable utilities, characteristic function), games with coalition structures, optimization theory (Kuhn-Tucker conditions), probability theory (value of information), mechanism design of contracts.

**Theoretical and practical significance**

In this thesis, the research is focused on implementation of stability analysis which theoretical significance is justified as a part of a complete study of examined systems or processes. The theoretical results are within the theory of stability analysis including the concepts of Nash and individual stability, mechanism design of transfer schemes in case of transferable utilities, taxation scheme, and design of the set of feasible coalitions. One of the theoretical contribution of the thesis is a proposition of the trade-off objective function for the players in pollution control problems with two players, another one deals with examining the influence of uncertain information in parameters modeling of the systems.

The practical significance of the work is manifested in the research on creating environmental agreements and solution of pollution control problems which are the most significant problems for the last decades. The international community strongly supports cooperation in solving environmental problems. By exploiting the cooperative opportunity, the idea of having a grand coalition or "one big coalition plus many singletons" does not find popularity in some societies due to restrictions on cooperation and this leads to considering multiple-coalition cases in this thesis. Moreover, we testify the possibility of "buying cooperation" through various methods to maintain the stability of particular coalition structures. The support of "buying cooperation" is a meaningful demonstration of a trading principle. The application of models with uncertain information is wide and not limited by pollution control problems. Certain information in the economic activities is definitely important for a decision maker to take the right actions. Naturally, the stability of decision making can be affected if the information is uncertain, and in this thesis the latter case is carefully examined.

Two critical concepts: Nash stability and individual stability are defined in detail in Chapters 1 and 2. In particular, Chapter 1 examines a static model in which players generate all possible structures including two-coalition scenarios. Three mechanisms are proposed as a theoretical tool to make the desirable structure stable, and each mechanism implementation is demonstrated on a numerical example. The dynamic model of a pollution control problem is clearly described in Chapter 2. Starting with two-player trade-off mechanism inspired by the contract design from supply chain theory, we also consider a three-player differential game for which a transfer scheme and subgame consistency are investigated. In the last chapter, the analysis of uncertain information in pollution control problems is revealed. Specifically, Chapter 3 discloses three scenarios showing the disturbance brought to decision making measured by NVI.

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**Brief description of the thesis structure**

The thesis consists of an introduction, three chapters, conclusions and bibliography. Each chapter starts with model and basic definition description, and problem formulation. The results including propositions, numerical examples for clear illustration of the theory are presented at the end of each chapter. The thesis contains 117 pages (129 pages in a Russian version) including 19 tables and 14 figures. The bibliography cites 112 items listed in alphabetical order.

The first chapter is devoted to studying stability mechanisms designed for every form of a coalition structure including grand coalition, partial cooperation (one coalition plus several singletons and multiple coalitions), and the structure composed by many singletons. Section 1.1 properly describes a static model with four players. The reason for choosing a static model is to reduce the complexity for calculating the equilibria. In Section 1.2, equilibria for all possible coalition structures are obtained. The stability analysis based on nontransferable utilities is presented in Section 1.3 after defining Nash stability and individual stability concepts. Since nontransferable payoffs cannot meet stability conditions, we propose other methods of making some desired structures stable. Three mechanisms including transfer scheme, taxation scheme, and design of feasible coalitions are described in Section 1.4. In some cases, we theoretically verify the existence of a stable coalition structure, but in some cases it is demonstrated on numerical examples. Finally, Section 1.5 briefly concludes the first chapter.

Different from a static setting examined in the first chapter, the second chapter approaches a dynamic setting. Section 2.1 proposes a trade-off mechanism borrowed from supply chain contract theory. Compared with a fully cooperative scenario, the trade-off mechanism does not require entire coordination of players' behavior. In this section, advantages of a trade-off mechanism are analyzed through its comparison with cooperative and noncooperative scenarios in terms of the pollution level. The results are revealed in numerical examples. In Section 2.2, a differential game with three asymmetric players different in their attitude to the pollution reduction is examined. Three types of cooperation scenarios are presented and the Nash stability and individual stability concepts are applied in a dynamic case. Then identification of stable structures is conducted under cases of nontransferable and transferable utilities. Furthermore, we determine the boundary of transferable utilities for developed countries to "buy cooperation" with developing country and maintain a stable cooperation scenario. Theoretical results are supported by two numerical examples. In conclusion, a short summary is given in Section 2.3.

In contrast to the internal elements of stability described in Chapters 1 and 2, the last chapter investigates the influence of external elements, in particular, uncertain information, on the system stability. Section 3.1 explores VI for uncertainty about terminal costs. Specifically, the following situation is examined. When a player finds out that he will have terminal costs, the original strategy constructed without this information can be possibly affected and then changed by a better one. Section 3.2 examines VI in case of uncertainty about possible adjustment of a control upper boundary. Similarly, once the information about the change of control upper boundary is revealed, it is expected that the stability of strategy trajectory will be altered consequently. In Section 3.3, VI about uncertainty in estimation of initial pollution stock is investigated. When the initial pollution stock is overestimated or underestimated, the deviation will be enlarged in the final payoff. Under cooperative and noncooperative scenarios, players' benefits or losses are determined by making inaccurate estimations. Section 3.4 contains a brief conclusion to Chapter 3.

At the end of the thesis, a general conclusion of the whole thesis is given. Possible directions for future research are also discussed in the final conclusion.

**Results submitted for defense**

1. The Nash and individual stability conditions are defined for static and dynamic pollution control games. Conditions for stable coalition structures are

in particular identified for three- and four-player games in an explicit form.

2. Transfer scheme is designed for making a particular coalition structure stable by transferring the share of profits between developed and developing countries to sustain cooperation when the utilities are transferable.

3. Taxation scheme is proposed to increase the cost of deviation from the coalition structure in which players' deviations are nonprofitable. In detail, a uniform transition tax based on the highest benefit from deviation is implemented.

4. Design of the set of feasible coalitions is proposed by implementing restrictions on coalition formation. Defining the set of feasible coalitions is equivalent to implementing restrictions on players' deviation possibilities.

5. The existence conditions for stable coalition structures are proved for a dynamic model of pollution control and justified on numerical examples. In the static model, existence can be theoretically verified under the transfer scheme and feasible coalitions setting.

6. Trade-off mechanism inspired by supply chain contract theory is proposed as a different approach to sustain cooperation by trading the profit of a developed country to a developing one in return for its responsibility in taking pollution reduction costs into account. This idea is realized by designing modified objective functions for the players.

7. The value of information is calculated for pollution control problems in case of uncertain terminal costs, uncertain adjustment of control upper boundary, and inaccurate estimation of initial pollution stock.

**Verification of results**

The main results obtained in the thesis were presented at the International Conferences "Game Theory and Management" (Saint Petersburg, 2021, 2023); International Conference "Game Theory and Applications" (Saint Petersburg, 2022); International Conference "Mathematical Optimization Theory and Operations Research" (online, 2022; Yekaterinburg, 2023), and at the seminars of Department of Mathematical Game Theory and Statistical Decisions at Saint Petersburg State University.

**Publications**

Based on the results of the thesis, the following works were published: [21, 22, 87, 88, 89]. And [90] is under review.

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# Chapter 1

# Stable Agreements in Static Games of Pollution Control

In this chapter, we conduct the stability analysis for a static model of pollution control with four players. The Nash stability and individual stability are verified for all possible coalition structures including two-coalition scenarios. Moreover, for making a particular coalition structure stable, three mechanisms – transfer scheme, taxation scheme and designing the set of feasible coalitions are proposed. The existence of a stable coalition structure is proved for some parameter values.

## 1.1 Model

Let a set of countries or players be $N = \{1, 2, 3, 4\}$, and players be of two types: $I$ is a vulnerable player (developed country), and $II$ is an invulnerable player (developing country).[1] The player's type defines her strategy in pollution reduction activities and the attitude to environmental policies. Let players 1, 2 be of type $I$ and players 3, 4 be of type $II$.

The countries produce goods, and this production activity generates emissions. The strategy of player $i$ is $e_i$ which is the quantity of emission. Following the formulation in [33, 64], the pollution stock $S$ is given by

$$S = \mu \sum_{i \in N} e_i + \delta S_0, \ S_0 > 0, \tag{1.1}$$

where $\mu > 0$ is the marginal influence on pollution accumulation $S$ issued by the players' emissions, and $\delta > 0$ is the nature's absorption rate. The value $S_0$ is an initial level of pollution stock before the players choose their strategies.

[1] We examine the case of four players for simplicity of calculations, but the results and all the schemes we propose in the paper can be applied to the case with any number of players.

Assuming that invulnerable and vulnerable players are different in their attitudes to the pollution reduction policy in terms of the damage caused by their industrial activities, this is modeled by different payoff functions. A vulnerable player aims to maximize her payoff given by

$$\max_{e_i \geq 0} W_i = \alpha_i e_i - \frac{1}{2} e_i^2 - \frac{1}{2} \beta S^2, \tag{1.2}$$

where $\alpha_i > 0$, $\beta > 0$ is a per-unit damage cost parameter, whereas the objective function of an invulnerable player goes as

$$\max_{e_i \geq 0} W_i = \alpha_i e_i - \frac{1}{2} e_i^2. \tag{1.3}$$

We should notice that the payoff function given by (1.2) with $\beta = 0$ defines the payoff of any vulnerable player.

## 1.2 Equilibria under Different Scenarios

In this section, we assume that the players may cooperate and form coalitions of any size, so cooperation may be full when all the players join to form a unique coalition, or partial when coalitions of any sizes can be formed. Therefore, not only the grand coalition may be formed within a partially cooperative scenario, but also smaller coalitions are possibly formed implying the formation of specific coalition structures. By coalition structure $\pi$ we mean any partition of a set of players, that is, $\pi = \{B_1, \ldots, B_m\}$ such that $B_j \subset N$, $B_j \cap B_k = \varnothing$, $\cup_{j=1}^m B_j = N$. For example, the number of coalition structures that can be formed by four players is 15.

We specify the possible coalition structures or scenarios in a four-player game:

1. *Noncooperative scenario*: $\pi_1 = \{\{I\}, \{I\}, \{II\}, \{II\}\}$;

2. *Cooperative scenario*: $\pi_2 = \{\{I, I, II, II\}\}$;

3. *Partially cooperative scenarios*:

   (a) Case 1 $\{\{I, I\}, \{II\}, \{II\}\}, \{\{I, II\}, \{I\}, \{II\}\}$ (type "2 + 1 + 1": one coalition with two members, other players are singletons) :
   $\pi_{3_1} = \{\{1, 2\}, \{3\}, \{4\}\}, \pi_{3_2} = \{1, 3\}, \{2\}, \{4\}\}, \pi_{3_3} = \{\{1, 4\}, \{2\}, \{3\}\}$,
   $\pi_{3_4} = \{\{2, 3\}, \{1\}, \{4\}\}, \pi_{3_5} = \{\{2, 4\}, \{1\}, \{3\}\}, \pi_{3_6} = \{\{3, 4\}, \{1\}, \{2\}\}$;

(b) Case 2 $\{\{I,I\},\{II,II\}\},\{\{I,II\},\{I,II\}\}$ (type "$2+2$": two coalitions with two members in each coalition):
$\pi_{4_1} = \{\{1,2\},\{3,4\}\}, \pi_{4_2} = \{\{1,3\},\{2,4\}\}, \pi_{4_3} = \{\{1,4\},\{2,3\}\}$;

(c) Case 3 $\{\{I,I,II\},\{II\}\},\{\{I,II,II\},\{I\}\}$ (type "$3+1$": one coalition with three members):
$\pi_{5_1} = \{\{1,2,3\},\{4\}\}, \pi_{5_2} = \{\{1,2,4\},\{3\}\}$,
$\pi_{5_3} = \{\{1,3,4\},\{2\}\}, \pi_{5_4} = \{\{2,3,4\},\{1\}\}$.

We examine the game described in the previous section when the set of players is partitioned into structure $\pi$. There are assumptions about the players' behavior:

1. The players belonging to a coalition choose their strategies maximizing the payoff of this coalition, which is defined as the sum of the payoffs of the players belonging to this coalition. Therefore, the coalition is considered as a singleton.

2. The coalitions behave noncooperatively with respect to each other, and the Nash equilibrium is considered as an equilibrium concept in the game between coalitions.

The following series of propositions provide the conditions of the Nash equilibria in the game under different coalition structures.

### 1.2.1 Noncooperaitve scenario

**Proposition 1.1.** *In the noncooperative scenario $\pi_1$, the Nash equilibrium is given by*

$$e_i^{nc} = \frac{\alpha_i + \beta\mu^2\alpha_i - \beta\mu^2\sum_{k\in M\setminus i}\alpha_k - \beta\mu\delta S_0}{1+2\beta\mu^2},\ \ i = 1,2,$$
$$e_j^{nc} = \alpha_j,\ \ j = 3,4,$$

*when the equilibrium strategies are nonnegative, the equilibrium pollution stock is*

$$S^{nc} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+2\beta\mu^2}, \tag{1.4}$$

*where $\alpha_{1234} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.*

*Proof.* First, the objective of player 3 and player 4 does not depend on the stock variable and other players' strategies, so we can easily obtain that the maximal values of their objectives $W_3^{\pi_1} = \frac{\alpha_3^2}{2}$ and $W_4^{\pi_1} = \frac{\alpha_4^2}{2}$ are reached at $e_3 = \alpha_3$ and $e_4 = \alpha_4$ respectively.

Second, Player 1 is of type $I$. Her optimization problem is

$$W_1^{\pi_1} = \alpha_1 e_1 - \frac{1}{2}e_1^2 - \frac{1}{2}\beta S^2 \to \max_{e_1 \geq 0}.$$

Substituting the stock of form (1.1), we rewrite the optimization problem as follows:

$$W_1^{\pi_1} = \alpha_1 e_1 - \frac{1}{2}e_1^2 - \frac{1}{2}\beta(\delta S_0 + \mu\alpha_{34} + \mu e_1 + \mu e_2)^2 \to \max_{e_1 \geq 0},$$

where $\alpha_{34} = \alpha_3 + \alpha_4$.

We let the first derivative of $W_1^{\pi_1}$ on $e_1$ be equal to zero and determine that player 1 can obtain the maximal profit when

$$e_1^{nc} = \frac{\alpha_1 - \beta\mu(\delta S_0 + \mu\alpha_{34} + \mu e_2^{nc})}{1 + \beta\mu^2}$$

based on the sign of the second derivative on $e_1^{nc}$ is negative. Similarly, for player 2 we obtain her best reply strategy $e_2^{nc} = \frac{\alpha_2 - \beta\mu(\delta S_0 + \mu\alpha_{34} + \mu e_1^{nc})}{1+\beta\mu^2}$. Solving these two linear equations, we get equilibrium strategies.

Then, by substituting them into expression (1.1) of the stock, we get the value of equilibrium stock (1.4). □

### 1.2.2 Cooperative scenario

**Proposition 1.2.** *In the cooperative scenario $\pi_2$, the optimal players' strategies are given by*

$$e_i^c = \frac{\alpha_i + 6\beta\mu^2\alpha_i - 2\beta\mu^2\sum_{j\in N\setminus i}\alpha_j - 2\beta\mu\delta S_0}{1 + 8\beta\mu^2}, \quad i \in \{1,2,3,4\},$$

*when the optimal strategies are nonnegative, the pollution stock under scenario $\pi_2$ is*

$$S^c = \frac{\delta S_0 + \alpha_{1234}\mu}{1 + 8\beta\mu^2}. \tag{1.5}$$

*Proof.* The optimization problem of the grand coalition $M$ is given by

$$W^{\pi_2} = \sum_{i=1}^{4}(\alpha_i e_i - \frac{1}{2}e_i^2) - \beta S^2 \to \max_{e_i \geq 0, i\in N}. \tag{1.6}$$

After finding the first derivative on each $e_i$ and verifying the negative sign of the second derivative, we obtain optimal strategies:

$$e_i^c = \frac{\alpha_i - 2\beta\mu(\delta S_0 + \mu\sum_{j\in N\setminus i} e_j^c)}{1 + 2\beta\mu^2}. \tag{1.7}$$

Then we get the final result by solving these four linear equations. By substituting expressions of these strategies into (1.1), we get the optimal value of stock (1.5) for a cooperative scenario. □

### 1.2.3 Partially Cooperative Scenarios

In this section, we examine the players' equilibrium behavior under partial cooperation.

**Proposition 1.3.** *Under partially cooperative scenarios, the coalitional Nash equilibrium and corresponding emission stock are as follows:*

1. $\pi_{3_1} = \{\{1,2\},\{3\},\{4\}\}$*: the Nash equilibrium is given by*

$$e_i^{3_1} = \frac{\alpha_i + 2\beta\mu^2\alpha_i - 2\beta\mu^2\sum_{k\in N\backslash i}\alpha_k - 2\beta\mu\delta S_0}{1+4\beta\mu^2}, \ i=1,2$$

$$e_j^{3_1} = \alpha_j, \ j=3,4.$$

*The pollution stock is*

$$S^{3_1} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+4\beta\mu^2}.$$

2. $\pi_{3_2} = \{\{1,3\},\{2\},\{4\}\}$*: the Nash equilibrium is given by*

$$e_i^{3_2} = \frac{\alpha_i + 2\beta\mu^2\alpha_i - \beta\mu^2\sum_{k\in N\backslash i}\alpha_k - \beta\mu\delta S_0}{1+3\beta\mu^2}, \ i=1,2,3,$$

$$e_4^{3_2} = \alpha_4.$$

*The pollution stock is*

$$S^{3_2} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+3\beta\mu^2}. \tag{1.8}$$

3. $\pi_{3_3} = \{\{1,4\},\{2\},\{3\}\}$*: the Nash equilibrium is given by*

$$e_i^{3_3} = \frac{\alpha_i + 2\beta\mu^2\alpha_i - \beta\mu^2\sum_{k\in N\backslash i}\alpha_k - \beta\mu\delta S_0}{1+3\beta\mu^2}, \ i=1,2,4,$$

$$e_3^{3_3} = \alpha_3.$$

*The pollution stock is*

$$S^{3_3} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+3\beta\mu^2}. \tag{1.9}$$

*4.* $\pi_{4_2} = \{\{1,3\},\{2,4\}\}$*: the Nash equilibrium is given by*

$$e_i^{4_2} = \frac{\alpha_i + 3\beta\mu^2\alpha_i - \beta\mu^2 \sum_{k\in N\setminus i}\alpha_k - \beta\mu\delta S_0}{1+4\beta\mu^2}, \ i \in M.$$

*The pollution stock is*

$$S^{4_2} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+4\beta\mu^2}. \tag{1.10}$$

*5.* $\pi_{5_1} = \{\{1,2,3\},\{4\}\}$*: the Nash equilibrium is given by*

$$e_i^{5_1} = \frac{\alpha_i + 4\beta\mu^2\alpha_i - 2\beta\mu^2 \sum_{k\in N\setminus i}\alpha_k - 2\beta\mu\delta S_0}{1+6\beta\mu^2}, \ i = 1,2,3,$$

$$e_4^{5_1} = \alpha_4.$$

*The pollution stock is*

$$S^{5_1} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+6\beta\mu^2}. \tag{1.11}$$

*6.* $\pi_{5_2} = \{\{1,2,4\},\{3\}\}$*: the Nash equilibrium is given by*

$$e_i^{5_2} = \frac{\alpha_i + 4\beta\mu^2\alpha_i - 2\beta\mu^2 \sum_{k\in N\setminus i}\alpha_k - 2\beta\mu\delta S_0}{1+6\beta\mu^2}, \ i = 1,2,4,$$

$$e_3^{5_2} = \alpha_3.$$

*The pollution stock is*

$$S^{5_2} = \frac{\delta S_0 + \alpha_{1234}\mu}{1+6\beta\mu^2}. \tag{1.12}$$

*In all items we assume that equilibrium strategies are nonnegative.*

*Proof.* Omitted to save the space. □

We also summarize the Nash equilibrium players' strategies and equilibrium values of stock for any given scenario in Table 1.1.

### 1.2.4 Analysis of different scenarios

Using Proposition 1.1–1.3, we first compare the scenarios in terms of equilibrium pollution stock and then in terms of players' payoffs. In Table 1.2, in the second column we rank the equilibrium stock from the minimal one denoted as $S$ - $0$ (corresponding to the cooperative scenario $\pi_2$) to the maximal one denoted by $S$ - $4$ (corresponding to the noncooperative scenario $\pi_1$ and partially cooperative scenario $\pi_{3_6}$). We should notice that the pollution stock under partially cooperative scenarios $\pi_{5_1}$ and $\pi_{5_2}$ is the second to the best. We summarize the comparative results on comparison of pollution stocks in Corollary 1.1.

Table 1.1: The Nash equilibrium players' strategies and equilibrium values of stock for all scenarios

| Scenario | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $S$ |
|---|---|---|---|---|---|
| $\pi_1$, $\pi_{3_6}$ | $\frac{\alpha_1+\beta\mu^2\alpha_1-\beta\mu^2\alpha_{234}-\beta\mu\delta S_0}{1+2\beta\mu^2}$ | $\frac{\alpha_2+\beta\mu^2\alpha_2-\beta\mu^2\alpha_{134}-\beta\mu\delta S_0}{1+2\beta\mu^2}$ | $\alpha_3$ | $\alpha_4$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+2\beta\mu^2}$ |
| $\pi_2$ | $\frac{\alpha_1+6\beta\mu^2\alpha_1-2\beta\mu^2\alpha_{234}-2\beta\mu\delta S_0}{1+8\beta\mu^2}$ | $\frac{\alpha_2+6\beta\mu^2\alpha_2-2\beta\mu^2\alpha_{134}-2\beta\mu\delta S_0}{1+8\beta\mu^2}$ | $\frac{\alpha_3+6\beta\mu^2\alpha_3-2\beta\mu^2\alpha_{124}-2\beta\mu\delta S_0}{1+8\beta\mu^2}$ | $\frac{\alpha_4+6\beta\mu^2\alpha_4-2\beta\mu^2\alpha_{123}-2\beta\mu\delta S_0}{1+8\beta\mu^2}$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+8\beta\mu^2}$ |
| $\pi_{3_1}$, $\pi_{4_1}$ | $\frac{\alpha_1+2\beta\mu^2\alpha_1-2\beta\mu^2\alpha_{234}-2\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\frac{\alpha_2+2\beta\mu^2\alpha_2-2\beta\mu^2\alpha_{134}-2\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\alpha_3$ | $\alpha_4$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+4\beta\mu^2}$ |
| $\pi_{3_2}$, $\pi_{3_4}$ | $\frac{\alpha_1+2\beta\mu^2\alpha_1-\beta\mu^2\alpha_{234}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\frac{\alpha_2+2\beta\mu^2\alpha_2-\beta\mu^2\alpha_{134}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\frac{\alpha_3+2\beta\mu^2\alpha_3-\beta\mu^2\alpha_{124}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\alpha_4$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+3\beta\mu^2}$ |
| $\pi_{3_3}$, $\pi_{3_5}$ | $\frac{\alpha_1+2\beta\mu^2\alpha_1-\beta\mu^2\alpha_{234}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\frac{\alpha_2+2\beta\mu^2\alpha_2-\beta\mu^2\alpha_{134}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\alpha_3$ | $\frac{\alpha_4+2\beta\mu^2\alpha_4-\beta\mu^2\alpha_{123}-\beta\mu\delta S_0}{1+3\beta\mu^2}$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+3\beta\mu^2}$ |
| $\pi_{4_2}$, $\pi_{4_3}$, $\pi_{5_3}$, $\pi_{5_4}$ | $\frac{\alpha_1+3\beta\mu^2\alpha_1-\beta\mu^2\alpha_{234}-\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\frac{\alpha_2+3\beta\mu^2\alpha_2-\beta\mu^2\alpha_{134}-\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\frac{\alpha_3+3\beta\mu^2\alpha_3-\beta\mu^2\alpha_{124}-\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\frac{\alpha_4+3\beta\mu^2\alpha_4-\beta\mu^2\alpha_{123}-\beta\mu\delta S_0}{1+4\beta\mu^2}$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+4\beta\mu^2}$ |
| $\pi_{5_1}$ | $\frac{\alpha_1+4\beta\mu^2\alpha_1-2\beta\mu^2\alpha_{234}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\frac{\alpha_2+4\beta\mu^2\alpha_2-2\beta\mu^2\alpha_{134}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\frac{\alpha_3+4\beta\mu^2\alpha_3-2\beta\mu^2\alpha_{124}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\alpha_4$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+6\beta\mu^2}$ |
| $\pi_{5_2}$ | $\frac{\alpha_1+4\beta\mu^2\alpha_1-2\beta\mu^2\alpha_{234}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\frac{\alpha_2+4\beta\mu^2\alpha_2-2\beta\mu^2\alpha_{134}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\alpha_3$ | $\frac{\alpha_3+4\beta\mu^2\alpha_3-2\beta\mu^2\alpha_{123}-2\beta\mu\delta S_0}{1+6\beta\mu^2}$ | $\frac{\delta S_0+\alpha_{1234}\mu}{1+6\beta\mu^2}$ |

**Remark 1.1.** *We use the following notation:* $\alpha_S = \sum_{i\in S}\alpha_i$ *for any* $S \subset N$.

**Corollary 1.1.** *The equilibrium emission stock corresponding to different scenarios satisfies the following conditions:*

$$S^c < S^{5_1} = S^{5_2} < S^{3_1} = S^{4_1} = S^{4_2} = S^{4_3}$$
$$= S^{5_3} = S^{5_4} < S^{3_2} = S^{3_3} = S^{3_4} = S^{3_5} < S^{nc} = S^{3_6}.$$

*Proof.* The result immediately follows from the comparison of emission stocks corresponding to possible scenarios, for which the equilibrium values of stock are given in Propositions 1.1–1.3. □

Table 1.2: Equilibrium pollution stock and players' payoffs under different scenarios

| Scenario | Pollution ($S$) | VP 1 ($W_1$) | VP 2 ($W_2$) | InvP 3 ($W_3$) | InvP 4 ($W_4$) |
|---|---|---|---|---|---|
| $\pi_1, \pi_{3_6}$ | S - 4 | $W_1$ - 5 | $W_2$ - 5 | $W_3$ - 0 | $W_4$ - 0 |
| $\pi_2$ | S - 0 | $W_1$ - 0 | $W_2$ - 0 | $W_3$ - m | $W_4$ - m |
| $\pi_{3_1}, \pi_{4_1}$ | S - 2 | $W_1$ - 4 | $W_2$ - 4 | $W_3$ - 0 | $W_4$ - 0 |
| $\pi_{3_2}, \pi_{3_4}$ | S - 3 | $W_1$ - 3 | $W_2$ - 3 | $W_3$ - k | $W_4$ - 0 |
| $\pi_{3_3}, \pi_{3_5}$ | S - 3 | $W_1$ - 3 | $W_2$ - 3 | $W_3$ - 0 | $W_4$ - k |
| $\pi_{4_2}, \pi_{4_3}, \pi_{5_3}, \pi_{5_4}$ | S - 2 | $W_1$ - i | $W_2$ - i | $W_3$ - 1 | $W_4$ - 1 |
| $\pi_{5_1}$ | S - 1 | $W_1$ - j | $W_2$ - j | $W_3$ - 4 | $W_4$ - 0 |
| $\pi_{5_2}$ | S - 1 | $W_1$ - j | $W_2$ - j | $W_3$ - 0 | $W_4$ - 4 |

Now we compare scenarios in terms of players' payoffs given in Table 1.2 (the last four columns). For example, consider the vulnerable player 1. We rank the player's payoffs from the maximal one indicated as $W_1$ - 0 (corresponding to cooperative scenario $\pi_2$) to the minimal one $W_1$ - 5 (corresponding to scenarios $\pi_1$ and $\pi_{3_6}$). We should notice that there are ranks $i, j, m, k$ in the table. They take the values $i, j \in \{1, 2\}$, and $m, k \in \{2, 3\}$, and these values depend on the parameters as follows:

$$\begin{cases} i = 1, j = 2, & \text{if } 28\beta^2\mu^4 - 1 > 0, \\ i = 2, j = 1, & \text{if } 28\beta^2\mu^4 - 1 < 0, \\ i = j = 1, & \text{if } 28\beta^2\mu^4 - 1 = 0. \end{cases} \tag{1.13}$$

$$\begin{cases} m = 2, k = 3, & \text{if } 3 + 8\beta\mu^2 - 28\beta^2\mu^4 < 0, \\ m = 3, k = 2, & \text{if } 3 + 8\beta\mu^2 - 28\beta^2\mu^4 > 0, \\ m = k = 2, & \text{if } 3 + 8\beta\mu^2 - 28\beta^2\mu^4 = 0. \end{cases} \tag{1.14}$$

**Remark 1.2.** *The non-negativity of players' Nash equilibrium or optimal strategies should also be satisfied, which indicates we need to ensure that the least Nash equilibrium or optimal strategies in Table 1.1 are nonnegative. Therefore,*

1. *For vulnerable player 1, we have* $\alpha_1 + 2\beta\mu\big(\mu(\alpha_1 - \alpha_{234}) - \delta S_0\big) \geq 0,$

2. *For vulnerable player 2, we have* $\alpha_2 + 2\beta\mu\big(\mu(\alpha_2 - \alpha_{134}) - \delta S_0\big) \geq 0,$

3. *For invulnerable player 3, we have* $\alpha_3 + \beta\mu\big(\mu(2\alpha_3 - \alpha_{124}) - \delta S_0\big) \geq 0,$

4. *For invulnerable player 4, we have* $\alpha_4 + \beta\mu\big(\mu(2\alpha_4 - \alpha_{123}) - \delta S_0\big) \geq 0.$

*i.e., we need to satisfy the following inequalities:*

$$\alpha_i \geq \frac{2\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j \in N \backslash i} \alpha_j), \quad i = 1, 2; \tag{1.15}$$

$$\alpha_i \geq \frac{\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j = N \backslash i} \alpha_j), \quad i = 3, 4. \tag{1.16}$$

*Furthermore, these inequalities can be simplified in a way that only for two players* $x, y$*,* $\arg\min_x \alpha_x, x \in \{1, 2\}$ *and* $\arg\min_y \alpha_y, y \in \{3, 4\}$*, their inequalities should be met, because assume* $x' = \{1, 2\}\backslash x$ *and* $y' = \{3, 4\}\backslash y$*, we have*

$$\alpha_{x'} \geq \alpha_x \geq \frac{2\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j \in N \backslash x} \alpha_j) \geq \frac{2\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j \in N \backslash x'} \alpha_j),$$

$$\alpha_{y'} \geq \alpha_y \geq \frac{\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j \in N \backslash y} \alpha_j) \geq \frac{\beta\mu}{1 + 2\beta\mu^2}(\delta S_0 + \mu \sum_{j \in N \backslash y'} \alpha_j).$$

*Thus, once the player* $x, y$ *satisfy the inequalities in (1.15) and (1.16), all inequalities stand true.*

Besides, we make the following conclusions after comparing different scenarios of cooperation in the game defined by (1.1)–(1.3):

1. The vulnerable players can obtain their maximal (minimal) payoffs only under a cooperative (noncooperative) scenario.

2. The invulnerable players can obtain their maximal payoffs when they are acting alone or in a homogeneous coalition containing only invulnerable players.

3. The pollution stock reaches the minimal level under a cooperative scenario, while under a noncooperative scenario the pollution stock level is the worst.

4. An invulnerable player gets the least payoff when she cooperates with two vulnerable players but not with the other invulnerable player.

## 1.3 Identification of Stable Coalition Structures

In this section, we assume that the players' payoffs are nontransferable, i.e., players obtain their payoffs according to the given payoff functions even if they form a coalition. Vector $W^\pi = (W_1^\pi, \ldots, W_m^\pi) \in \mathbb{R}^m$ denotes the corresponding players' payoffs when coalition structure $\pi$ is formed.

We examine the stability of all possible scenarios or coalition structures. A stable coalition structure is a candidate to be formed. There are different concepts of the coalition structure stability proposed for nondynamic games. A coalition structure $\pi = \{B_1, \ldots, B_m\}$, such that $B_1 \cup \cdots \cup B_m = N$ and $B_i \cap B_j = \varnothing$ for all $i, j = 1, \ldots, m, i \neq j$ is stable when any player cannot increase her payoff if she changes this structure in an individual way. We should notice that we consider two possibilities for a deviating player: (i) she can join any possible coalition without any restrictions (see Section 1.3.1), (ii) the coalition which the deviating player would like to join can block the player's joining if there exists at least one coalition member who can lose by accepting such a player (see Section 1.3.2).

### 1.3.1 Nash-Stable scenarios with nontransferable payoffs

The first definition of a stable coalition structure assumes that all individual deviations of the players are possible.

**Definition 1.1.** *A coalition structure $\pi = \{B_1, \ldots, B_m\}$ is Nash stable (or, simply, stable) if for any player $i \in N$ it holds that*

$$W_i^\pi \geq W_i^{\pi'} \text{ for all } \pi' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-\{B(i), B_j\}}\},$$

*where $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-\{B(i),B_j\}} = \pi\backslash\{B(i), B_j\}$, and $W^\pi$, $W^{\pi'}$ denote the vectors of players' payoffs under coalition structures $\pi$ and $\pi'$ respectively.*

In Definition 1.1, any player can deviate from her current coalition joining another existing coalition or becoming a singleton.

**Proposition 1.4.** *In the game of pollution control given by* (1.1)–(1.3) *with nontransferable payoffs, there is no Nash-stable coalition structure or scenario.*

*Proof.* Verifying the stability conditions given in Definition 1.1 and taking into account the equilibrium players' payoffs under different scenarios (see Table 1.1), we can conclude that for any scenario, there exists at least one player whose deviation is profitable. Therefore, a stable scenario does not exist. □

### 1.3.2 Individually stable coalition scenarios with nontransferable payoffs

In this section, we investigate another stability concept for coalition structures. We assume that the players in a coalition can refuse to cooperate with another player willing to join them in case this player can bring the loss to anyone's profit inside the coalition. This means that not all deviations of a player are possible. Therefore, we give another definition of a stable coalition structure with a reasonable block of external entries.

**Definition 1.2.** *A coalition structure* $\pi = \{B_1, \ldots, B_m\}$ *is individually stable if for any player* $i \in N$ *it holds that*

$$W_i^{\pi} \geq W_i^{\pi''} \text{ for all } \pi'' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-\{B(i),B_j\}}\} \text{ such that}$$
$$W_k^{\pi''} \geq W_k^{\pi} \text{ for all } k \in B_j,$$

*where* $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-\{B(i),B_j\}} = \pi\backslash\{B(i), B_j\}$, *and* $W^{\pi}$, $W^{\pi''}$ *denote the vectors of players' payoffs under the coalition structures* $\pi$ *and* $\pi''$ *respectively.*

Obviously, the set of individually stable coalition structures contains the set of the Nash stable coalition structures [91]. The following proposition characterizes the conditions of individually stable coalition structures.

**Proposition 1.5.** *In the game of pollution control defined by* (1.1)–(1.3) *with nontransferable payoffs, only coalition structures or scenarios* $\pi_{3_1}$ *and* $\pi_{4_1}$ *are individually stable.*

*Proof.* Verifying the stability conditions given in Definition 1.2 and taking into account the equilibrium players' payoffs under different scenarios (see Table 1.1), we can easily obtain that only two coalition structures satisfy them. We demonstrate which inequalities are satisfied for scenarios $\pi_{3_1}$ and $\pi_{4_1}$ in Table 1.3. They are colored in blue. □

## 1.4 Designing mechanisms to make scenarios stable

In this section, we propose three mechanisms to make the coalition structure stable in case it is not stable when the payoffs to the players are nontransferable, i.e., it does not satisfy Definition 1.1. These mechanisms are based on: (i) adopting a payment

Table 1.3: Conditions to verify individual stability of coalition structures $\pi_{3_1}$ and $\pi_{4_1}$

| Scenario | Vul. Player 1 | Vul. Player 2 | Invul. Player 3 | Invul. Player 4 |
|---|---|---|---|---|
| $\pi_{3_1} = \{\{1,2\},\{3\},\{4\}\}$ | $\begin{cases} W_1^{\pi_{3_1}} \geq W_1^{\pi_1} \\ \left[\begin{array}{l} \begin{cases} W_3^{\pi_{3_1}} < W_3^{\pi_{3_2}} \\ W_1^{\pi_{3_1}} \geq W_1^{\pi_{3_2}} \end{cases} \\ \text{or } W_3^{\pi_{3_1}} \geq W_3^{\pi_{3_2}} \end{array}\right. \\ \left[\begin{array}{l} \begin{cases} W_4^{\pi_{3_1}} < W_4^{\pi_{3_3}} \\ W_1^{\pi_{3_1}} \geq W_1^{\pi_{3_3}} \end{cases} \\ \text{or } W_4^{\pi_{3_1}} \geq W_4^{\pi_{3_3}} \end{array}\right. \end{cases}$ | $\begin{cases} W_2^{\pi_{3_1}} \geq W_2^{\pi_1} \\ \left[\begin{array}{l} \begin{cases} W_3^{\pi_{3_1}} < W_3^{\pi_{3_4}} \\ W_2^{\pi_{3_1}} \geq W_2^{\pi_{3_4}} \end{cases} \\ \text{or } W_3^{\pi_{3_1}} \geq W_3^{\pi_{3_4}} \end{array}\right. \\ \left[\begin{array}{l} \begin{cases} W_4^{\pi_{3_1}} < W_4^{\pi_{3_5}} \\ W_2^{\pi_{3_1}} \geq W_2^{\pi_{3_5}} \end{cases} \\ \text{or } W_4^{\pi_{3_1}} \geq W_4^{\pi_{3_5}} \end{array}\right. \end{cases}$ | $\begin{cases} \left[\begin{array}{l} \begin{cases} W_1^{\pi_{3_1}} < W_1^{\pi_{5_1}} \\ W_2^{\pi_{3_1}} < W_2^{\pi_{5_1}} \\ W_3^{\pi_{3_1}} \geq W_3^{\pi_{5_1}} \end{cases} \\ \text{or } W_1^{\pi_{3_1}} \geq W_1^{\pi_{5_1}} \\ \text{or } W_2^{\pi_{3_1}} \geq W_2^{\pi_{5_1}} \end{array}\right. \\ \left[\begin{array}{l} \begin{cases} W_4^{\pi_{3_1}} < W_4^{\pi_{4_1}} \\ W_3^{\pi_{3_1}} \geq W_3^{\pi_{4_1}} \end{cases} \\ \text{or } W_4^{\pi_{3_1}} \geq W_4^{\pi_{4_1}} \end{array}\right. \end{cases}$ | $\begin{cases} \left[\begin{array}{l} \begin{cases} W_1^{\pi_{3_1}} < W_1^{\pi_{5_2}} \\ W_2^{\pi_{3_1}} < W_2^{\pi_{5_2}} \\ W_4^{\pi_{3_1}} \geq W_4^{\pi_{5_2}} \end{cases} \\ \text{or } W_1^{\pi_{3_1}} \geq W_1^{\pi_{5_2}} \\ \text{or } W_2^{\pi_{3_1}} \geq W_2^{\pi_{5_2}} \end{array}\right. \\ \left[\begin{array}{l} \begin{cases} W_3^{\pi_{3_1}} < W_3^{\pi_{4_1}} \\ W_4^{\pi_{3_1}} \geq W_4^{\pi_{4_1}} \end{cases} \\ \text{or } W_3^{\pi_{3_1}} \geq W_3^{\pi_{4_1}} \end{array}\right. \end{cases}$ |
| $\pi_{4_1} = \{\{1,2\},\{3,4\}\}$ | $\begin{cases} W_1^{\pi_{4_1}} \geq W_1^{\pi_{3_6}} \\ \left[\begin{array}{l} \begin{cases} W_3^{\pi_{4_1}} < W_3^{\pi_{5_3}} \\ W_4^{\pi_{4_1}} < W_4^{\pi_{5_3}} \\ W_1^{\pi_{4_1}} \geq W_1^{\pi_{5_3}} \end{cases} \\ \text{or } W_3^{\pi_{4_1}} \geq W_3^{\pi_{5_3}} \\ \text{or } W_4^{\pi_{4_1}} \geq W_4^{\pi_{5_3}} \end{array}\right. \end{cases}$ | $\begin{cases} W_2^{\pi_{4_1}} \geq W_2^{\pi_{3_6}} \\ \left[\begin{array}{l} \begin{cases} W_3^{\pi_{4_1}} < W_3^{\pi_{5_4}} \\ W_4^{\pi_{4_1}} < W_4^{\pi_{5_4}} \\ W_2^{\pi_{4_1}} \geq W_2^{\pi_{5_4}} \end{cases} \\ \text{or } W_3^{\pi_{4_1}} \geq W_3^{\pi_{5_4}} \\ \text{or } W_4^{\pi_{4_1}} \geq W_4^{\pi_{5_4}} \end{array}\right. \end{cases}$ | $\begin{cases} W_3^{\pi_{4_1}} \geq W_3^{\pi_{3_1}} \\ \left[\begin{array}{l} \begin{cases} W_1^{\pi_{4_1}} < W_1^{\pi_{5_1}} \\ W_2^{\pi_{4_1}} < W_2^{\pi_{5_1}} \\ W_3^{\pi_{4_1}} \geq W_3^{\pi_{5_1}} \end{cases} \\ \text{or } W_1^{\pi_{4_1}} \geq W_1^{\pi_{5_1}} \\ \text{or } W_2^{\pi_{4_1}} \geq W_1^{\pi_{5_1}} \end{array}\right. \end{cases}$ | $\begin{cases} W_4^{\pi_{4_1}} \geq W_4^{\pi_{3_1}} \\ \left[\begin{array}{l} \begin{cases} W_1^{\pi_{4_1}} < W_1^{\pi_{5_2}} \\ W_2^{\pi_{4_1}} < W_2^{\pi_{5_2}} \\ W_4^{\pi_{4_1}} \geq W_4^{\pi_{5_2}} \end{cases} \\ \text{or } W_1^{\pi_{4_1}} \geq W_1^{\pi_{5_2}} \\ \text{or } W_2^{\pi_{4_1}} \geq W_1^{\pi_{5_2}} \end{array}\right. \end{cases}$ |

scheme defined by a cooperative solution such as the Shapley value, the CIS-value, etc. (Section 1.4.1); (ii) implementing taxation scheme for any player's deviations in case the scenario is not stable and at least one player can benefit deviating from it (Section 1.4.2); (iii) implementing the set of feasible coalitions, the set of coalitions that can be formed in the game, by making restrictions on formation of coalitions that can destabilize the scenario (Section 1.4.3).

### 1.4.1 Nash stability of coalition structures with transferable payoffs

In this section, we examine the Nash stability of scenarios when players' payoffs are transferable between the coalition members. The transfers between players can be defined in different ways, e.g., by adopting any cooperative solutions such as the Shapley value, the CIS-value, and the core [3]. These solutions applied to the games with coalition structures satisfy the efficiency property, according to which the sum of the payments to the players is equal to the payoff of the coalition they belong to. The payoffs that players obtain after making transfers with respect to allocation $\xi^\pi = (\xi_i^\pi : i \in S, S \in \pi)$ are represented in Table 1.4. In this table, we use the following notations: $\xi_S^\pi = \sum_{i \in S} \xi_i^\pi$ and $W_S^\pi = \sum_{i \in S} W_i^\pi$.

We demonstrate the mechanism of making transfers based on the CIS-value [29], but any other cooperative solution can be adopted in a similar way. For any player

Table 1.4: The players' payoffs under various coalition structures in a transferable case

| Scenarios | VP 1 | VP 2 | InvP 3 | InvP 4 |
|---|---|---|---|---|
| $\pi_1 = \{\{1\},\{2\},\{3\},\{4\}\}$ | $W_1^{\pi_1}$ | $W_2^{\pi_1}$ | $W_3^{\pi_1}$ | $W_4^{\pi_1}$ |
| $\pi_2 = \{\{1,2,3,4\}\}$ | $\xi_1^{\pi_2}$ | $\xi_2^{\pi_2}$ | $\xi_3^{\pi_2}$ | $W_{1234}^{\pi_2} - \xi_{123}^{\pi_2}$ |
| $\pi_{3_1} = \{\{1,2\},\{3\},\{4\}\}$ | $\xi_1^{\pi_{3_1}}$ | $W_{12}^{\pi_{3_1}} - \xi_1^{\pi_{3_1}}$ | $W_3^{\pi_{3_1}}$ | $W_4^{\pi_{3_1}}$ |
| $\pi_{3_2} = \{1,3\},\{2\},\{4\}\}$ | $\xi_1^{\pi_{3_2}}$ | $W_2^{\pi_{3_2}}$ | $W_{13}^{\pi_{3_2}} - \xi_1^{\pi_{3_2}}$ | $W_4^{\pi_{3_2}}$ |
| $\pi_{3_3} = \{\{1,4\},\{2\},\{3\}\}$ | $\xi_1^{\pi_{3_3}}$ | $W_2^{\pi_{3_3}}$ | $W_3^{\pi_{3_3}}$ | $W_{14}^{\pi_{3_3}} - \xi_1^{\pi_{3_3}}$ |
| $\pi_{3_4} = \{\{2,3\},\{1\},\{4\}\}$ | $W_1^{\pi_{3_4}}$ | $\xi_2^{\pi_{3_4}}$ | $W_{23}^{\pi_{3_4}} - \xi_2^{\pi_{3_4}}$ | $W_4^{\pi_{3_4}}$ |
| $\pi_{3_5} = \{\{2,4\},\{1\},\{3\}\}$ | $W_1^{\pi_{3_5}}$ | $\xi_2^{\pi_{3_5}}$ | $W_3^{\pi_{3_5}}$ | $W_{24}^{\pi_{3_5}} - \xi_2^{\pi_{3_5}}$ |
| $\pi_{3_6} = \{\{3,4\},\{1\},\{2\}\}$ | $W_1^{\pi_{3_6}}$ | $W_2^{\pi_{3_6}}$ | $\xi_3^{\pi_{3_6}}$ | $W_{34}^{\pi_{3_6}} - \xi_3^{\pi_{3_6}}$ |
| $\pi_{4_1} = \{\{1,2\},\{3,4\}\}$ | $\xi_1^{\pi_{4_1}}$ | $W_{12}^{\pi_{4_1}} - \xi_1^{\pi_{4_1}}$ | $\xi_3^{\pi_{4_1}}$ | $W_{34}^{\pi_{4_1}} - \xi_3^{\pi_{4_1}}$ |
| $\pi_{4_2} = \{\{1,3\},\{2,4\}\}$ | $\xi_1^{\pi_{4_2}}$ | $\xi_2^{\pi_{4_2}}$ | $W_{13}^{\pi_{4_2}} - \xi_1^{\pi_{4_2}}$ | $W_{24}^{\pi_{4_2}} - \xi_2^{\pi_{4_2}}$ |
| $\pi_{4_3} = \{\{1,4\},\{2,3\}\}$ | $\xi_1^{\pi_{4_3}}$ | $\xi_2^{\pi_{4_3}}$ | $W_{23}^{\pi_{4_3}} - \xi_2^{\pi_{4_3}}$ | $W_{14}^{\pi_{4_3}} - \xi_1^{\pi_{4_3}}$ |
| $\pi_{5_1} = \{\{1,2,3\},\{4\}\}$ | $\xi_1^{\pi_{5_1}}$ | $\xi_2^{\pi_{5_1}}$ | $W_{123}^{\pi_{5_1}} - \xi_{12}^{\pi_{5_1}}$ | $W_4^{\pi_{5_1}}$ |
| $\pi_{5_2} = \{\{1,2,4\},\{3\}\}$ | $\xi_1^{\pi_{5_2}}$ | $\xi_2^{\pi_{5_2}}$ | $W_3^{\pi_{5_2}}$ | $W_{124}^{\pi_{5_2}} - \xi_{12}^{\pi_{5_2}}$ |
| $\pi_{5_3} = \{\{1,3,4\},\{2\}\}$ | $\xi_1^{\pi_{5_3}}$ | $W_2^{\pi_{5_3}}$ | $\xi_3^{\pi_{5_3}}$ | $W_{134}^{\pi_{5_3}} - \xi_{13}^{\pi_{5_3}}$ |
| $\pi_{5_4} = \{\{2,3,4\},\{1\}\}$ | $W_1^{\pi_{5_4}}$ | $\xi_2^{\pi_{5_4}}$ | $\xi_3^{\pi_{5_4}}$ | $W_{234}^{\pi_{5_4}} - \xi_{23}^{\pi_{5_4}}$ |

$i \in S$ such that $S \in \pi$, the $i$th component of the CIS-value is determined by

$$\xi_i^{\pi} = \hat{W}_i + \frac{W_S^{\pi} - \sum\limits_{j \in S} \hat{W}_j}{|S|}, \tag{1.17}$$

where $\hat{W}_i$ is the player $i$'s payoff that she obtains by individual deviation from coalition $S$ and becomes a singleton under assumption that all other players remain in the coalitions they belong to before the deviation. Formally, $\hat{W}_i = W_i^{\pi'}$, where $\pi' = \{\pi \setminus S, S \setminus \{i\}, \{i\}\}$. The value $\hat{W}_i$ can be interpreted as the guaranteed payoff of player $i$ if she decides to act individually while all other players do not change the coalitions they belong to.

**Proposition 1.6.** *In the game defined by* (1.1)–(1.3)*, when the players' payoffs are transferable and defined by the CIS-value given by* (1.17)*, there exists a stable coalition structure if* $\beta\mu^2 \leq 1.9549$*. Moreover,*

1. *If* $\beta\mu^2 \in (0, 0.125]$*, then coalition structure* $\pi_2$ *is Nash stable;*
2. *If* $\beta\mu^2 \in [0.125, 0.333)$*, then coalition structures* $\pi_{5_3}$ *and* $\pi_{5_4}$ *are Nash stable;*
3. *If* $\beta\mu^2 \in [0.333, 1.2071)$*, then coalition structures* $\pi_{3_1}$*,* $\pi_{4_1}$*,* $\pi_{5_3}$*, and* $\pi_{5_4}$ *are Nash stable;*
4. *If* $\beta\mu^2 \in [1.2071, 1.9549]$*, then coalition structure* $\pi_{3_1}$ *is Nash stable.*

5. *If* $\beta\mu^2 \in (1.9549, +\infty)$, *there is no Nash stable coalition structure.*

*Proof.* We substitute the values of $W_i^\pi$ for any $i \in M$ and $\pi$ from Table 1.1 into formula (1.17), then we verify if the conditions of Definition 1.1 are satisfied. We come to the following conclusions:

1. The coalition structures $\pi_1, \pi_{3_2}, \pi_{3_3}, \pi_{3_4}, \pi_{3_5}, \pi_{3_6}, \pi_{4_2}, \pi_{4_3}$ are never Nash stable, which can immediately be identified by comparing players' payoffs under given structures and their payoffs when they deviate.

2. The coalition structure $\pi_2$ is Nash stable if

$$8\beta\mu^2 - 1 \leq 0,$$

   and taking into account that $\beta, \mu \geq 0$, we obtain that it is equivalent to the condition:

$$\beta\mu^2 \in (0, 0.125].$$

   It holds true below and for a red curve in the cyan area drawn in Fig. 1.1.

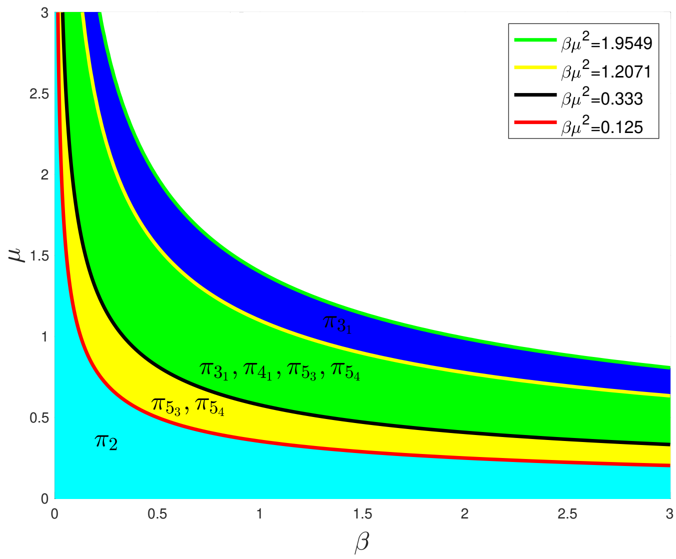


Figure 1.1: Stable coalition structures from Proposition 1.6 for colored areas: cyan for $\pi_2$; yellow for $\pi_{5_3}$, $\pi_{5_4}$; green for $\pi_{3_1}$, $\pi_{4_1}$, $\pi_{5_3}$, $\pi_{5_4}$, and blue for $\pi_{3_1}$

3. The coalition structure $\pi_{3_1}$ is Nash stable if

$$1 + 5\beta\mu^2 + 5\beta^2\mu^4 - 4\beta^3\mu^6 \geq 0,$$
$$3\beta\mu^2 - 1 \geq 0,$$

which is equivalent to condition

$$\beta\mu^2 \in [0.333, 1.9549].$$

It holds true in the green and blue areas, and their borders in Fig. 1.1.

4. The coalition structure $\pi_{4_1}$ is Nash stable if

$$1 + 4\beta\mu^2 - 4\beta^2\mu^4 \geq 0,$$
$$3\beta\mu^2 - 1 \geq 0,$$

which is equivalent to condition

$$\beta\mu^2 \in [0.333, 1.2071].$$

It holds true in the green area and its border represented in Fig. 1.1.

5. The coalition structures $\pi_{5_1}, \pi_{5_2}$ are Nash stable when the three inequalities

$$156\beta^3\mu^6 + 80\beta^2\mu^4 + 7\beta\mu^2 - 1 \leq 0,$$
$$3\beta\mu^2 - 1 \leq 0,$$
$$8\beta\mu^2 - 1 \geq 0,$$

hold true. However, the system has no solution.

6. The coalition structures $\pi_{5_3}, \pi_{5_4}$ are Nash stable when the two inequalities

$$1 + 4\beta\mu^2 - 4\beta^2\mu^4 \leq 0,$$
$$8\beta\mu^2 - 1 \geq 0,$$

hold true. This system is equivalent to the condition

$$\beta\mu^2 \in [0.125, 1.2071],$$

which holds true in the yellow and green areas, and their borders shown in Fig. 1.1.

Therefore, for any values of $\beta > 0$ and $\mu > 0$, such that $\beta\mu^2 \leq 1.9549$, there always exists at least one Nash stable scenario. □

We should notice that in the case of nontransferable payoffs, the least polluted coalition structure $\pi_2$ (see Table 1.2) is always unstable. However, coalition structure

$\pi_2$ can be stable when the players' payoffs are transferable through the CIS-value as indicated in Proposition 1.6. Therefore, it is reasonable to adopt a mechanism of payment transfers through the CIS-value if one has an incentive to make the most environmentally friendly scenario stable. The scheme of transfers is also adopted for environmental agreements between developing and developed countries in a dynamic context in [33, 64, 89].

### 1.4.2 Transition taxation mechanism

In this section, we propose another mechanism to make a coalition structure Nash or individually stable by implementing transition costs when a player deviates from the coalition she belongs to. Let us describe this mechanism through an example. Consider a coalition structure $\pi_{5_1}$ which is actually not stable, but the authority or government is willing to make it stable. We prove in Proposition 1.4 and 1.6 that $\pi_{5_1}$ is not Nash stable either with nontransferable payoffs, or transferable payoffs with adopted transfers based on the CIS-value, we can involve taxation mechanism for the stability of this scenario. We define a uniform taxation on players' payoffs in case of a deviation from scenario $\pi = \{B_1, \dots, B_m\}$ as follows:

$$T^{\pi} = \max_{i \in N} \left\{ \frac{\left[ \max\limits_{\pi'} \hat{W}_i^{\pi'} - \hat{W}_i^{\pi} \right]^+}{\hat{W}_i^{\pi}} \right\} \times 100\%, \tag{1.18}$$

where $\pi' = \{B(i) \backslash \{i\}, B_j \cup \{i\}, \pi_{-\{B(i), B_j\}}\}$; $\hat{W}_i^{\pi'}$ is a payoff to player $i$ under scenario $\pi'$ after adopting transfers if needed; operator $[A]^+$ equals $A$ if $A \geq 0$, and equals zero otherwise.

We give some remarks about the value of taxes $T^{\pi}$ applied for any scenario $\pi$ given by (1.18) and the proposed taxation mechanism in general:

1. The difference $\max\limits_{\pi'} \hat{W}_i^{\pi'} - \hat{W}_i^{\pi}$ is the maximal value that player $i$ can obtain by individual deviation from the coalition she belongs to. As in Definition 1.1, we assume that player $i$ can join any coalition in $\pi$ or become a singleton. We can also determine the taxation mechanism in case of defining individual stability given in Definition 1.2 in a similar way.

2. If scenario $\pi$ is stable, then $T^{\pi}$ equals zero.

3. Formula (1.18) gives a value of a uniform taxation, i.e., a unique tax is applied for any player for a given scenario $\pi$. We can also provide a scheme of individual

taxation defining different taxes to different players depending on how much they can benefit from deviations. In the latter case, we need to delete maximum in formula (1.18).

4. Formula (1.18) is written for the case of positive payoffs $\hat{W}_i^\pi$ given in the denominator.

5. An authority or coordination center is needed to implement the taxation scheme. This mechanism can be involved in a contract determining the players' payoffs and taxes when players are realizing a particular scenario.

Now we give a definition of stability of a scenario $\pi$ with transferable payoffs and an applied taxation scheme.

**Definition 1.3.** *A coalition structure $\pi = \{B_1, \ldots, B_m\}$ is stable in the game with transferable payoffs and an applied taxation scheme if for any player $i \in N$ it holds that*

$$\hat{W}_i^\pi \geq \hat{W}_i^{\pi'} - \frac{T^\pi}{100}\hat{W}_i^\pi \quad \text{for all} \quad \pi' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-\{B(i),B_j\}}\},$$

*where $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-\{B(i),B_j\}} = \pi\backslash\{B(i), B_j\}$, and $\hat{W}^\pi$ and $\hat{W}^{\pi'}$ denote the vectors of players' payoffs after adopting transfers under coalition structures $\pi$ and $\pi'$ respectively.*

**Proposition 1.7.** *If the taxes for any player $i$ and given scenario $\pi$ are equal to $T^\pi$ given by formula* (1.18)*, then scenario $\pi$ is stable in terms of Definition 1.3.*

*Proof.* The result immediately follows from substituting the values of taxes given by (1.18) into stability condition in Definition 1.3. □

**Example 1.1.** *Let the parameters of the game satisfied the inequalities given in (1.15) and (1.16) be as follows:*

$$\alpha_1 = 10, \ \alpha_2 = 9, \ \alpha_3 = 4, \ \alpha_4 = 3,$$
$$\beta = 1, \ \mu = 0.4, \ \delta = 0.2, \ S_0 = 3.$$

*Consider scenario $\pi_{5_1} = \{\{1, 2, 3\}, \{4\}\}$ which is not stable that is proved in Proposition 1.6. We define the taxes to the players in this scenario by formula* (1.18)*. For the given parameters, $\beta\mu^2 = 0.16$, and by Proposition 1.6, coalition structures $\pi_{5_3} = \{\{1, 3, 4\}, \{2\}\}$ and $\pi_{5_4} = \{\{2, 3, 4\}, \{1\}\}$ are stable after adopting*

*CIS-values. In Table 1.5, we provide the payoffs to the players after adopting the CIS-value in* $\pi_{5_1}$ *and their payoffs obtained by individual deviations from* $\pi_{5_1}$*. For example, player 1 can deviate to scenario* $\pi_{4_3}$ *if she joins player* 4*, or to scenario* $\pi_{3_4}$ *if she becomes a singleton. The corresponding payoffs to player 1 in these deviations are* 17.9602 *and* 19.1340*. Similar calculations can be done for other players. Then, the uniform tax assigned to each player in scenario* $\pi_{5_1}$ *is* $T^{\pi_{5_1}} = 4.24\%$ *according to (1.18).*

Table 1.5: Players' payoffs in $\pi_{5_1} = \{\{1,2,3\},\{4\}\}$ and in possible deviations

| | $\hat{W}_i^{\pi_{5_1}}$ | $\{\hat{W}_i^{\pi'}, \forall \pi'\}$ | $\left[\max_{\pi'} \hat{W}_i^{\pi'} - \hat{W}_i^{\pi}\right]^+$ |
|---|---|---|---|
| Player 1 | 18.7418 | $\{17.9602; 19.1340\}$ | 0.3922 |
| Player 2 | 9.2418 | $\{8.4602; 9.6340\}$ | 0.3922 |
| Player 3 | 8.7816 | $\{8.0; 8.0\}$ | 0 |
| Player 4 | 4.5 | $\{4.2790\}$ | 0 |

### 1.4.3 Designing the set of feasible coalition structures

In Section 1.4.1, we showed that in case of transferable payoffs we can define the system of transfers based on any cooperative solution. After realization of this system of transfers, we obtain new stable coalition structures in comparison with a non-transferable case. Nevertheless, some "desirable" coalition structures may be still unstable. In this section, we provide a mechanism of designing the set of feasible coalition structures by defining the set of restricted coalitions. When there are restrictions on some coalition formations, the deviations after which these coalitions are formed cannot be materialized. Therefore, the corresponding players' deviations are also blocked meaning that more scenarios are stable. Knowing profitable deviations we can design the set of restricted coalitions to prevent undesirable deviations.

In this section, we introduce a feasible coalition structure by defining a set of restricted coalitions $\Omega = \{R_1, \ldots, R_\ell\}$, where $R_j \subset N$ for $j = 1, \ldots, \ell$. Then we can define a feasible coalition structure or feasible scenario $\pi = \{B_1, \ldots, B_m\}$ such that $\pi \cap \Omega = \varnothing$, $B_1 \cup \cdots \cup B_m = N$ and $B_i \cap B_j = \varnothing$ for all $i, j = 1, \ldots, m, i \neq j$. Now we define a stable feasible coalition structure by implementing restrictions on players' deviations from the current scenario, i.e., a player cannot individually deviate to the unfeasible scenario. Therefore, the set of players' possible deviations is restricted with respect to the one given in Definition 1.1 when all player's deviations can be materialized.

**Definition 1.4.** *A feasible coalition structure* $\pi = \{B_1, \ldots, B_m\}$ *is Nash stable (or simply, stable) in the game with restricted cooperation if for any player* $i \in N$ *it holds that*

$$W_i^{\pi} \geq W_i^{\pi'} \text{ for all } \pi' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-\{B(i),B_j\}}\}, \text{ such that } \{B(i)\backslash\{i\}, B_j \cup \{i\}\} \cap \Omega = \varnothing,$$

*where* $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-\{B(i),B_j\}} = \pi \setminus \{B(i), B_j\}$, *and* $W^{\pi}$, $W^{\pi'}$ *denote the vectors of players' payoffs under coalition structures* $\pi$ *and* $\pi'$ *respectively.*

We can also define individually stable feasible coalition structures assuming that in the condition from Definition 1.2 only deviations implying feasible scenarios are possible.

**Definition 1.5.** *A feasible coalition structure* $\pi = \{B_1, \ldots, B_m\}$ *is individually stable if for any player* $i \in N$ *it holds that*

$$W_i^{\pi} \geq W_i^{\pi''} \text{ for all } \pi'' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-\{B(i),B_j\}}\} \text{ such that } \{B(i) \setminus \{i\}, B_j \cup \{i\}\} \cap \Omega = \varnothing \text{ and } W_k^{\pi''} \geq W_k^{\pi} \text{ for all } k \in B_j,$$

*where* $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-\{B(i),B_j\}} = \pi \setminus \{B(i), B_j\}$, *and* $W^{\pi}$, $W^{\pi''}$ *denote the vectors of players' payoffs under the coalition structures* $\pi$ *and* $\pi''$ *respectively.*

We give some remarks about stable and individually stable feasible scenarios and the proposed mechanism of implementing restricted coalitions to make initially unstable scenarios stable:

1. We should notice that if coalition $S$ is restricted, it does not imply that coalition $S' \supset S$ is restricted. For example, if cooperation of two developing countries is restricted, $\{3, 4\} \in \Omega$, from this it does not follow that any coalition containing this set is restricted. Then, coalition $\{1, 3, 4\}$ is feasible if $\{1, 3, 4\} \notin \Omega$.

2. By adding a coalition to the set of restricted coalitions $\Omega$, we decrease the number of feasible scenarios and the number of possible deviations from a feasible scenario. Following Definitions 1.4 and 1.5, a player cannot deviate to the unfeasible scenario. This positively influences the stability of the scenario as less deviations from this scenario can be materialized.

3. To implement the mechanism of feasible scenarios, an authority or coordination center is needed. The restrictions on formation of some coalitions can be implemented by the rules or laws while creating contracts supporting cooperation. We do not focus our research on the conditions when this mechanism can be technically implemented, but we provide a mathematical model of such a mechanism implementation.

We demonstrate the adoption of this mechanism when the coalition of two developing countries is restricted due to the irrationality of the joint work of the two developing countries. Therefore, $\Omega = \{\{3,4\}\}$ is the set of restricted coalitions. Therefore, the scenarios $\pi_{3_6} = \{\{3,4\},\{1\},\{2\}\}$ and $\pi_{4_1} = \{\{3,4\},\{1,2\}\}$ are not feasible. We demonstrate how the sets of stable and individually stable coalitions change by implementing this restriction for nontransferable and transferable payoff cases.

**Proposition 1.8.** *In the game of pollution control given by* (1.1)–(1.3) *with nontransferable payoffs with the set of restricted coalitions* $\Omega = \{\{3,4\}\}$, *there is no Nash-stable feasible coalition structure or scenario, but there exists a unique individually stable feasible scenario* $\pi_{3_1} = \{\{1,2\},\{3\},\{4\}\}$.

*Proof.* The proof follows from Propositions 1.4 and 1.5 by verifying conditions given in Definitions 1.4 and 1.5. □

Now we provide the conditions for stability of feasible scenarios when the players' payoffs are transferable and defined by the CIS-value.

**Proposition 1.9.** *In the game defined by* (1.1)–(1.3), *when the players' payoffs are transferable and defined by the CIS-value given by* (1.17), *there always exists a stable feasible coalition structure. Moreover,*

1. *if* $\beta\mu^2 \in (0, 0.125]$, *then feasible coalition structure* $\pi_2$ *is stable;*

2. *if* $\beta\mu^2 \in [0.125, 0.333) \cup (1.9549, +\infty)$, *then feasible coalition structures* $\pi_{5_3}$ *and* $\pi_{5_4}$ *are stable;*

3. *if* $\beta\mu^2 \in [0.333, 1.9549]$, *then feasible coalition structures* $\pi_{3_1}$, $\pi_{5_3}$, *and* $\pi_{5_4}$ *are stable.*

*Proof.* We substitute the values of $W_i^{\pi}$ for any $i \in M$ and $\pi$ from Table 1.1 into formula (1.17), then we verify if the conditions of Definition 1.8 are satisfied. We come to the following conclusions:

1. The feasible coalition structures $\pi_1, \pi_{3_2}, \pi_{3_3}, \pi_{3_4}, \pi_{3_5}, \pi_{4_2}, \pi_{4_3}$ are never stable, which immediately follows from comparison of the players' payoffs under given structures and their payoffs when they deviate.

2. The feasible coalition structure $\pi_2$ is stable if

$$8\beta\mu^2 - 1 \leq 0,$$

and it is equivalent to the condition

$$\beta\mu^2 \in (0, 0.125],$$

taking into account that $\beta, \mu \geq 0$. This condition holds true below and for a red curve in Fig. 1.2, i.e., in the blue area.

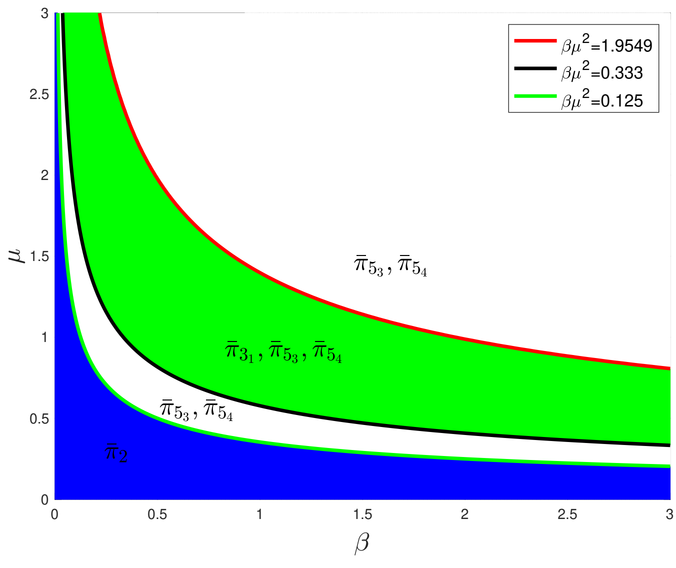


Figure 1.2: Stable feasible coalition structures from Proposition 1.9: blue for $\pi_2$; white for $\pi_{5_3}$, $\pi_{5_4}$; green for $\pi_{3_1}$, $\pi_{5_3}$, $\pi_{5_4}$

3. The feasible coalition structure $\pi_{3_1}$ is stable if

$$1 + 5\beta\mu^2 + 5\beta^2\mu^4 - 4\beta^3\mu^6 \geq 0,$$
$$3\beta\mu^2 - 1 \geq 0,$$

which is equivalent to condition

$$\beta\mu^2 \in [0.333, 1.9549].$$

It holds true in the green area and its borders in Fig. 1.2.

4. The feasible coalition structures $\pi_{5_1}, \pi_{5_2}$ are stable when the three inequalities

$$156\beta^3\mu^6 + 80\beta^2\mu^4 + 7\beta\mu^2 - 1 \le 0,$$
$$3\beta\mu^2 - 1 \le 0,$$
$$8\beta\mu^2 - 1 \ge 0,$$

hold true. However, this system has no solution.

5. The feasible coalition structures $\pi_{5_3}, \pi_{5_4}$ are Nash stable when the inequality

$$8\beta\mu^2 - 1 \ge 0,$$

holds true. This system is equivalent to the condition

$$\beta\mu^2 \in [0.125, \infty),$$

which is satisfied in the white and green areas shown in Fig. 1.2.

Therefore, for any values of $\beta > 0$ and $\mu > 0$, there always exists at least one Nash stable feasible scenario. □

We compare stable scenarios for a transferable payoff case when there are no restricted coalitions (Proposition 1.6) and when coalition $\{3, 4\}$ is restricted (Proposition 1.9):

- In the case of restricted coalition $\{3, 4\}$, there always exists a stable scenario, while with no restrictions on cooperation, for $\beta\mu^2 > 1.9549$ there is no stable coalition structure. With the restriction, in this interval scenarios $\pi_{5_3}$ and $\pi_{5_4}$ are stable.

- The range of parameters, where cooperative scenario $\pi_2$ is stable, does not change, it is when $\beta\mu^2 \in (0, 0.125]$. It can be easily explained by the fact that coalition $\{3, 4\}$ cannot be formed from $\pi_2$ by individual deviations.

- There are changes in the set of stable scenarios when $\beta\mu^2 \in [1.2071, 1.9549]$. With adding a restriction on formation of coalition $\{3, 4\}$, the scenarios $\pi_{5_3}$ and $\pi_{5_4}$ become stable, while they are not stable for the case without restrictions on the coalition formation.

If one is aiming to make a particular scenario stable in case it is not stable by Definition 1.1, even after making transfer payments defined in Section 1.4.1, the mechanism of designing the set of restricted coalitions or the set of feasible scenarios may help. We need to verify the profitable deviations and then design the set of restricted coalitions to prevent players from deviations. This can definitely be done with the presence of an authority or a regulator.

## 1.5 Conclusion to Chapter 1

In this chapter, the research focuses on the investigation of stability analysis in a static model of pollution control. In this static model, we show the analysis for the case of four players so that the coalition structure formed by them may be nontrivial, i.e., we may examine the partial cooperation where multiple coalitions can be generated.

After enumerating all possible coalition structures, the Nash stability and individual stability are given for a static model. When the utilities are nontransferable, we find that there is no Nash stable coalition structure, and only two coalition structures are individually stable. Therefore, we turn to the case of transferable payoffs in order to achieve much more environment-friendly stable coalition structures.

In total, three mechanisms are proposed to make some particular coalition structures stable. The payment transfer scheme is the most common technique. In this research, we use the CIS-value to redistribute the payoffs among all players and the result indicates that the grand coalition which generates the least pollution can be Nash stable under some conditions. As for the taxation scheme, we are manually building some obstacles for players to stay in the original coalition structures. The uniform tax applied to the players is determined as the the highest benefit that a player can achieve by deviation. Finally, the set of feasible coalitions is designed to further eliminate some desirable coalition formations. In this way, some coalition structures have a higher possibility to be stable. The examples are given for each mechanism for better understanding.

# Chapter 2

# Stable Agreements in Dynamic Games of Pollution Control

In this chapter, we investigate the design and existence of stable self-enforcing agreements in dynamic games of pollution reduction compared with static games discussed in the first chapter. In comparison to cooperative and noncooperative solutions, we firstly propose a trade-off mechanism [89], i.e., a contract, in which two players are generally behaving cooperatively but with less restriction on the coordination of their actions. Later, a Nash stability and individual stability approach is defined for identifying stable self-enforcing agreements for more than two players in terms of coalition structures [88].

## 2.1 Trade-off mechanism approach

In this section, we assume that the set of players $N = \{1, 2\}$ consists of two countries: developed and developing. The asymmetry between two countries is represented by their nonequivalent vulnerabilities to the pollution problem, i.e., the players are of two types: player 1 is a vulnerable player (developed country), and player 2 is an invulnerable player (developing country).

Following the model represented in [33, 64], the dynamics of the pollution stock $S$ are given by

$$\dot{S}(t) = \mu \sum_{i \in N} e_i(t) - \varepsilon S(t), \; S(0) = S_0, \tag{2.1}$$

where $e_i(t)$ denotes the quantity of emissions generated by player $i$, $\mu > 0$ is the marginal influence on pollution accumulation of the players' emissions, and $\varepsilon > 0$ is the nature's absorption rate.

Vulnerable and invulnerable players vary in their attitudes to the pollution reduction policy. The reactions to different attitudes are indicated in the model with adjustment of damage-cost item, i.e., an invulnerable player maximizes her payoff given by

$$\max_{e_2>0} W_2 = \int_0^\infty e^{-\rho t}(\alpha_2 e_2 - \frac{1}{2}e_2^2)dt, \tag{2.2}$$

whereas the objective function of a vulnerable player goes as

$$\max_{e_1>0} W_1 = \int_0^\infty e^{-\rho t}(\alpha_1 e_1 - \frac{1}{2}e_1^2 - \frac{1}{2}\beta_1 S^2)dt, \tag{2.3}$$

where $\rho > 0$ is a discount rate, and $\alpha_i, \beta_i$ are positive constants. The damage cost $\frac{1}{2}\beta_1 S^2$ is missed in (2.2), which conveys that an invulnerable player does not make efforts for pollution reduction on nature.

We consider three possible scenarios in terms of players' behavior/willingness to cooperation. In a noncooperative scenario, both players individually maximize their profits. Such a behavior is not environmentally friendly, i.e., does not overcome the pollution issue. In a cooperative scenario, players maximize their joint profit, which allows to address environmental problem, i.e. to reduce the pollution stock and to achieve the largest joint payoff. There are several problems concerning the realization of a cooperative scenario, and among them: (i) how to allocate fairly the joint profit, and (ii) how to achieve a full cooperative behavior, especially, when full coordination of the players' behavior is questionable. In the paper, we consider the third scenario, in which the cooperation is different from a fully cooperative scenario, but it is carried through a trade-off mechanism (see [24]), usually used in supply-chain coordination [25, 56]. This mechanism is a form of cooperative behavior proposed to find an efficient solution to mitigate the pollution damage, but it does not require full coordination of players' behavior over time. In the proposed trade-off mechanism, although two players are still acting by maximizing their own profits, there is a trade of pollution disposal between them. A vulnerable player compensates an invulnerable player's costs on involving her in the production reduction by transferring the share of her profits to the latter.

### 2.1.1 Equilibria under different scenarios

In this section, we examine the Nash equilibria in a two-player differential game under a noncooperative scenario, find the solution of the players' joint optimization problem in a cooperative scenario and obtain the Nash equilibrium strategies in a trade-off mechanism scenario.

**Noncooperaitve scenario**

Under a noncooperative scenario, the two players behave as singletons individually maximizing their profits given by (2.2) and (2.3) subject to the state dynamics (2.1).

**Proposition 2.1.** *Assuming an interior solution, in a noncooperative scenario, the feedback-Nash equilibrium in two-player differential game defined by objective functions (2.2) and (2.3) s.t. (2.1), is given by*

$$e_1^{nc}(t) = \alpha_1 + \mu(x_{nc}S^{nc}(t) + y_{nc}),$$
$$e_2^{nc}(t) = \alpha_2,$$

*where*

$$x_{nc} = \frac{\rho + 2\varepsilon - \sqrt{(\rho+2\varepsilon)^2 + 4\mu^2\beta_1}}{2\mu^2} < 0,$$
$$y_{nc} = \frac{\mu(\alpha_1+\alpha_2)x_{nc}}{\rho + \varepsilon - \mu^2 x_{nc}} < 0,$$
$$z_{nc} = \frac{(\alpha_1 + \mu y_{nc})^2 + 2\mu y_{nc}\alpha_2}{2\rho}.$$

*The corresponding equilibrium state trajectory is*

$$S^{nc}(t) = \frac{\mu(\alpha_1+\alpha_2) + \mu^2 y_{nc}}{\mu^2 x_{nc} - \varepsilon}(e^{(\mu^2 x_{nc}-\varepsilon)t} - 1) + e^{(\mu^2 x_{nc}-\varepsilon)t}S_0.$$

*The steady state stock of emissions is*

$$S_\infty^{nc} = \frac{\mu(\alpha_1+\alpha_2)(\rho+\varepsilon)}{(\varepsilon - \mu^2 x_{nc})(\rho + \varepsilon - \mu^2 x_{nc})},$$

*which is globally asymptotically stable when* $\mu^2 x_{nc} - \varepsilon < 0$.

*The Nash equilibrium players' payoffs are*

$$V_1^{nc} = \frac{1}{2}x_{nc}S_0^2 + y_{nc}S_0 + z_{nc},$$
$$V_2^{nc} = \frac{\alpha_2^2}{2\rho}.$$

*Proof.* See [33]. □

**Cooperative scenario**

In a cooperative scenario, the two players jointly maximize their total payoff, i.e., they solve the following optimization problem:

$$\max_{\substack{e_i \geq 0 \\ i\in N}} \sum_{i\in N} W_i(e_1, e_2),$$

subject to the state dynamics (2.1), and the players' payoff functions are given by (2.2) and (2.3).

**Proposition 2.2.** *Assuming an interior solution, in a cooperative scenario, the optimal feedback strategies in two-player differential game defined by objective functions (2.2) and (2.3) s.t. (2.1), are given by*

$$e_i^c(t) = \alpha_i + \mu(xS^c(t) + y), \qquad i = 1, 2,$$

*where*

$$\begin{aligned} x &= \frac{2\varepsilon + \rho - \sqrt{(2\varepsilon + \rho)^2 + 8\mu^2\beta_1}}{4\mu^2} < 0, \\ y &= \frac{(\alpha_1 + \alpha_2)\mu x}{\rho + \varepsilon - 2x\mu^2} < 0, \\ z &= \frac{(\alpha_1 + \mu y)^2 + (\alpha_2 + \mu y)^2}{2\rho}. \end{aligned}$$

*The corresponding cooperative state trajectory is*

$$S^c(t) = \frac{\mu(\alpha_1 + \alpha_2) + 2\mu^2 y}{2\mu^2 x - \varepsilon}(e^{(2\mu^2 x - \varepsilon)t} - 1) + e^{(2\mu^2 x - \varepsilon)t} S_0.$$

*The steady state stock of emissions is*

$$S_\infty^c = \frac{\mu(\alpha_1 + \alpha_2)(\rho + \varepsilon)}{(\varepsilon - 2\mu^2 x)(\rho + \varepsilon - 2\mu^2 x)},$$

*which is globally asymptotically stable when* $2\mu^2 x - \varepsilon < 0$.

*The joint players' payoff is*

$$V_{12}^c = \frac{1}{2}xS_0^2 + yS_0 + z.$$

*Proof.* See [33]. □

### Trade-off mechanism scenario

In this section, we represent the third scenario, in which players cooperate by making an agreement about the trade-off mechanism of payments/costs over time. The mechanism assumes that players agree on two parameters: (i) the compensation coefficient $0 < \tau < 1$ showing the profit share given by a vulnerable player to an invulnerable player for persuading the latter to deal with pollution problem, (ii) the cost coefficient $0 < \theta < 1$ indicating the magnitude of pollution amount which an invulnerable player should be responsible for. In this case, an invulnerable player's payoff function takes the form:

$$\max_{e_2>0} W_2 = \int_0^{\infty} e^{-\rho t}\Big(\alpha_2 e_2(t) + \tau\alpha_1 e_1(t) - \frac{1}{2}e_2^2(t) - \frac{1}{2}\beta_1\theta S^2(t)\Big)dt, \qquad (2.4)$$

while a vulnerable player's payoff function is

$$\max_{e_1>0} W_1 = \int_0^{\infty} e^{-\rho t}\Big((1-\tau)\alpha_1 e_1(t) - \frac{1}{2}e_1^2(t) - \frac{1}{2}\beta_1(1-\theta)S^2(t)\Big)dt. \qquad (2.5)$$

The parameters $(\tau, \theta)$ can be interpreted as a contract between two players and can be negotiated. In this formulation, we do consider these parameters as exogenously given, but one can assume them as decision variables of the players in a negotiation process. Obviously, the feedback-Nash equilibrium significantly depends on the values of $(\tau, \theta)$.

In the trade-off mechanism scenario, the two players individually maximize their own profits similar to what we have been described in a noncooperative scenario. However, the objective functions (2.4) and (2.5) are dependent not only on the state variable but also on the decision variables $e_1$, $e_2$.

**Proposition 2.3.** *Assuming an interior solution, in a trade-off mechanism scenario, the feedback-Nash equilibrium in a two-player differential game defined by objective functions (2.4) and (2.5) s.t. (2.1), is given by*

$$e_1^{ToM}(t) = \alpha_1(1-\tau) + \mu(x_1 S^{ToM}(t) + y_1),$$
$$e_2^{ToM}(t) = \alpha_2 + \mu(x_2 S^{ToM}(t) + y_2),$$

*where $x_1$, $x_2$, $y_1$, and $y_2$ are the solutions of the system of the equations* (2.12) *given in the proof.*

*The corresponding equilibrium state trajectory is*

$$S^{ToM}(t) = \frac{\mu B + \mu^2 y_{12}}{\mu^2 x_{12} - \varepsilon}(e^{(\mu^2 x_{12}-\varepsilon)t} - 1) + e^{(\mu^2 x_{12}-\varepsilon)t} S_0, \qquad (2.6)$$

*where $x_{12} = x_1 + x_2$, $y_{12} = y_1 + y_2$, and $B = \alpha_1(1-\tau) + \alpha_2$.*

*The steady state stock of emissions is*

$$S_{\infty}^{ToM} = \frac{\mu B + \mu^2 y_{12}}{\varepsilon - \mu^2 x_{12}}, \qquad (2.7)$$

*which is globally asymptotically stable when $\mu^2 x_{12} - \varepsilon < 0$.*

*The Nash equilibrium players' payoffs are*

$$V_1^{ToM} = \frac{1}{2}x_1 S_0^2 + y_1 S_0 + z_1,$$
$$V_2^{ToM} = \frac{1}{2}x_2 S_0^2 + y_2 S_0 + z_2,$$

*where $z_1$ and $z_2$ are defined in the proof.*

*Proof.* The optimization problem for each player is

$$W_1^{ToM} = \int_0^\infty e^{-\rho t}\Big(\alpha_1 e_1(t)(1-\tau) - \frac{1}{2}e_1^2 - \frac{1}{2}\beta_1(1-\theta)S^2(t)\Big)dt \to \max_{e_1\geq 0}, \tag{2.8}$$

$$W_2^{ToM} = \int_0^\infty e^{-\rho t}\Big(\alpha_2 e_2(t) + \tau(\alpha_1 e_1(t)) - \frac{1}{2}e_2^2(t) - \frac{1}{2}\beta_1\theta S^2(t)\Big)dt \to \max_{e_2\geq 0}. \tag{2.9}$$

Assuming the linear-quadratic form of the value functions $V_1(S) = \frac{1}{2}x_1 S^2 + y_1 S + z_1$ and $V_2(S) = \frac{1}{2}x_2 S^2 + y_2 S + z_2$, we write down the HJB equations for (2.8) and (2.9):

$$\rho V_1(S) = \max_{e_1}\Big\{\alpha_1 e_1(1-\tau) - \frac{1}{2}e_1^2 - \frac{1}{2}\beta_1(1-\theta)S^2 + V_1'(S)[\mu(e_1+e_2) - \varepsilon S]\Big\}, \tag{2.10}$$

$$\rho V_2(S) = \max_{e_2}\Big\{\alpha_2 e_2 + \tau(\alpha_1 e_1) - \frac{1}{2}e_2^2 - \frac{1}{2}\beta_1\theta S^2 + V_2'(S)[\mu(e_1+e_2) - \varepsilon S]\Big\}. \tag{2.11}$$

Maximizing the expression in RHS in (2.10), we obtain that $e_1 = \alpha_1 + \mu V_1'(S)$, and maximizing the expression in RHS in (2.11), we obtain that $e_2 = \alpha_2 + \mu V_2'(S)$. Taking into account the derivatives $V_1'(S) = x_1 S + y_1$, $V_2'(S) = x_2 S + y_2$, and substituting these expressions into (2.10), we obtain an equation:

$$\begin{aligned}\rho\Big(\frac{1}{2}x_1 S^2 + y_1 S + z_1\Big) &= \alpha_1(1-\tau)[\alpha_1(1-\tau) + \mu(x_1 S + y_1)] - \\ &- \frac{1}{2}[\alpha_1(1-\tau) + \mu(x_1 S + y_1)]^2 - \frac{1}{2}\beta_1(1-\theta)S^2 + \\ &+ (x_1 S + y_1)\Big(\mu[\alpha_1(1-\tau) + \alpha_2 + \mu(x_1 S + y_1 + x_2 S + y_2)] - \varepsilon S\Big).\end{aligned}$$

Taking into account the derivative $V_2'(S) = x_2 S + y_2$, and substituting the expressions into (2.11), we obtain an equation:

$$\begin{aligned}\rho\Big(\frac{1}{2}x_2 S^2 + y_2 S + z_2\Big) &= \alpha_2[\alpha_2 + \mu(x_2 S + y_2)] + \\ &+ \tau\alpha_1[\mu(x_1 S + y_1) + \alpha_1(1-\tau)] - \frac{1}{2}[\alpha_2 + \mu(x_2 S + y_2)]^2 - \frac{1}{2}\beta_1\theta S^2 + \\ &+ (x_2 S + y_2)\Big(\mu[\mu(x_1 S + y_1) + \alpha_1(1-\tau) + \alpha_2 + \mu(x_2 S + y_2)] - \varepsilon S]\Big).\end{aligned}$$

By identification, two linear quadratic equations containing $x_1, x_2$ can be written as

$$\mu^2 x_1^2 + 2\mu^2 x_1 x_2 - 2\varepsilon x_1 - \rho x_1 - \beta_1(1-\theta) = 0,$$
$$\mu^2 x_2^2 + 2\mu^2 x_1 x_2 - 2\varepsilon x_2 - \rho x_2 - \beta_1\theta = 0.$$

Rewriting these equations which should be solved to find $x_1$ and $x_2$, and summarizing with the rest of equations, we obtain the system:

$$\begin{cases} 3\mu^4 x_1^4 - 4\mu^2(2\varepsilon+\rho)x_1^3 + \big((2\varepsilon+\rho)^2 + 6\mu^2\beta_1\theta - 2\mu^2\beta_1\big)x_1^2 - (1-\theta)^2\beta_1^2 = 0, \\ 3\mu^4 x_2^4 - 4\mu^2(2\varepsilon+\rho)x_2^3 + \big((2\varepsilon+\rho)^2 - 6\mu^2\beta_1\theta + 4\beta_1\mu^2\big)x_2^2 - \beta_1^2\theta^2 = 0, \\ y_1 = \dfrac{\mu^3 x_1[(x_2 B + \tau\alpha_1 x_1)A - \mu^2 x_1 x_2 B]}{A(A^2 - \mu^4 x_1 x_2)} - \dfrac{\mu x_1 B}{A}, \\ y_2 = \dfrac{\mu^3 x_1 x_2 B - \mu(x_2 B + \tau\alpha_1 x_1)A}{A^2 - \mu^4 x_1 x_2}, \\ z_1 = \dfrac{2\mu y_1 B + \alpha_1^2(1-\tau)^2 + \mu^2 y_1^2 + 2\mu^2 y_1 y_2}{2\rho}, \\ z_2 = \dfrac{2\mu y_2 B + \alpha_2^2 + 2\alpha_1^2\tau(1-\tau) + \mu^2 y_2^2 + 2\mu^2 y_1 y_2 + \tau\alpha_1\mu y_1}{2\rho}, \end{cases} \tag{2.12}$$

where $A = \mu^2 x_1 + \mu^2 x_2 - \rho - \varepsilon$ and $B = \alpha_1(1-\tau) + \alpha_2$.

In the system (2.12), we need to solve the first two equations, then substituting $x_1$ and $x_2$ into the rest four equations we find $y_1$, $y_2$, $z_1$, and $z_2$. We should notice that we require that $x_1, x_2$ be negative to prove the stability of the steady state.

The expression of the equilibrium stock $S^{ToM}(t)$ is obtained as a solution of equation (2.1) and it is given by (2.6). If $t$ tends to infinity in (2.1), we obtain the steady state of emission stock given by (2.7), which globally asymptotically stable when $\mu^2 x_{12} - \varepsilon < 0$. □

### 2.1.2 Comparison of scenarios

In this section, we investigate the performance of trade-off mechanism with various set values ($\tau$, $\theta$) by comparing the pollution stock, the players' strategies and payoffs under this scenario with noncooperative and cooperative scenarios. We are interested in finding a set of values ($\tau$, $\theta$) which both players are keen on accepting the trade-off mechanism, i.e., their profits in this scenario are larger than they could obtain in a noncooperative case.

Table 2.1: Types of the set $(\tau, \theta)$ (trade-off mechanism vs noncooperation)

| | Vul. Player 1 | Invul. Player 2 | Steady State |
|---|---|---|---|
| (i) Profitable for invulnerable player | $W_1^{ToM} < W_1^{nc}$ | $W_2^{ToM} \geq W_2^{nc}$ | $S_\infty^{ToM} < S_\infty^{nc}$ |
| (ii) Profitable for vulnerable player | $W_1^{ToM} \geq W_1^{nc}$ | $W_2^{ToM} < W_2^{nc}$ | $S_\infty^{ToM} < S_\infty^{nc}$ |
| **(iii) Profitable for both players** | $W_1^{ToM} \geq W_1^{nc}$ | $W_2^{ToM} \geq W_2^{nc}$ | $S_\infty^{ToM} < S_\infty^{nc}$ |
| (iv) Nonprofitable for both players | $W_1^{ToM} < W_1^{nc}$ | $W_2^{ToM} < W_2^{nc}$ | $S_\infty^{ToM} < S_\infty^{nc}$ |
| (v) Not acceptable | — | — | $S_\infty^{ToM} \geq S_\infty^{nc}$ |

**Noncooperative scenario vs trade-off mechanism**

In the differential game described above, the set of parameters $(\tau, \theta)$, where $\tau \in (0,1)$, $\theta \in (0,1)$, defining the trade-off mechanism can be classified into five types by comparing players' payoffs and the steady-state emission level in noncooperative and trade-off mechanism scenarios: (i) profitable for invulnerable player when only invulnerable player gets a larger payoff in a trade-off mechanism scenario than in a noncooperative one, and the steady-state stock in a trade-off mechanism scenario is lower than in a noncooperative one; (ii) profitable for vulnerable player, when only vulnerable player benefits from a trade-off mechanism scenario with respect to noncooperation, and the steady-state stock with the trade-off mechanism is lower than in noncooperation; (iii) profitable for both players, i.e., both players will obtain higher payoffs adopting the trade-off mechanism; (iv) nonprofitable for both players, namely, this set is not profitable for both players, but the steady-state stock with the trade-off mechanism is lower than in noncooperation; (v) not acceptable, i.e., the two players generate more pollution than in a noncooperative scenario.

Apparently, both players accept the set $(\tau, \theta)$ if and only if both of them will benefit from it and the steady-state pollution stock is less than in a noncooperative scenario. The types of $(\tau, \theta)$ and the corresponding inequalities for the profits and the steady state are given in Table 2.1.

It is impossible to verify the inequalities in Table 2.1 in a general-form game, thus we demonstrate these types on a numerical example in Section 2.1.3.

**Cooperative scenario vs trade-off mechanism**

In this section, we compare the players' payoffs and steady-state pollution stock in the trade-off mechanism and cooperative scenarios. Repeating the same classifica-

Table 2.2: Types of the set $(\tau, \theta)$ (trade-off mechanism vs cooperation)

| | Vul. Player 1 | Invul. Player 2 | Steady State |
|---|---|---|---|
| (i) Profitable for invulnerable player | $W_1^{ToM} < W_1^c$ | $W_2^{ToM} \geq W_2^c$ | $S_\infty^{ToM} < S_\infty^c$ |
| (ii) Profitable for vulnerable player | $W_1^{ToM} \geq W_1^c$ | $W_2^{ToM} < W_2^c$ | $S_\infty^{ToM} < S_\infty^c$ |
| **(iii) Profitable for both players** | $W_1^{ToM} \geq W_1^c$ | $W_2^{ToM} \geq W_2^c$ | $S_\infty^{ToM} < S_\infty^c$ |
| (iv) Nonprofitable for both players | $W_1^{ToM} < W_1^c$ | $W_2^{ToM} < W_2^c$ | $S_\infty^{ToM} < S_\infty^c$ |
| (v) Not acceptable | — | — | $S_\infty^{ToM} \geq S_\infty^c$ |

tion given in Section 2.1.2, we present five types of parameters $(\tau, \theta)$ in Table 2.2, in which we compare the trade-off mechanism and a cooperative scenario. We again are interested in the subset of $(\tau, \theta)$ such that both players are beneficial from the trade-off mechanism, but we expect that we cannot find such a subset comparing this scenario with the cooperative one.

We verify the inequalities given in Table 2.2 in a numerical example in Section 2.1.3.

### 2.1.3 Numerical example

In this section, we present a numerical example to illustrate the performance of a trade-off mechanism with respect to the values of $(\tau, \theta)$. The parameters of the game are

$$\beta_1 = 1,\ \alpha_1 = 9,\ \alpha_2 = 4,$$
$$\varepsilon = 0.4,\ \mu = 0.35,\ \rho = 0.1,\ S_0 = 1.$$

As shown in Fig. 2.1, the set $(\tau, \theta)$ is divided into five areas corresponding to the particular types described in Table 2.1. The black area represents the subset of $(\tau, \theta)$ under which the players in a trade-off mechanism pollute more than in a noncooperative scenario. The red (blue) area corresponds to the subset of $(\tau, \theta)$ when only vulnerable (involnurable) player performs better in a trade-off mechanism polluting less (in total) than in a noncooperative scenario while the green area gives the Pareto-improving values of $(\tau, \theta)$, when both players benefit from adopting a trade-off mechanism vs noncooperative scenario. The yellow area indicates that both players are not interested in a trade-off mechanism scenario while they reduce the steady-state pollution stock.

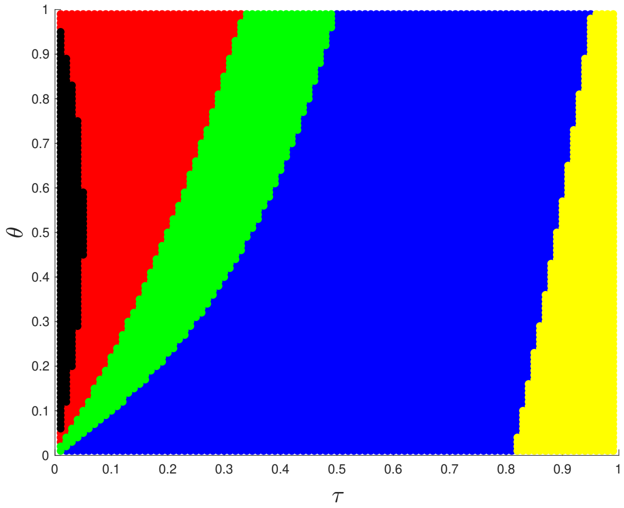


Figure 2.1: Trade-off mechanism vs noncooperative scenario. Black: type (v) in Table 2.1; red: type (ii) in Table 2.1; blue: type (i) in Table 2.1; yellow: type (iv) in Table 2.1; green: type (iii) in Table 2.1 corresponding to Pareto-improving pairs of values $(\tau, \theta)$

We should notice that for any $\tau \in (0, 0.49)$ there exists a nonempty interval for $\theta$ such that the values $(\tau, \theta)$ define a Pareto-improving trade-off mechanism for the players, i.e. both players are interested in adopting it. Although it is clear that both players are beneficial by choosing $(\tau, \theta)$ from the green area in Fig. 2.1, the question is this: how much can they improve their payoffs by a trade-off mechanism? We can calculate the players' benefits (in percentage) obtained by adopting a trade-off mechanism in comparison with a noncooperative scenario using a coefficient

$$M_i = \left| \frac{W_i^{nc} - W_i^{ToM}}{W_i^{nc}} \right| \times 100\%, \; i = 1, 2. \tag{2.13}$$

We represent some values of Pareto optimal set $(\tau, \theta)$ (from the green area in Fig. 2.1) and the improvement coefficients $M_1$ and $M_2$ for vulnerable and invulnerable players respectively in Table 2.3. In a numerical example, the maximal percentage of improvement for player 1 (vulnerable player) is 75.61% and for player 2 (invulnerable player) is 87.8%. It is expected that player 1 is more beneficial with low $\tau$ and high $\theta$, while player 2 is interested in high $\tau$.

We should notice that the trade-off mechanism is useful to outperform a cooperative scenario in terms of the pollution level (the pollution stock in the trade-off mechanism can be lower than in a cooperative scenario) as indicated in Fig. 2.2 by nonblack area. In this figure we can see the presence of three areas including a large black area in which the pollution in the trade-off mechanism is larger than in a cooperative scenario. In the blue area, the trade-off mechanism is beneficial only

Table 2.3: Benefits from adopting trade-off mechanism vs noncooperative scenario for the Pareto-improving values of $(\tau, \theta)$

| $(\tau, \theta)$ | $(0.31, 0.89)$ | $(0.49, 0.96)$ | $(0.4, 0.97)$ | $(0.34, 0.89)$ | $(0.2, 0.23)$ |
|---|---|---|---|---|---|
| $M_1$ | 75.61% | 0.08% | 40.23% | 60.22% | 1.44% |
| $M_2$ | 0.72% | 87.8% | 46.33% | 21.05% | 47.35% |

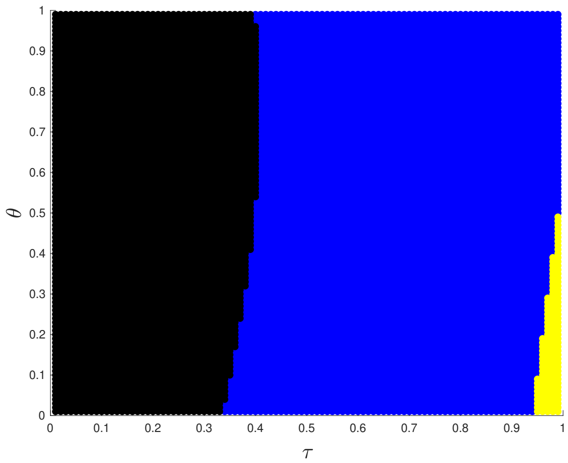


Figure 2.2: Trade-off mechanism vs cooperative scenario. Black: type (v) from Table 2.2; blue: type (i) in Table 2.2; yellow: type (iv) in Table 2.2

for an invulnerable player. But constructing Fig. 2.2, in a cooperative scenario we do not use any profit-allocation mechanism to redefine players' payoffs. Therefore, the areas may change after adopting any cooperative payoff allocation rule.

Apart from the benefit brought to the players when they adopt the trade-off mechanism, the comparisons of players' strategies and pollution stock under different scenarios are also of a particular interest. As demonstrated in Fig. 2.3, we select three representative sets $(0.31, 0.89)$, $(0.49, 0.96)$, $(0.4, 0.97)$ from the Pareto optimal set of pairs $(\tau, \theta)$. The first (second) pair of parameters benefits player 1 (player 2) at the highest level, while the third one gives more or less the same improvement to both players (40.23% for player 1 and 46.33% for player 2) as shown in Table. 2.3. In Fig. 2.3, we observe that after applying trade-off mechanism proposed above, the emission quantity or the strategy of a vulnerable player becomes almost constant (relatively horizontal line), and for an invulnerable player, the quantity of emissions has dropped a lot in comparison with the noncooperative level. More importantly, the pollution stock indicates that trade-off mechanism is capable of reducing pollution even more than a cooperative scenario (see Fig. 2.3) and brings benefits to both

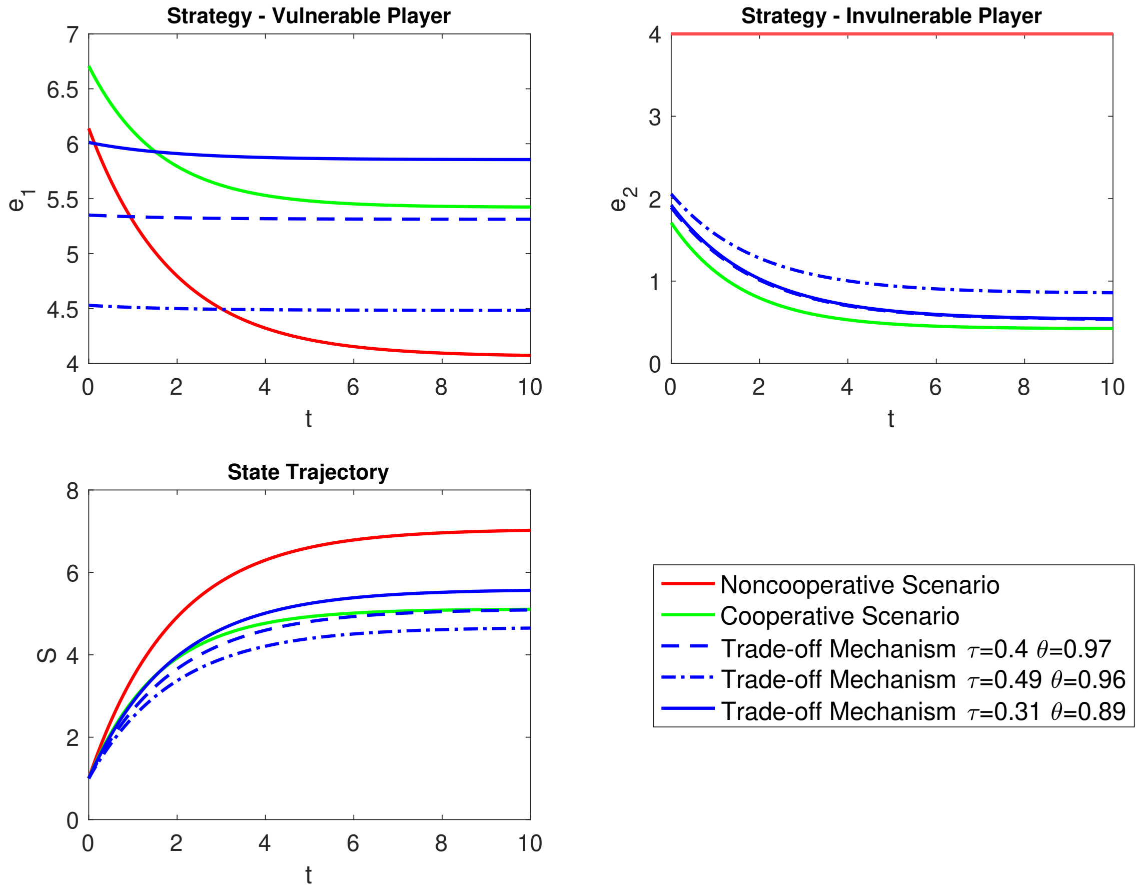


Figure 2.3: Strategies of vulnerable and invulnerable players, and state trajectory (pollution stock) under different scenarios: noncooperative, cooperative and three trade-off mechanisms with parameters $(\tau, \theta) \in \{(0.31, 0.89), (0.49, 0.96), (0.4, 0.97)\}$

players in comparison with the Nash equilibrium. For instance, when the players adopt a trade-off mechanism with parameters $(\tau, \theta) = (0.49, 0.96)$, then the emission stock is less than in a cooperative scenario, and the players' payoffs increase by 0.08% and 87.8% for player 1 and 2 respectively.

## 2.2 The Nash stability and individual stability approach of different cooperative scenarios

In this section, we consider three neighboring industries or countries called players producing goods and generating pollution that damages the environment. The set of players is $N = \{1, 2, 3\}$. The three players are asymmetric in terms of their profit functions implied by different environmental behavior with respect to emission concern. The players are of two types: $I$ is a vulnerable player (or developed country), and $II$ is a invulnerable player (or developing country). The vulnerable player has an environmental concern and follows the environmentally friendly policy. The invulnerable player does not care about the emission stock, i.e. she is following an environmentally unfriendly policy. Let player 1 be of type $II$ and players 2 and 3 are of type $I$. The set of players in terms of types is defined as $N = \{I, I, II\}$ containing two developed and one developing countries.

The countries produce goods, and this production activity generates emissions. Inspired by the model introduced in [33], we define the dynamical system of the pollution stock $S$ as

$$\dot{S}(t) = \mu \sum_{i \in N} e_i(t) - \varepsilon S(t), \ S(0) = S_0, \tag{2.14}$$

where $e_i(t)$ denotes the quantity of emissions produced by player $i$, $\mu > 0$ is the marginal influence on pollution accumulation $S$ issued by the players' emissions, and $\varepsilon > 0$ is the nature's absorption rate.

The invulnerable player is not concerned by the damage done to the environment and cares only about the revenues from her industrial activities, i.e., this player maximizes her payoff given by

$$W_i = \int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t))dt, \tag{2.15}$$

where $\rho > 0$ is the discount rate, while, due to the obvious reasons, the vulnerable players have the strength to overtake the responsibility for pollutant reduction. Therefore, the objective function of a vulnerable player goes as

$$W_i = \int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt, \tag{2.16}$$

where $\beta_i > 0$, and the last term under the integral in (2.16) represents the damage costs.

To summarize, we can say that all players have the objective function (2.16) but for player 1 parameter $\beta_1$ equals zero, so the term $\frac{1}{2}\beta_i S^2$ is omitted in the payoff function (2.15), while for players 2 and 3 parameters $\beta_2, \beta_3$ are strictly positive.

**Remark 2.1.** *In a general case, two vulnerable players 2 and 3 are asymmetric in the parameters, i.e. parameters $\alpha$ and $\beta$ are assumed to be different for these players.*

### 2.2.1 Equilibria under different scenarios

In this section, we examine the cooperative setting of the described game assuming that cooperation may be partial. Therefore, not only the grand coalition may be formed, but also smaller coalitions can be formed implying the formation of specific coalition structures. The following theorems provide the conditions of the Nash equilibria in the three-player differential game when the objectives of the players given by (2.16) are subject to the state dynamics (2.14) under different coalition partitions. We consider the possible coalition structures or scenarios:

1. *Noncooperative scenario*, $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$;
2. *Cooperative scenario*, $\pi_2 = \{\{I, I, II\}\}$;
3. *Partially cooperative scenarios*:
   (a) Case 1 (two developed countries cooperate): $\pi_3 = \{\{I, I\}, \{II\}\}$;
   (b) Case 2 (one developing and one developed country cooperate): $\pi_4 = \{\{I, II\}, \{I\}\}$. This scenario has two variants, these are $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ and $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$. Hereinafter, we refer to a general form of the coalition structure $\pi_4$ if the result is true for both structures $\pi_{4_1}$ and $\pi_{4_2}$.

#### Noncooperative scenario

In this scenario, the three players act as singletons maximizing (2.16) with respect to state dynamics (2.14), i. e., the coalition structure is $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$. In the following proposition we characterize the Nash equilibrium for a noncooperative scenario.

**Proposition 2.4.** *Assuming an interior solution, in the noncooperative scenario* $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$, *the feedback-Nash equilibrium is given by*

$$e_1^{nc}(t) = \alpha_1,$$
$$e_j^{nc}(t) = \alpha_j + \mu(x_j S^{nc}(t) + y_j), \; j = 2, 3,$$

*where $x_j, y_j, z_j$ for $j = 2, 3$ satisfy the following system:*

$$
\begin{aligned}
&3\mu^4 x_j^4 - 4(2\varepsilon + \rho)\mu^2 x_j^3 + [4\mu^2\beta_{5-j} + (2\varepsilon+\rho)^2 - 2\mu^2\beta_j]x_j^2 - \beta_j^2 = 0, \ x_j < 0\\
&y_j = \frac{\mu\alpha_{123}x_j(\rho+\varepsilon-\mu^2 x_{23} + \mu^2 x_{5-j})}{(\rho+\varepsilon-\mu^2 x_{23})^2 - \mu^4 x_j x_{5-j}} < 0,\\
&z_j = \frac{\alpha_j^2 + 2\mu y_j \alpha_{123} + 2\mu^2 y_j y_{5-j}}{2\rho},
\end{aligned}
$$

*where $\alpha_{23} = \alpha_2 + \alpha_3, \alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, $x_{23} = x_2 + x_3$.*

*The corresponding equilibrium state trajectory is*

$$
S^{nc}(t) = \frac{\mu\alpha_{123} + \mu^2 y_{23}}{\mu^2 x_{23} - \varepsilon}(e^{(\mu^2 x_{23}-\varepsilon)t} - 1) + e^{(\mu^2 x_{23}-\varepsilon)t} S_0, \tag{2.17}
$$

*where $y_{23} = y_2 + y_3$.*

*The steady-state emission stock is*

$$
S_\infty^{nc} = \frac{\mu\alpha_{123} + \mu^2 y_{23}}{\varepsilon - \mu^2 x_{23}}, \tag{2.18}
$$

*which is globally asymptotically stable if $\mu^2 x_{23} - \varepsilon < 0$.*

*Proof.* First, the objective of player 1 does not depend on the stock variable and we can easily obtain that the maximal value of her objective is reached when $e_1 = \alpha_1$.

Second, players 2 and 3 are of type $I$, we take the second player to illustrate calculations. The player 2's optimization problem is

$$
W_2^{\pi_1} = \int_0^\infty e^{-\rho t}(\alpha_2 e_2(t) - \frac{1}{2}e_2^2(t) - \frac{1}{2}\beta_2 S^2(t))dt \to \max_{e_2(t)\geq 0}. \tag{2.19}
$$

Assuming the linear-quadratic form of the value functions $V_2(S) = \frac{1}{2}x_2 S^2 + y_2 S + z_2$ and $V_3(S) = \frac{1}{2}x_3 S^2 + y_3 S + z_3$, we write down the HJB equation for (2.19), which looks like

$$
\rho V_2(S) = \max_{e_2}\Big\{\big(\alpha_2 e_2 - \frac{1}{2}e_2^2 - \frac{1}{2}\beta_2 S^2\big) + V_2'(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \tag{2.20}
$$

Maximizing the expression in the RHS of equation (2.20), we obtain $e_2 = \alpha_2 + \mu V_2'(S)$, and the corresponding strategy for player 3 is $e_3 = \alpha_3 + \mu V_3'(S)$. Taking into account the derivatives $V_j'(S) = x_j S + y_j, j = 2, 3$, and substituting these expressions into (2.20), we obtain an equation:

$$
\begin{aligned}
\rho\Big(\frac{1}{2}x_2 S^2 + y_2 S + z_2\Big) &= \frac{1}{2}[\alpha_2 + \mu(Sx_2 + y_2)]^2 + \mu\alpha_1(x_2 S + y_2) +\\
&+ \mu(x_2 S + y_2)[\alpha_3 + \mu(x_3 S + y_3)] - \frac{1}{2}\beta_2 S^2 - \varepsilon S(x_2 S + y_2).
\end{aligned}
$$

By identification, two linear quadratic equations containing $x_2, x_3$ can be written as

$$\mu^2 x_2^2 - (2\varepsilon+\rho)x_2 + 2\mu^2 x_2 x_3 - \beta_2 = 0,$$
$$\mu^2 x_3^2 - (2\varepsilon+\rho)x_3 + 2\mu^2 x_2 x_3 - \beta_3 = 0.$$

Correspondingly, the expressions for $x_2$ and $x_3$ are the solutions of the following equations:

$$3\mu^4 x_2^4 - 4(2\varepsilon+\rho)\mu^2 x_2^3 + [4\mu^2\beta_3 + (2\varepsilon+\rho)^2 - 2\mu^2\beta_2]x_2^2 - \beta_2^2 = 0, \tag{2.21}$$
$$3\mu^4 x_3^4 - 4(2\varepsilon+\rho)\mu^2 x_3^3 + [4\mu^2\beta_2 + (2\varepsilon+\rho)^2 - 2\mu^2\beta_3]x_3^2 - \beta_3^2 = 0. \tag{2.22}$$

We cannot write an explicit solution of the system of (2.21) and (2.22) with respect to $x_2, x_3$, therefore, we introduce this system with new variable values:

$$\begin{aligned}
y_2 &= \frac{\mu\alpha_{123}x_2(\rho+\varepsilon-\mu^2 x_{23}+\mu^2 x_3)}{(\rho+\varepsilon-\mu^2 x_{23})^2 - \mu^4 x_2 x_3},\\
y_3 &= \frac{\mu\alpha_{123}x_3(\rho+\varepsilon-\mu^2 x_{23}+\mu^2 x_2)}{(\rho+\varepsilon-\mu^2 x_{23})^2 - \mu^4 x_2 x_3},\\
z_2 &= \frac{\alpha_2^2 + 2\mu y_2\alpha_{123} + 2\mu^2 y_2 y_3}{2\rho},\\
z_3 &= \frac{\alpha_3^2 + 2\mu y_3\alpha_{123} + 2\mu^2 y_2 y_3}{2\rho},
\end{aligned}$$

where $x_{23} = x_2 + x_3$.

The global stability of the steady state requires $\mu^2 x_{23} - \varepsilon < 0$, therefore the negative roots $x_2$ and $x_3$ from (2.21) and (2.22) are chosen. Subsequently, the expression of the equilibrium stock $S^{nc}(t)$ is obtained as a solution of (2.14) and it is given by formula (2.17). The steady state of stock is shown in (2.18) by setting the equation in (2.14) equal to zero. □

In the noncooperative scenario, the invulnerable player has a constant emission level equal to the coefficient of her linear profits from production, and her instant payoff is equal to $\frac{\alpha_1^2}{2}$ with the whole-game payoff $\frac{\alpha_1^2}{2\rho}$. Obviously, the equilibrium strategy of the invulnerable player in noncooperative scenario is larger than her strategy in cooperative scenario and partially cooperative scenario, when she cooperates with a vulnerable player. This will be proved in Section 3.3.

**Cooperative scenario**

In the cooperative scenario, the coalition structure is $\pi_2 = \{\{I, I, II\}\}$, and the three players coordinate their strategies to maximize their total payoff, that is

$$\max_{\substack{e_i \geq 0 \\ i \in N}} \sum_{i \in N} W_i(e_1, e_2, e_3), \tag{2.23}$$

subject to the state dynamics (2.14) with initial condition $S(0) = S_0$. The subsequent proposition demonstrates the necessary conditions for the optimal solution.

**Proposition 2.5.** *Assuming an interior solution, in the cooperative scenario, when $\pi_2 = \{I, I,\ II\}$, the players' optimal feedback strategies are given by*

$$e_i^c(t) = \alpha_i + \mu(x_c S^c(t) + y_c),\ \ i \in N, \tag{2.24}$$

*where*

$$\begin{aligned}
x_c &= \frac{2\varepsilon + \rho - \sqrt{(2\varepsilon + \rho)^2 + 12\mu^2\beta_{123}}}{6\mu^2} < 0, \\
y_c &= \frac{\mu x_c \alpha_{123}}{\rho + \varepsilon - 3\mu^2 x_c} < 0, \\
z_c &= \frac{\sum_{i=1}^{3}(\alpha_i + \mu y_c)^2}{2\rho},
\end{aligned}$$

*and* $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_{123} = \beta_1 + \beta_2 + \beta_3$.

*The cooperative state trajectory is*

$$S^c(t) = \frac{\mu\alpha_{123} + 3\mu^2 y_c}{3\mu^2 x_c - \varepsilon}(e^{(3\mu^2 x_c - \varepsilon)t} - 1) + e^{(3\mu^2 x_c - \varepsilon)t} S_0.$$

*The steady-state emission stock is*

$$S_\infty^c = \frac{(\rho + \varepsilon)\mu\alpha_{123}}{(\varepsilon - 3\mu^2 x_c)(\rho + \varepsilon - 3\mu^2 x_c)},$$

*which is globally asymptotically stable if* $3\mu^2 x_c - \varepsilon < 0$.

*Proof.* In the cooperative scenario, since all players jointly maximize the total profit (2.23), the optimization problem is given by

$$\begin{aligned}
W^{\pi_2} &= W_1^{\pi_2} + W_2^{\pi_2} + W_3^{\pi_2} \\
&= \sum_{i=1}^{3} \int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{\substack{e_i \geq 0 \\ i \in N}}.
\end{aligned} \tag{2.25}$$

To solve an optimization problem (2.25), the HJB equation can be written as

$$\rho V_c(S) = \max_{e_1,e_2,e_3} \Big\{ \sum_{i=1}^{3}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V_c'(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \quad (2.26)$$

Maximizing the expression in the RHS of equation (2.26), we write the first-order condition and find the optimal control $e_i = \alpha_i + \mu V_c'(S)$. Assuming the linear-quadratic form of $V_c(S)$, we set $V_c(S) = \frac{1}{2}x_c S^2 + y_c S + z_c$. Then substituting the corresponding variables in (2.26) brings

$$\rho\Big(\frac{1}{2}x_c S^2 + y_c S + z_c\Big) = \frac{1}{2}[\alpha_1 + \mu(x_c S + y_c)]^2 - \frac{1}{2}(\beta_1+\beta_2+\beta_3)S^2 - \varepsilon S(x_c S + y_c) + \frac{1}{2}[\alpha_2 + \mu(x_c S + y_c)]^2 + \frac{1}{2}[\alpha_3 + \mu(x_c S + y_c)]^2.$$

By the procedure of identification, we obtain the system of equations:

$$\begin{aligned} & 3\mu^2 x_c^2 - (2\varepsilon + \rho)x_c - \beta_{123} = 0, \\ & y_c = \frac{\mu x_c \alpha_{123}}{\rho + \varepsilon - 3\mu^2 x_c}, \\ & z_c = \frac{\sum_{i=1}^{3}(\alpha_i + \mu y_c)^2}{2\rho}. \end{aligned} \quad (2.27)$$

Since the global stability of the steady state is always satisfied under $3\mu^2 x_c - \varepsilon < 0$, then we take the negative root from (2.27), that is

$$x_c = \frac{2\varepsilon + \rho - \sqrt{(2\varepsilon+\rho)^2 + 12\mu^2\beta_{123}}}{6\mu^2}.$$

Substituting all necessary variables into (2.14), and solving this differential equation, we obtain the expression for $S^c(t)$ and $S^c_\infty$ as presented in the proposition. □

We should notice that the strategy of invulnerable player 1, that is $e_1^c(t)$, is less than her strategy in a noncooperative case $e_1^{nc}(t)$ for any $t$. It follows from negativity of coefficients $x_c$ and $y_c$ in (2.24) and the form of $e_1^{nc}(t)$, which is equal to $\alpha_1$ for any $t > 0$. Obviously, the payoff of player 1 in a cooperative scenario is less than in a noncooperative one. Therefore, it is not profitable for an invulnerable player to cooperate with vulnerable players if they do not compensate switching from noncooperative to cooperative behavior to player 1 (see [33]). The system of transfers between three players to make their cooperation stable is discussed in Section 5.

**Partially cooperative scenarios**

**Case 1: $\{\{I,I\},\{II\}\}$.** In this subsection, we examine the equilibrium behavior of the players under partial cooperation. We start with the scenario when coalition structure $\{\{I,I\},\{II\}\}$ is formed, in which two vulnerable players form a coalition, while the invulnerable player acts as a singleton. For convenience, we instantiate the coalition structure in this case as $\{\{1\},\{2,3\}\}$. The objective of coalition $\{2,3\}$ is given by

$$\max_{e_2,e_3} \sum_{i=2}^{3} W_i(e_1,e_2,e_3)$$

constrained to the objective function (2.16) of the invulnerable player and the state dynamics (2.14).

**Proposition 2.6.** *Assuming an interior solution, under partially cooperative scenario with coalition structure $\pi_3 = \{\{1\},\{2,3\}\}$, the feedback-Nash equilibrium is given by*

$$\begin{aligned} e_1^{pc_1}(t) &= \alpha_1, \\ e_2^{pc_1}(t) &= \alpha_2 + \mu(x_{c_1} S^{pc_1}(t) + y_{c_1}), \\ e_3^{pc_1}(t) &= \alpha_3 + \mu(x_{c_1} S^{pc_1}(t) + y_{c_1}), \end{aligned}$$

*where*

$$\begin{aligned} x_{c_1} &= \frac{2\varepsilon + \rho - \sqrt{(2\varepsilon+\rho)^2 + 8\mu^2\beta_{23}}}{4\mu^2} < 0, \\ y_{c_1} &= \frac{\mu x_{c_1}\alpha_{123}}{\rho + \varepsilon - 2\mu^2 x_{c_1}} < 0, \\ z_{c_1} &= \frac{(\alpha_2 + \mu y_{c_1})^2 + (\alpha_3 + \mu y_{c_1})^2 + 2\mu y_{c_1}\alpha_1}{2\rho}, \end{aligned}$$

*and $\beta_{23} = \beta_2 + \beta_3$.*

*The corresponding Nash equilibrium trajectory under partially cooperation scenario (case 1) is*

$$S^{pc_1}(t) = \frac{\mu\alpha_{123} + 2\mu^2 y_{c_1}}{2\mu^2 x_{c_1} - \varepsilon}(e^{(2\mu^2 x_{c_1} - \varepsilon)t} - 1) + e^{(2\mu^2 x_{c_1} - \varepsilon)t} S_0.$$

*The steady-state emission stock is*

$$S_\infty^{pc_1} = \frac{(\rho+\varepsilon)\mu\alpha_{123}}{(\varepsilon - 2\mu^2 x_{c_1})(\rho + \varepsilon - 2\mu^2 x_{c_1})},$$

*which is globally asymptotically stable if $2\mu^2 x_{c_1} - \varepsilon < 0$.*

*Proof.* Since the invulnerable player 1 maximizes her own payoff, she behaves in the same way as in the noncooperative scenario since her objective does not depend on the stock variable. Next, we consider the payoff function of a coalition of the two other players, who are vulnerable players, and this coalition solves the problem:

$$W^{\pi_3} = W_2^{\pi_3} + W_3^{\pi_3} = \sum_{i=2}^{3} \int_0^{\infty} e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{e_2,e_3}.$$

The HJB equation in this case is as follows:

$$\rho V_{c_1}(S) = \max_{e_2,e_3}\Big\{\sum_{i=2}^{3}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V'_{c_1}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \quad (2.28)$$

Maximizing the expression in the RHS of equation (2.28), we obtain strategies: $e_j = \alpha_j + \mu V'_{c_1}(S), j = 2, 3$. Assuming a linear-quadratic form of $V_{c_1}$, i.e., $V_{c_1}(S) = \frac{1}{2}x_{c_1}S^2 + y_{c_1}S + z_{c_1}$, we get:

$$\rho\Big(\frac{1}{2}x_{c_1}S^2 + y_{c_1}S + z_{c_1}\Big) = \frac{1}{2}[\alpha_2 + \mu(Sx_{c_1} + y_{c_1})^2 + \frac{1}{2}[\alpha_3 + \mu(Sx_{c_1} + y_{c_1}]^2 +$$
$$+ \mu\alpha_1(x_{c_1}S + y_{c_1}) - \frac{1}{2}\beta_{23}S^2 - \varepsilon S(x_{c_1}S + y_{c_1}).$$

By identification, we obtain

$$\begin{aligned} &2\mu^2 x_{c_1}^2 - (2\varepsilon + \rho)x_{c_1} - \beta_{23} = 0, \quad (2.29)\\ &y_{c_1} = \frac{\mu x_{c_1}\alpha_{123}}{\rho + \varepsilon - 2\mu^2 x_{c_1}},\\ &z_{c_1} = \frac{(\alpha_2 + \mu y_{c_1})^2 + (\alpha_3 + \mu y_{c_1})^2 + 2\mu y_{c_1}\alpha_1}{2\rho}. \end{aligned}$$

Here we also need to take a negative root of $x_{c_1}$ from (2.29) for satisfying the global stability of the solution, thus $x_{c_1} = \frac{2\varepsilon+\rho-\sqrt{(2\varepsilon+\rho)^2+8\mu^2\beta_{23}}}{4\mu^2}$, then $S^{pc_1}(t)$ is obtained as a solution of (2.14) with initial condition $S(0) = S_0$. □

**Case 2:** $\{\{I, II\}, \{I\}\}$**.** In this section, we analyze a partially cooperation scenario when the coalition between a developed country and a developing country is formed, and the other developed country acts as a singleton. We suppose that an invulnerable player 1 and a vulnerable player 2 play cooperatively. Then the optimization problem for a coalition $\{1, 2\}$ is

$$\max_{e_1,e_2}\sum_{i=1}^{2} W_i(e_1, e_2, e_3),$$

where the payoff function $W_i, i = 1, 2$ is given by (2.15) and (2.16) respectively.

**Proposition 2.7.** *Assuming an interior solution, in the partially cooperative scenario (case 2) with coalition structure* $\pi_{4_1} = \{\{1,2\},\{3\}\}$*, the feedback-Nash equilibrium is given by*

$$\begin{aligned} e_1^{pc_2}(t) &= \alpha_1 + \mu(x_{c_2} S^{pc_2}(t) + y_{c_2}), \\ e_2^{pc_2}(t) &= \alpha_2 + \mu(x_{c_2} S^{pc_2}(t) + y_{c_2}), \\ e_3^{pc_2}(t) &= \alpha_3 + \mu(x_{3_{c2}} S^{pc_2}(t) + y_{3_{c2}}), \end{aligned} \tag{2.30}$$

*where* $x_{c_2}, x_{3_{c_2}}, y_{c_2}, y_{3_{c_2}}, z_{c_2}, z_{3_{c_2}}$ *satisfy the following system*

$$\begin{aligned} &12\mu^4 x_{c_2}^4 - 8(2\varepsilon+\rho)\mu^2 x_{c_2}^3 + \Big((2\varepsilon+\rho)^2 + 4\mu^2\beta_3 - 4\mu^2\beta_2\Big)x_{c_2}^2 - \beta_2^2 = 0, \\ &3\mu^4 x_{3_{c_2}}^4 - 4(2\varepsilon+\rho)\mu^2 x_{3_{c_2}}^3 + \Big((2\varepsilon+\rho)^2 + 8\mu^2\beta_2 - 2\mu^2\beta_3\Big)x_{3_{c_2}}^2 - \beta_3^2 = 0, \\ &y_{c_2} = \frac{\mu\alpha_{123} x_{c_2}(\rho+\varepsilon-2\mu^2 x_{c_2})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2} x_{3_{c_2}}}, \\ &y_{3_{c_2}} = \frac{\mu\alpha_{123} x_{c_2}(\rho+\varepsilon-\mu^2 x_{3_{c_2}})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2} x_{3_{c_2}}}, \\ &z_{c_2} = \frac{\alpha_1^2 + \alpha_2^2 + 2\mu y_{c_2}(\alpha_{123} + \mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho}, \\ &z_{3_{c_2}} = \frac{\alpha_3^2 + \mu y_{3_{c_2}}(2\alpha_{123} + 4\mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho}. \end{aligned}$$

*The corresponding state trajectory is given by*

$$S^{pc_2}(t) = \frac{\mu\alpha_{123} + \mu^2(2y_{c_2} + y_{3_{c_2}})}{\mu^2(2x_{c_2} + x_{3_{c_2}}) - \varepsilon}\left(e^{[\mu^2(2x_{c_2}+x_{3_{c_2}})-\varepsilon]t} - 1\right) + e^{[\mu^2(2x_{c_2}+x_{3_{c_2}})-\varepsilon]t} S_0.$$

*The steady-state emission stock is*

$$S_\infty^{pc_2} = \frac{\mu\alpha_{123} + \mu^2 y_{3_{c_2}} + 2\mu^2 y_{c_2}}{\varepsilon - 2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}}},$$

*which is globally asymptotically stable if* $\mu^2(2x_{c_2} + x_{3_{c_2}}) - \varepsilon < 0$*.*

*Proof.* We consider $\{\{I, II\},\{I\}\}$, in which player 3 acts as a singleton. There are two optimization problems to solve. First, for the coalition of players 1 and 2, we formulate their joint optimization problem:

$$W^{\pi_{4_1}} = W_1^{\pi_{4_1}} + W_2^{\pi_{4_1}} = \sum_{i=1}^{2}\int_0^\infty e^{-\rho t}(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i S^2(t))dt \to \max_{e_1,e_2}.$$

Player 3 aims to maximize

$$W_3^{\pi_{4_1}} = \int_0^\infty e^{-\rho t}(\alpha_3 e_3(t) - \frac{1}{2}e_3^2(t) - \frac{1}{2}\beta_3 S^2(t))dt \to \max_{e_3}.$$

Following the method of previous cases, we write the HJB equation:

$$\rho V_{c_2}(S) = \max_{e_1,e_2}\Big\{\sum_{i=1}^{2}(\alpha_i e_i - \frac{1}{2}e_i^2 - \frac{1}{2}\beta_i S^2) + V'_{c_2}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}, \quad (2.31)$$

$$\rho V_{3_{c_2}}(S) = \max_{e_3}\Big\{(\alpha_3 e_3 - \frac{1}{2}e_3^2 - \frac{1}{2}\beta_3 S^2) + V'_{3_{c_2}}(S)[\mu(e_1+e_2+e_3) - \varepsilon S]\Big\}. \quad (2.32)$$

We infer that $V_{c_2}(S) = \frac{1}{2}x_{c_2}S^2 + y_{c_2}S + z_{c_2}$, $V_{3_{c_2}}(S) = \frac{1}{2}x_{3_{c_2}}S^2 + y_{3_{c_2}}S + z_{3_{c_2}}$, consequently, the optimal strategies can be constructed as

$$e_j(t) = \alpha_j + \mu(x_{c_2}S(t) + y_{c_2}), \quad j \in 1,2,$$
$$e_3(t) = \alpha_3 + \mu(x_{3_{c_2}}S(t) + y_{3_{c_2}}).$$

Then by substituting $V_{c_2}(S), V'_{c_2}(S), V_{3_{c_2}}(S), V'_{3_{c_2}}(S)$ into (2.31) and (2.32), we obtain

$$\rho\Big(\frac{1}{2}x_{c_2}S^2 + y_{c_2}S + z_{c_2}\Big) = \frac{1}{2}[\alpha_1 + \mu(Sx_{c_2} + y_{c_2}]^2 + \frac{1}{2}[\alpha_2 + \mu(Sx_{c_2} + y_{c_2}]^2$$
$$+ \mu(x_{c_2}S + y_{c_2})[\alpha_3 + \mu(x_{3_{c_2}}S + y_{3_{c_2}})] - \frac{1}{2}\beta_{12}S^2 - \varepsilon S(x_{c_2}S + y_{c_2}),$$
$$\rho\Big(\frac{1}{2}x_{3_{c_2}}S^2 + y_{3_{c_2}}S + z_{3_{c_2}}\Big) = \frac{1}{2}[\alpha_3 + \mu(Sx_{3_{c_2}} + y_{3_{c_2}})]^2 - -\varepsilon S(x_{3_{c_2}}S + y_{3_{c_2}})$$
$$+ \mu(x_{3_{c_2}}S + y_{3_{c_2}})[\alpha_{12} + 2\mu(x_{c_2}S + y_{c_2})] - \frac{1}{2}\beta_3 S^2.$$

The following system is obtained by identification method:

$$12\mu^4 x_{c_2}^4 - 8(2\varepsilon+\rho)\mu^2 x_{c_2}^3 + \Big((2\varepsilon+\rho)^2 + 4\mu^2\beta_3 - 4\mu^2\beta_2\Big)x_{c_2}^2 - \beta_2^2 = 0,$$
$$3\mu^4 x_{3_{c_2}}^4 - 4(2\varepsilon+\rho)\mu^2 x_{3_{c_2}}^3 + \Big((2\varepsilon+\rho)^2 + 8\mu^2\beta_2 - 2\mu^2\beta_3\Big)x_{3_{c_2}}^2 - \beta_3^2 = 0,$$
$$y_{c_2} = \frac{\mu\alpha_{123}x_{c_2}(\rho+\varepsilon-2\mu^2 x_{c_2})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2}x_{3_{c_2}}},$$
$$y_{3_{c_2}} = \frac{\mu\alpha_{123}x_{c_2}(\rho+\varepsilon-\mu^2 x_{3_{c_2}})}{(\rho+\varepsilon-2\mu^2 x_{c_2} - \mu^2 x_{3_{c_2}})^2 - 2\mu^4 x_{c_2}x_{3_{c_2}}},$$
$$z_{c_2} = \frac{\alpha_1^2 + \alpha_2^2 + 2\mu y_{c_2}(\alpha_{123} + \mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho},$$
$$z_{3_{c_2}} = \frac{\alpha_3^2 + \mu y_{3_{c_2}}(2\alpha_{123} + 4\mu y_{c_2} + \mu y_{3_{c_2}})}{2\rho}.$$

We get the negative roots of $x_{c_2}$ and $x_{3_{c_2}}$ as usual, and obtain $S^{pc_2}(t)$ and $S^{pc_2}_\infty$ afterwards. □

**Remark 2.2.** *In Proposition 2.7, we consider the case 2 with coalition structure* $\pi_{4_1} = \{\{1,2\},\{3\}\}$ *and omit the case of coalition structure* $\pi_{4_2} = \{\{1,3\},\{2\}\}$*, for which the equilibrium can be easily found by Proposition 2.7 by replacing a vulnerable player 2 in coalition* $\{1,2\}$ *by player 3.*

We should notice that the strategy of invulnerable player 1 in a partially cooperative scenario $e_1^{pc_1}(t)$ (case 1) is equal to her strategy in a noncooperative case $e_1^{nc}(t)$. It is true as in both these cases player 1's payoff is not affected by the stock variable. In the meantime, her strategy in a partially cooperative scenario $e_1^{pc_2}(t)$ (case 2) is less than her strategy in a noncooperative case $e_1^{nc}(t)$ for any $t$. It follows from negativity of coefficients $x_{c_2}$ and $y_{c_2}$ in (2.30) and the form of $e_1^{nc}(t)$, which is equal to $\alpha_1$ for any $t > 0$. Obviously, the payoff of player 1 in partially cooperative scenario (case 2) is less than in a noncooperative one. Again, it is not profitable for an invulnerable player to cooperate with a vulnerable player if the latter does not compensate switching from noncooperative to partially cooperative behavior to player 1.

**Remark 2.3.** *The Nash equilibrium or optimal strategies obtained above need to be nonnegative, which requires the following inequalities need to be satisfied:*

$$\begin{aligned}
&\alpha_i + \mu(x_i S^{nc}(t) + y_i) \geq 0, i = 2,3,\\
&\alpha_i + \mu(x_c S^{c}(t) + y_c) \geq 0, i = 1,2,3,\\
&\alpha_i + \mu(x_{c_1} S^{pc_1}(t) + y_{c_1}) \geq 0, i = 2,3,\\
&\alpha_i + \mu(x_{c_2} S^{pc_2}(t) + y_{c_2}) \geq 0, i = 1,2,\\
&\alpha_3 + \mu(x_{3_{c2}} S^{pc_2}(t) + y_{3_{c_2}}) \geq 0,\\
&\alpha_i + \mu(x_{c_3} S^{pc_3}(t) + y_{c_3}) \geq 0, i = 1,3,\\
&\alpha_2 + \mu(x_{2_{c3}} S^{pc_3}(t) + y_{2_{c_3}}) \geq 0.
\end{aligned} \tag{2.33}$$

*And since some parameters can't be obtained in the explicit form, these inequalities can only be verified in the numerical examples.*

### 2.2.2 Identification of stable coalition structures

We examine all possible scenarios or coalition structures on stability. A stable coalition structure is a candidate to be formed from some perspective. There are different concepts of a coalition structure stability proposed for nondynamic games (see discussion in Introduction). A coalition structure $\pi = \{B_1, \ldots, B_m\}$, such that $B_1 \cup \cdots \cup B_m = N$ and $B_i \cap B_j = \varnothing$ for all $i, j = 1, \ldots, m, i \neq j$ is stable when any player does not increase her payoff if she changes this structure in an individual way. We should notice that we consider two possibilities for a deviating player: (i) she can join any possible coalition without any restrictions (see Section 2.2.2), (ii) the coalition which the deviating player would like to join can block the deviation if there exists at least one member who can lose by accepting the deviator (see Section 2.2.2). In Section 2.2.2 we examine a situation when players' payoffs are nontransferable, i.e. the payoff of any player acting as a member of a coalition is equal to her payoff defined by her initially given payoff function. The vector $W^\pi = (W_1^\pi, \ldots, W_n^\pi) \in \mathbb{R}^n$ represents the corresponding payoffs to the players in coalition structure $\pi$. We highlight that the players' payoffs are defined by the given payoff functions in the Nash equilibrium calculated for the corresponding coalition structure (see Propositions 2.4–2.7 in Section 2.2.1).

#### Nash-stable coalition structures

The first definition of a stable coalition structure assumes that all individual deviations of the players are possible.

**Definition 2.1.** *A coalition structure $\pi = \{B_1, \ldots, B_m\}$ is Nash stable (or simply, stable) if for any player $i \in N$ it holds that*

$$W_i^\pi \geq W_i^{\pi'} \text{ for all } \pi' = \{B(i) \backslash \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\},$$

*where $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-B(i) \cup B_j} = \pi \backslash \{B(i), B_j\}$, and $W^\pi$, $W^{\pi'}$ denotes the vectors of players' payoffs under coalition structures $\pi$ and $\pi'$ respectively.*

In Definition 2.1, any player can deviate from her current coalition joining another existing coalition or becoming a singleton.

**Remark 2.4.** *If the inequality in Definition 2.1 is strict, i.e., $W_i^\pi > W_i^{\pi'}$ for all $\pi' = \{B(i) \backslash \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$, the coalition structure $\pi = \{B_1, \ldots, B_m\}$ is called strictly stable.*

The following proposition characterizes the conditions of the Nash-stable coalition structures in the differential game defined by (2.14)–(2.16).

**Proposition 2.8.** *In the differential game given by* (2.14)–(2.16), *the following coalition structures or scenarios:*

*1)* $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$ *(noncooperative scenario);*

*2)* $\pi_2 = \{\{I, I, II\}\}$ *(cooperative scenario);*

*3)* $\pi_3 = \{\{I, I\}, \{II\}\}$ *(partially cooperative scenario when an invulnerable player is acting as a singleton);*

*4)* $\pi_4 = \{\{I, II\}, \{I\}\}$ *(partially cooperative scenario when vulnerable and invulnerable players cooperate), including* $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ *and* $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$

*are stable or Nash stable if and only if the corresponding conditions given in Table 2.4 are satisfied. Each row corresponds to a particular scenario.*

Table 2.4: Stability conditions for the Nash stable coalition structures

| | Invul. Player 1 | Vul. Player 2 | Vul. Player 3 |
|---|---|---|---|
| $\pi_1$ | $\begin{cases} W_1^{\pi_1} \geq W_1^{\pi_{4_1}} \\ W_1^{\pi_1} \geq W_1^{\pi_{4_2}} \end{cases}$ | $\begin{cases} W_2^{\pi_1} \geq W_2^{\pi_3} \\ W_2^{\pi_1} \geq W_2^{\pi_{4_1}} \end{cases}$ | $\begin{cases} W_3^{\pi_1} \geq W_3^{\pi_3} \\ W_3^{\pi_1} \geq W_3^{\pi_{4_2}} \end{cases}$ |
| $\pi_2$ | $W_1^{\pi_2} \geq W_1^{\pi_3}$ | $W_2^{\pi_2} \geq W_2^{\pi_{4_1}}$ | $W_3^{\pi_2} \geq W_3^{\pi_{4_2}}$ |
| $\pi_3$ | $W_1^{\pi_3} \geq W_1^{\pi_2}$ | $\begin{cases} W_2^{\pi_3} \geq W_2^{\pi_1} \\ W_2^{\pi_3} \geq W_2^{\pi_{4_1}} \end{cases}$ | $\begin{cases} W_3^{\pi_3} \geq W_3^{\pi_1} \\ W_3^{\pi_3} \geq W_3^{\pi_{4_2}} \end{cases}$ |
| $\pi_{4_1}$ | $\begin{cases} W_1^{\pi_{4_1}} \geq W_1^{\pi_1} \\ W_1^{\pi_{4_1}} \geq W_1^{\pi_{4_2}} \end{cases}$ | $\begin{cases} W_2^{\pi_{4_1}} \geq W_2^{\pi_1} \\ W_2^{\pi_{4_1}} \geq W_2^{\pi_3} \end{cases}$ | $W_3^{\pi_{4_1}} \geq W_3^{\pi_2}$ |
| $\pi_{4_2}$ | $\begin{cases} W_1^{\pi_{4_2}} \geq W_1^{\pi_1} \\ W_1^{\pi_{4_2}} \geq W_1^{\pi_{4_1}} \end{cases}$ | $W_2^{\pi_{4_2}} \geq W_2^{\pi_2}$ | $\begin{cases} W_3^{\pi_{4_2}} \geq W_3^{\pi_1} \\ W_3^{\pi_{4_2}} \geq W_3^{\pi_3} \end{cases}$ |

It is impossible to write down the conditions given in Proposition 2.8 in an explicit form. We demonstrate them on the numerical examples in Section 2.2.2.

### Individually stable coalition structures

In this section, we investigate another stability concept. Now we assume that the players in a coalition can refuse to cooperate with another player willing to join them in case this player can bring the loss in anyone's profit inside the coalition. Therefore, we give another definition of a stable coalition structure with a reasonable block of external entries.

**Definition 2.2.** *A coalition structure* $\pi = \{B_1, \ldots, B_m\}$ *is individually stable if for any player* $i \in N$ *it holds that*

$$W_i^{\pi} \geq W_i^{\pi''} \text{ for all } \pi'' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-B(i)\cup B_j}\} \text{ such that } W_k^{\pi''} \geq W_k^{\pi} \text{ for all } k \in B_j,$$

*where* $B_j \in \pi \cup \varnothing$, $B_j \neq B(i)$, $\pi_{-B(i)\cup B_j} = \pi \backslash \{B(i), B_j\}$, *and* $W^{\pi}$, $W^{\pi''}$ *denote the vectors of players' payoffs under the coalition structures* $\pi$ *and* $\pi''$ *respectively.*

Obviously, the set of individually stable coalition structures contains the set of the Nash-stable coalition structures [91].

**Remark 2.5.** *If the inequality in Definition 2 is strict, i.e.* $W_i^{\pi} > W_i^{\pi''}$ *for all* $\pi'' = \{B(i)\backslash\{i\}, B_j \cup \{i\}, \pi_{-B(i)\cup B_j}\}$, *then the coalition structure* $\pi = \{B_1, \dots, B_m\}$ *is said to be strictly individually stable.*

The following proposition characterizes the conditions of individually stable coalition structures in the differential game defined by (2.14)–(2.16).

**Proposition 2.9.** *In the differential game defined by* (2.14)–(2.16)*, the following coalition structures or scenarios:*

1) $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$ *(noncooperative scenario);*
2) $\pi_2 = \{\{I, I, II\}\}$ *(cooperative scenario);*
3) $\pi_3 = \{\{I, I\}, \{II\}\}$ *(partially cooperative scenario when an invulnerable player does not cooperate with other players);*
4) $\pi_4 = \{\{I, II\}, \{I\}\}$ *(partially cooperative scenario when vulnerable and invulnerable players cooperate), including* $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ *and* $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$

*are stable or individually stable if and only if the corresponding conditions given in Table 2.5 are satisfied. Each row corresponds to a particular scenario.*

It is impossible to write the conditions given in Proposition 2.9 in an explicit form. We demonstrate them on the numerical examples in Section 2.2.2.

#### Numerical simulations

For better understanding how to verify the stability of different scenarios, we provide two numerical examples.

In the first run, the parameters of the game satisfied the inequalities given in (2.33) are given as follows:

$$\beta_1 = 0,\ \beta_2 = 3,\ \beta_3 = 4,$$
$$\alpha_1 = 5,\ \alpha_2 = 6,\ \alpha_3 = 8,$$
$$\varepsilon = 0.6,\ \mu = 0.3,\ S_0 = 1.$$

Table 2.5: Stability conditions for individually stable coalition structures

| | Invul. Player 1 | Vul. Player 2 | Vul. Player 3 |
|---|---|---|---|
| $\pi_1$ | $\begin{cases}\left[\begin{array}{l}\begin{cases}W_2^{\pi_1} < W_2^{\pi_{4_1}}\\ W_1^{\pi_1} \geq W_1^{\pi_{4_1}}\end{cases}\\ \text{or } W_2^{\pi_1} \geq W_2^{\pi_{4_1}}\end{array}\right.\\ \left[\begin{array}{l}\begin{cases}W_3^{\pi_1} < W_3^{\pi_{4_2}}\\ W_1^{\pi_1} \geq W_1^{\pi_{4_2}}\end{cases}\\ \text{or } W_3^{\pi_1} \geq W_3^{\pi_{4_2}}\end{array}\right.\end{cases}$ | $\begin{cases}\left[\begin{array}{l}\begin{cases}W_3^{\pi_1} < W_3^{\pi_3}\\ W_2^{\pi_1} \geq W_2^{\pi_3}\end{cases}\\ \text{or } W_3^{\pi_1} \geq W_3^{\pi_3}\end{array}\right.\\ \left[\begin{array}{l}\begin{cases}W_1^{\pi_1} < W_1^{\pi_{4_1}}\\ W_2^{\pi_1} \geq W_2^{\pi_{4_1}}\end{cases}\\ \text{or } W_1^{\pi_1} \geq W_1^{\pi_{4_1}}\end{array}\right.\end{cases}$ | $\begin{cases}\left[\begin{array}{l}\begin{cases}W_2^{\pi_1} < W_2^{\pi_3}\\ W_3^{\pi_1} \geq W_3^{\pi_3}\end{cases}\\ \text{or } W_2^{\pi_1} \geq W_2^{\pi_3}\end{array}\right.\\ \left[\begin{array}{l}\begin{cases}W_1^{\pi_1} < W_1^{\pi_{4_2}}\\ W_3^{\pi_1} \geq W_3^{\pi_{4_2}}\end{cases}\\ \text{or } W_1^{\pi_1} \geq W_1^{\pi_{4_2}}\end{array}\right.\end{cases}$ |
| $\pi_2$ | $W_1^{\pi_2} \geq W_1^{\pi_3}$ | $W_2^{\pi_2} \geq W_2^{\pi_{4_1}}$ | $W_3^{\pi_2} \geq W_3^{\pi_{4_2}}$ |
| $\pi_3$ | $\left[\begin{array}{l}\begin{cases}W_2^{\pi_3} < W_2^{\pi_2}\\ W_3^{\pi_3} < W_3^{\pi_2}\\ W_1^{\pi_3} \geq W_1^{\pi_2}\end{cases}\\ \text{or } W_2^{\pi_3} \geq W_2^{\pi_2}\\ \text{or } W_3^{\pi_3} \geq W_3^{\pi_2}\end{array}\right.$ | $\begin{cases}W_2^{\pi_3} \geq W_2^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_1^{\pi_3} < W_1^{\pi_{4_1}}\\ W_2^{\pi_3} \geq W_2^{\pi_{4_1}}\end{cases}\\ \text{or } W_1^{\pi_3} \geq W_1^{\pi_{4_1}}\end{array}\right.\end{cases}$ | $\begin{cases}W_3^{\pi_3} \geq W_3^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_1^{\pi_3} < W_1^{\pi_{4_2}}\\ W_3^{\pi_3} \geq W_3^{\pi_{4_2}}\end{cases}\\ \text{or } W_1^{\pi_3} \geq W_1^{\pi_{4_2}}\end{array}\right.\end{cases}$ |
| $\pi_{4_1}$ | $\begin{cases}W_1^{\pi_{4_1}} \geq W_1^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_3^{\pi_{4_1}} < W_3^{\pi_{4_2}}\\ W_1^{\pi_{4_1}} \geq W_1^{\pi_{4_2}}\end{cases}\\ \text{or } W_3^{\pi_{4_1}} \geq W_3^{\pi_{4_2}}\end{array}\right.\end{cases}$ | $\begin{cases}W_2^{\pi_{4_1}} \geq W_2^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_3^{\pi_{4_1}} < W_3^{\pi_3}\\ W_2^{\pi_{4_1}} \geq W_2^{\pi_3}\end{cases}\\ \text{or } W_3^{\pi_{4_1}} \geq W_3^{\pi_3}\end{array}\right.\end{cases}$ | $\left[\begin{array}{l}\begin{cases}W_1^{\pi_{4_1}} < W_1^{\pi_2}\\ W_2^{\pi_{4_1}} < W_2^{\pi_2}\\ W_3^{\pi_{4_1}} \geq W_3^{\pi_2}\end{cases}\\ \text{or } W_1^{\pi_{4_1}} \geq W_1^{\pi_2}\\ \text{or } W_2^{\pi_{4_1}} \geq W_2^{\pi_2}\end{array}\right.$ |
| $\pi_{4_2}$ | $\begin{cases}W_1^{\pi_{4_2}} \geq W_1^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_2^{\pi_{4_2}} < W_2^{\pi_{4_1}}\\ W_1^{\pi_{4_2}} \geq W_1^{\pi_{4_1}}\end{cases}\\ \text{or } W_2^{\pi_{4_2}} \geq W_2^{\pi_{4_1}}\end{array}\right.\end{cases}$ | $\left[\begin{array}{l}\begin{cases}W_1^{\pi_{4_2}} < W_1^{\pi_2}\\ W_3^{\pi_{4_2}} < W_3^{\pi_2}\\ W_2^{\pi_{4_2}} \geq W_2^{\pi_2}\end{cases}\\ \text{or } W_1^{\pi_{4_2}} \geq W_1^{\pi_2}\\ \text{or } W_3^{\pi_{4_2}} \geq W_3^{\pi_2}\end{array}\right.$ | $\begin{cases}W_3^{\pi_{4_2}} \geq W_3^{\pi_1}\\ \left[\begin{array}{l}\begin{cases}W_2^{\pi_{4_2}} < W_2^{\pi_3}\\ W_3^{\pi_{4_2}} \geq W_3^{\pi_3}\end{cases}\\ \text{or } W_2^{\pi_{4_2}} \geq W_2^{\pi_3}\end{array}\right.\end{cases}$ |

Using Propositions 2.4–2.7, we can calculate the corresponding players' payoffs for all scenarios, which are represented in Table 2.6, where we highlight in bold the players' maximal payoffs among different scenarios. Verifying the Nash stability conditions

Table 2.6: Players' payoffs under different scenarios (first run)

| | Player 1 Invul. player | Player 2 Vul. player | Player 3 Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\},\{2\},\{3\}\}$ | **4.167** | 2.772 | 6.306 |
| $\pi_2 = \{\{1,2,3\}\}$ | 3.734 | **3.205** | **7.085** |
| $\pi_3 = \{\{1\},\{2,3\}\}$ | **4.167** | 2.810 | 6.581 |
| $\pi_{4_1} = \{\{1,2\},\{3\}\}$ | 4.069 | 2.976 | 6.596 |
| $\pi_{4_2} = \{\{1,3\},\{2\}\}$ | 3.995 | 3.043 | 1.994 |

given in Proposition 2.8 and individual stability conditions given in Proposition 2.9, we perceive that the current parameter setting does not meet any Nash-stable scenario, but there is a unique individually stable scenario, that is, $\pi_3 = \{\{I, I\}, \{II\}\}$, in which the developing country acts alone while two developed countries cooperate. The reason is in the concern of paying additional cost for the damage to the environment. Therefore, even if we do not define the system of transfer payments between players inside a coalition, we are able to find an individually stable scenario in the game.

We depict the players' equilibrium strategies for different scenarios in Fig. 2.4. The corresponding equilibrium state trajectories are also presented in Fig. 2.4. We can notice that the smallest (largest) pollution stock is observed with the cooperative (noncooperative) scenario corresponding to coalition structure $\pi_2$ ($\pi_1$), which is expected. It is interesting that the stock corresponding to coalition structure $\pi_3$, which is a unique individually stable scenario, is the lowest total pollution stock among other pollution stocks corresponding to the partially cooperative scenarios.

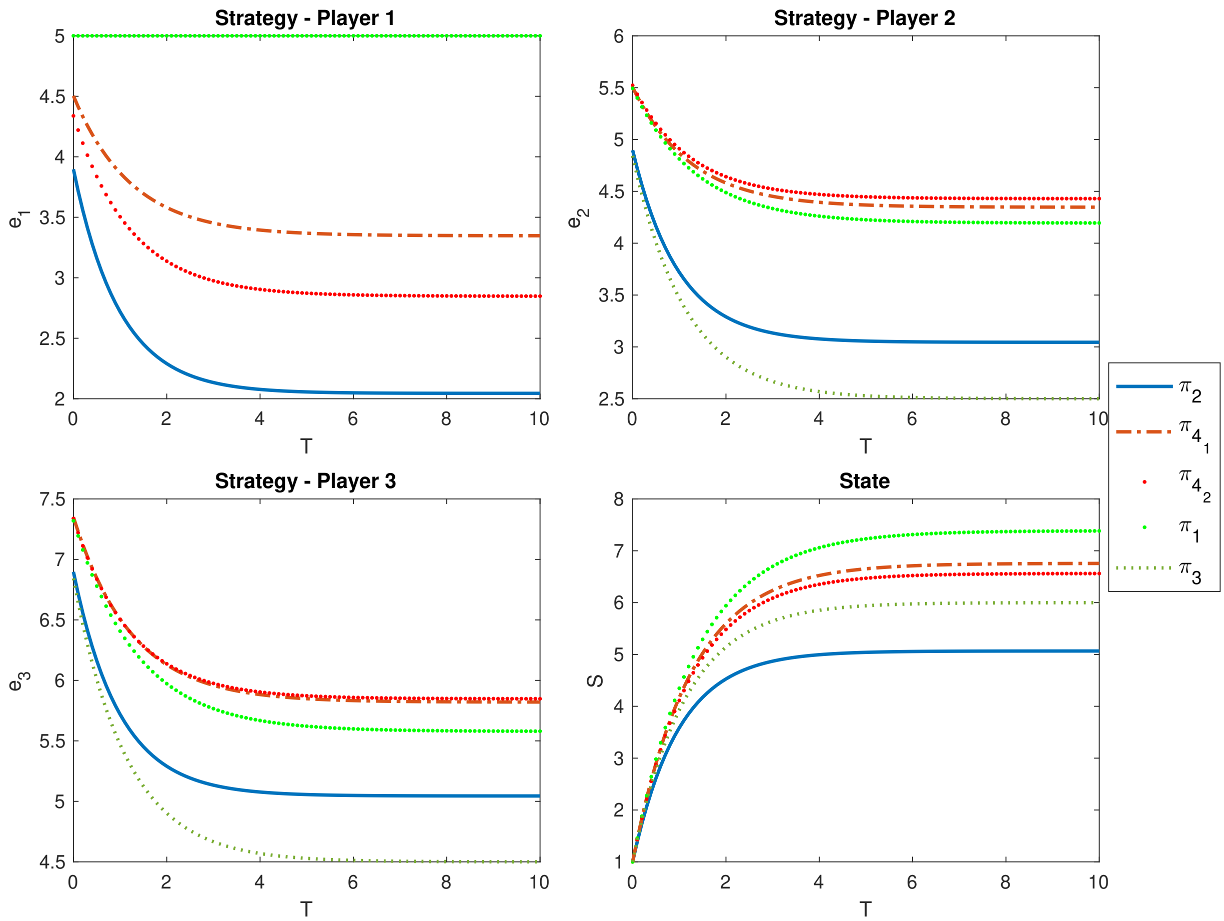


Figure 2.4: Equilibrium strategies and state trajectories under different scenarios (first run)

In the second run, the parameters of the game satisfied the inequalities given in

(2.33) are:

$$\beta_1 = 0,\ \beta_2 = 2,\ \beta_3 = 3,$$
$$\alpha_1 = 5,\ \alpha_2 = 6,\ \alpha_3 = 7,$$
$$\varepsilon = 0.6,\ \mu = 0.5,\ S_0 = 1.$$

Repeating the same calculations as in the first run, we obtain the players' payoffs which are represented in Table 2.7. The analysis of stability conditions from Propositions 2.8 and 2.9 shows that we again have no Nash stable coalition structures, but we have a unique individually stable scenario, that is, $\pi_1 = \{\{I\}, \{I\}, \{II\}\}$ when all players act alone. We should notice that it is different from the first run. We again obtain that the developing country can only get maximal payoff by acting alone, but as for the other two developed countries, in the second run the cooperation is not stable because player 2 has a profitable deviation when becoming a singleton.

Table 2.7: Players' payoffs under different scenarios (second run)

| | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | **4.167** | 2.056 | 2.029 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 3.279 | **2.859** | **3.898** |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | **4.167** | 2.048 | 2.813 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 3.965 | 2.470 | 2.736 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 3.724 | 2.672 | 1.180 |

As compared with the first run, the most distinguishable difference is recognized in the equilibrium strategy for player 3. For this player, the control trajectory in the noncooperative scenario, i.e., with stable coalition structure $\pi_1$, intersects her trajectory in the cooperative scenario at some instant of time and becomes lower after that time. As in the first run, the equilibrium emission stock in scenario $\pi_3$ is the lowest one among partially cooperative scenarios, but this scenario is not stable in this case contrary to the first run.

**Dynamically stable coalition structures**

In this section, we examine the stability of the coalition structures along the equilibrium trajectories. Assuming that a coalition structure is stable at the initial time $t = 0$, it may become unstable at some instant of time on the corresponding state

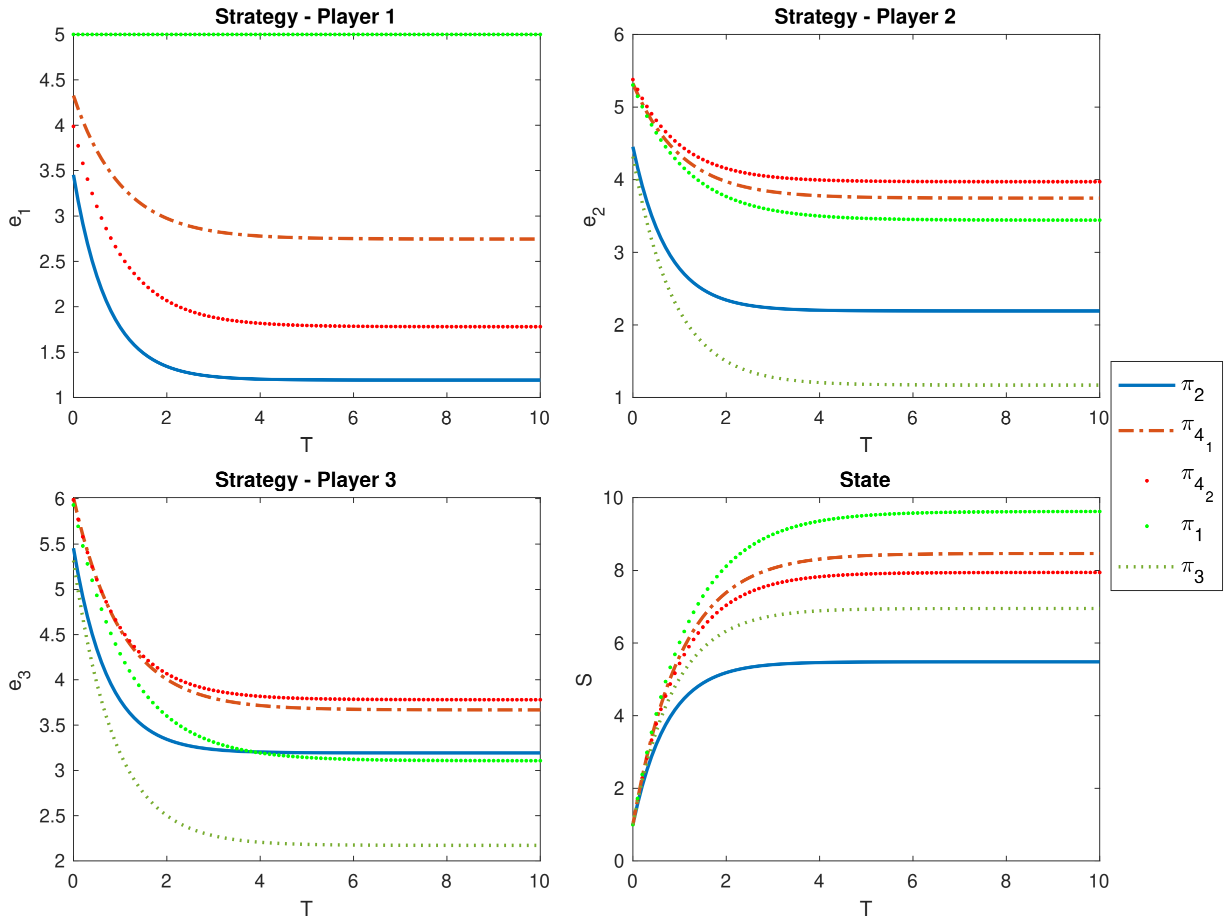


Figure 2.5: Equilibrium strategies and state trajectories under different scenarios (second run)

trajectory. To analyze it, we calculate the players' payoffs in the subgames starting at time $\bar{t} = 1, 5, 10$ under different scenarios. We also start two runs with the same set of parameters given in the previous section. The players' payoffs for these subgames under different coalition structures are collected in Tables 2.8 (the first run) and 2.9 (the second run). In Tables 2.8 and 2.9, there are many zero payoffs, but they are not precisely equal to zero, but are approximated to zero.

We make the following observations:

- Partially cooperative scenario corresponding to the coalition structure $\pi_3$ is Nash stable at any intermediate time $\bar{t} = 1, 5, 10$, and it is the unique individually stable scenario at these time instants.

- For the whole game, there are no Nash stable coalition structures in the initial time for both runs (see Definition 1). But at any intermediate time $\bar{t} = 1, 5, 10$, the coalition structure $\pi_3$ turns to be the unique Nash stable coalition structure.

Table 2.8: The players' payoffs in the subgames starting at time $\bar{t} = 1, 5, 10$ (first run)

| $\bar{t} = 1$ | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -6.681 | -6.382 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0.144 | -0.307 | -0.255 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | -0.495 | -0.499 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0.190 | -0.569 | -4.736 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0.177 | -5.082 | -0.786 |
| $\bar{t} = 5$ | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -20.911 | -25.467 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0 | 0 | 0 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | 0 | 0 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0 | 0 | -19.807 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0 | -15.464 | 0 |
| $\bar{t} = 10$ | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -21.817 | -26.682 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0 | 0 | 0 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | 0 | 0 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0 | 0 | -20.553 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0 | -15.936 | 0 |

### 2.2.3 Stable coalition structures under transfer payment schemes

The ultimate purpose of the developed countries is not unconditionally pursue their individual profit, as expected they are looking forward to promoting the cooperation with a developing country to further reduce the global pollution emissions. In the previous sections, we examine stable scenarios with nontransferable payoffs, and from the numerical examples we could notice that there may exist two types of stable scenarios $\pi_1$ and $\pi_3$ in which the developing country always acts as a singleton. But neither of these scenarios leads to the lowest emissions. Therefore, one can be interested in motivating two developed countries and a developing one to cooperate or to make cooperative scenario $\pi_2$ stable through defining the scheme of payment transfers between the players (see Section 2.2.3). We also define the payment transfer scheme to make partially cooperative scenarios stable (see Section 2.2.3). The latter problem may be actual when the full cooperative scenario cannot be realized by any reason (e.g., external restrictions on full cooperation).

Table 2.9: The players' payoffs in the subgames starting at time $\bar{t} = 1, 5, 10$ (second run)

| $\bar{t} = 1$ | Player 1 Invul. player | Player 2 Vul. player | Player 3 Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -10.364 | -16.938 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0.094 | -0.277 | -0.401 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | -0.578 | -0.820 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0.171 | -0.710 | -13.159 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0.132 | -6.830 | -1.051 |
| $\bar{t} = 5$ | Player 1 Invul. player | Player 2 Vul. player | Player 3 Vul. player |
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -25.443 | -39.911 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0 | 0 | 0 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | 0 | 0 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0 | 0 | -29.138 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0 | -15.550 | 0 |
| $\bar{t} = 10$ | Player 1 Invul. player | Player 2 Vul. player | Player 3 Vul. player |
| $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ | 4.167 | -25.998 | -40.755 |
| $\pi_2 = \{\{1, 2, 3\}\}$ | 0 | 0 | 0 |
| $\pi_3 = \{\{1\}, \{2, 3\}\}$ | 4.167 | 0 | 0 |
| $\pi_{4_1} = \{\{1, 2\}, \{3\}\}$ | 0 | 0 | -29.526 |
| $\pi_{4_2} = \{\{1, 3\}, \{2\}\}$ | 0 | -15.722 | 0 |

**Nash and individual stability of cooperative scenario**

To make the cooperative scenario stable, we need to define conditions to satisfy the system of inequalities from Proposition 2.8. Under these conditions no player has an incentive to deviate in an individual way. We should notice that for the cooperative scenario Nash and individual stability conditions are the same, which can be easily found from Propositions 2.8 and 2.9. Therefore, we need to define the payments to the players $\xi_i$, $i = 1, 2, 3$, such that

$$\begin{cases} \xi_1 + \xi_2 + \xi_3 = \sum_{i=1}^{3} W_i^{\pi_2}, \\ \xi_1 \geq W_1^{\pi_3}, \\ \xi_2 \geq W_2^{\pi_{4_2}}, \\ \xi_3 \geq W_3^{\pi_{4_1}}. \end{cases} \tag{2.34}$$

If there exists a solution of system (2.34), then the transfer payment to player $i \in N$ is defined by

$$\theta_i^{\pi_2} = \xi_i - W_i^{\pi_2}. \tag{2.35}$$

The transfer payment $\theta_i^{\pi_2}$ to player $i$ can have any sign: (i) positive, and it means that a player is paid by other players, (ii) negative, when a player pays other players to support cooperation, (iii) zero, if player $i$ is paid according to her initially given payoff function $W_i^{\pi_2}$.

**Example 2.1.** *We demonstrate a construction of a payment scheme for the numerical examples introduced in Section 2.2.2, using parameters for two runs. Substituting the players' payoffs under different scenarios into system* (2.34)*, we obtain that for both runs there exists a solution of this system, and it is a set-valued solution for both runs. To be precise, the solution is a triangle region drawn in Fig. 2.6a and 2.6b. As long as as the payment vector* $(\xi_1, \xi_2, \xi_3)$ *is located within the orange triangle region for the first run, and the green triangle region for the second run, the cooperative scenario is stable.*

*Additionally, according to (2.35) the transfer* $\theta_1$ *to the developing country stands positive for both runs, which means that two developed countries actually send a part of their profits to compensate cooperative and vulnerable behavior for the developing country. The term "buying" cooperation used in [33] is actual to describe such behavior of the developed countries.*

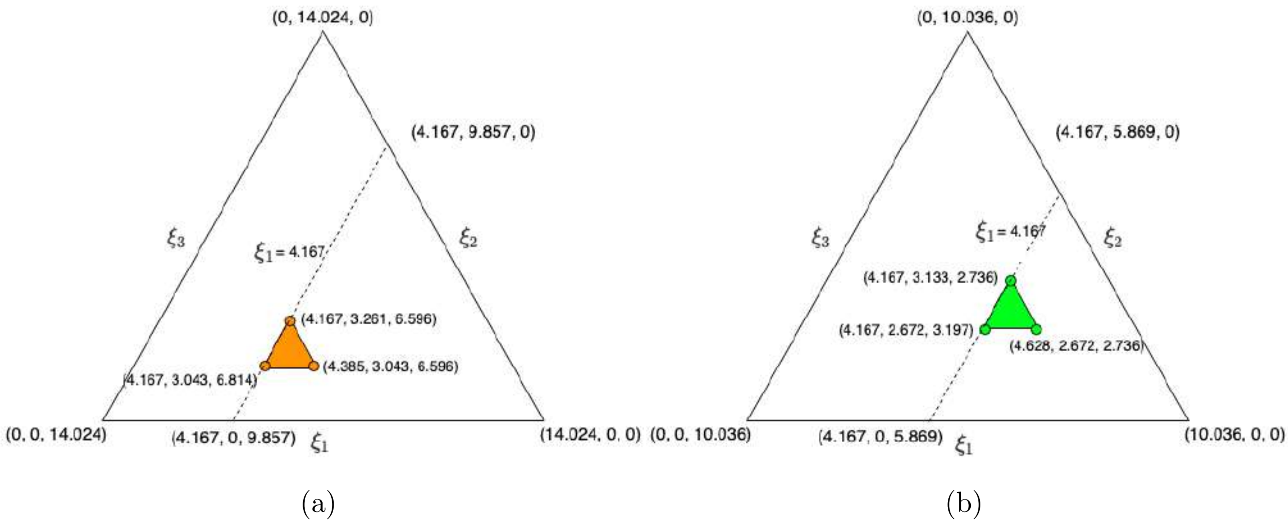


Figure 2.6: The set of payments to the players $(\xi_1, \xi_2, \xi_3)$ satisfying conditions (2.34) ((a): the first run, (b): the second run).

*The choice of the unique vector of payments is out of consideration in this research paper. But we refer the reader to the literature on cooperative games to define "reasonable" solution in the set of payments given by* (2.34)*. We may give an intuition support of a "wise" solution from the perspective of developed countries. They* *are aware of the boundary of saving the stability of a cooperative scenario, and it is reasonable that they would maximally conserve their interest, i.e., contribute the minimal compensation to the developing country. In Fig. 2.6a and 2.6b, the dashed line represents the set of payments with* $\xi_1 = 4.167$*, that is, the minimal payoff for developing country under which the cooperative scenario is stable. Therefore, for two developed countries, it may be reasonable to choose the payments from the intersection of the colored and dashed regions.*

**Nash stability of partially cooperative scenario**

Now we examine if the partially cooperative scenario, say $\pi_{4_1} = \{\{1,2\},\{3\}\}$, is stable when the system of transfers is applied. First, we define the players' payoffs under different scenarios in the case, when any coalition is allowed to make transfers. When the transfers are made, the efficiency conditions should be satisfied. In Table 2.10, the payoffs to the players with transfers are defined for all scenarios: Table

Table 2.10: Payoffs to the players with transfer payments

| | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\},\{2\},\{3\}\}$ | 4.167 | 2.772 | 6.306 |
| $\pi_2 = \{\{1,2,3\}\}$ | $\xi_1^{\pi_2}$ | $\xi_2^{\pi_2}$ | $14.024 - \xi_1^{\pi_2} - \xi_1^{\pi_2}$ |
| $\pi_3 = \{\{1\},\{2,3\}\}$ | 4.167 | $\xi_2^{\pi_3}$ | $10.290 - \xi_2^{\pi_3}$ |
| $\pi_{4_1} = \{\{1,2\},\{3\}\}$ | $\xi_1^{\pi_{4_1}}$ | $7.045 - \xi_1^{\pi_{4_1}}$ | 6.596 |
| $\pi_{4_2} = \{\{1,3\},\{2\}\}$ | $\xi_1^{\pi_{4_2}}$ | 3.043 | $5.991 - \xi_1^{\pi_{4_2}}$ |

2.10 contains five variables. To make scenario $\pi_{4_1}$ Nash stable we need to satisfy the following system of inequalities (see Proposition 2.8):

$$\begin{aligned}
&\xi_1^{\pi_{4_1}} \geq 4.167, \\
&\xi_1^{\pi_{4_1}} \geq \xi_1^{\pi_{4_2}}, \\
&7.045 - \xi_1^{\pi_{4_1}} \geq 2.772, \\
&7.045 - \xi_1^{\pi_{4_1}} \geq \xi_2^{\pi_3}, \\
&6.596 \geq 14.024 - \xi_1^{\pi_2} - \xi_2^{\pi_2}.
\end{aligned} \tag{2.36}$$

If the transfers are determined in such a way, that any coalition divides the surplus payoff equally between players, i.e. the cooperative solution concept is the CIS-value (see Table 2.11), then we can easily check if this system is satisfied for the CIS-value

[29]. In a blue color, we highlight the stable scenario. As we can easily see, scenario $\pi_{4_1}$ is not stable if the payments to the players are organized with the CIS-value.

Table 2.11: Payoffs to the players if any coalition uses the CIS-value to allocate the joint profit (scenario $\pi_2$ is stable)

| | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\},\{2\},\{3\}\}$ | 4.167 | 2.772 | 6.306 |
| $\pi_2 = \{\{1,2,3\}\}$ | 4.240 | 3.116 | 6.669 |
| $\pi_3 = \{\{1\},\{2,3\}\}$ | 4.167 | 3.378 | 6.912 |
| $\pi_{4_1} = \{\{1,2\},\{3\}\}$ | 4.220 | 2.825 | 6.596 |
| $\pi_{4_2} = \{\{1,3\},\{2\}\}$ | 1.926 | 3.043 | 4.065 |

Table 2.12: Payoffs to the players with the new transfer scheme (scenarios $\pi_2, \pi_{4_1}$ are stable)

| | Player 1<br>Invul. player | Player 2<br>Vul. player | Player 3<br>Vul. player |
|---|---|---|---|
| $\pi_1 = \{\{1\},\{2\},\{3\}\}$ | 4.167 | 2.772 | 6.306 |
| $\pi_2 = \{\{1,2,3\}\}$ | 4.240 | 3.189 ↑ | 6.596 ↓ |
| $\pi_3 = \{\{1\},\{2,3\}\}$ | 4.167 | 2.825 ↓ | 7.465 ↑ |
| $\pi_{4_1} = \{\{1,2\},\{3\}\}$ | 4.220 | 2.825 | 6.596 |
| $\pi_{4_2} = \{\{1,3\},\{2\}\}$ | 1.926 | 3.043 | 4.065 |

In the system (2.36), the last two inequalities are not satisfied. Therefore, we can change the transfer payments to players 2 and 3, e.g. in the cooperative scenario $\pi_2$ and partially cooperative scenario $\pi_3 = \{\{1\},\{2,3\}\}$. It can be done taking into account the efficiency of any coalition in any scenario. The new payments to the players are given in Table 2.12, in which the stable scenarios are colored in blue. We increase the payoff to player 2 in comparison with the CIS-value in scenario $\pi_2$ (3.189 vs. 3.116), but decrease the payoff to player 3 (6.596 vs. 6.669). The efficiency of coalition $\{1,2,3\}$ in this scenario is still satisfied after transfer changes, i.e. the joint payoff of the coalition is 14.024. We also increase the payoff to player 3 in comparison with the CIS-value in scenario $\pi_3$ (7.465 vs. 6.912), but decrease the payoff to player 2 (2.825 vs. 3.378). The efficiency of coalition $\{2,3\}$ in this scenario is satisfied, i.e. the joint payoff of coalition $\{2,3\}$ is 10.290.

## 2.3 Conclusion to Chapter 2

In this chapter, our research specifically encircles the design and stability verification of pollution reduction agreements in differential games with asymmetric players.

In the trade-off mechanism approach, a trade-off mechanism is proposed through a contract in which the vulnerable player contributes part of revenues to compensate the invulnerable player for the latter's effort in pollution reduction. The critical question arises in the determination of the optimal parameter set value, i.e., percent of profits given to the invulnerable player and percent of pollution reduction assigned to the invulnerable player. In our numerical example, we traverse every possible combination of the parameter set and compare the results with cooperative and noncooperative scenarios. We conclude that we can find the optimal parameter sets of the trade-off mechanism, i.e., stable agreement to completely outperform a noncooperative scenario. It is also clear that this mechanism is worse than a fully cooperative scenario in terms of improving both players' profits.

In the Nash stability and individual stability approaches, we consider the differential game with one invulnerable and two vulnerable players. In contrast to the trade-off mechanism mentioned above, the stable agreement is applied in coalition structures formed by three players. We examine different cooperative scenarios when players can partially cooperate on stability. To examine all possible coalition structures, we propose three types of scenarios: (i) cooperative, (ii) noncooperative, and (iii) partially cooperative, in which the coalition's profit is dependent on the outside players' behavior (in particular, it depends on if they form coalitions or not). The general conditions of Nash and individual stability of coalition structures or scenarios are determined. Two numerical examples demonstrate the procedure of finding a stable scenario. We also introduce the procedure of making a particular scenario stable (if possible) by defining a special transfer scheme.

# Chapter 3

# Value of Information in Differential Games of Pollution Control

In this chapter, we consider a differential game where the acquired information, i.e., a terminal cost [21], an exact value of the upper bound of control [21], and an initial pollution stock [87, 94] in a pollution control problem are unknown. We investigate the value of corresponding information by comparing the payoff with known information and the payoff without it through *Value of Information* (VI) initially defined in [80]. In the latter case when the initial pollution stock is unknown, we take into account its estimation.

## 3.1 Value of information for uncertainty about terminal costs

In this section, a problem of controlling the volumes of pollutants is being considered [27]. The total payoff is obtained with a terminal cost corresponding to a fine for the pollution at the last moment of the production period. Under this condition, we study the value of information, which shows how the awareness of terminal cost would affect the player's payoff.

A cooperative solution in a differential game of pollution control is found in [5]. The conditions for an optimal control solution are obtained in [45] and a cooperative solution in three-person differential games is provided in [43].

In this section, we formulate a model in a three-player setting for easy tractability of the results. We assume that the production is proportional to the volume of environmental pollution. The strategy of player $i$ is a pollution rate per unit time, $u_i \in [0, b_i]$.

The system dynamics are defined by equation:

$$\dot{x}(t) = \sum_{i=1}^{3} u_i(t), \quad x(0) = x_0, \quad t \in [0, T], \tag{3.1}$$

where $x(t)$ is the pollution stock. Suppose the penalty at the terminal moment is also proportional to the level of pollution $x$. Then the payoff of player $i$ is a mixed Bolza function:

$$K_i(x_0, T - t_0, u) = \int_{t_0}^{T} \left( \big(b_i - \frac{1}{2} u_i(t)\big) u_i(t) - h_i x \right) dt - D_i x(T), \tag{3.2}$$

where $u = (u_1, u_2, u_3)$ and $(b_i - \frac{1}{2} u_i(t)) u_i(t)$ denote the revenue function reflecting the gross profit the player $i$ could obtain at time $t$ with $b_i \geq 0$, $h_i x$ represents the expenses for elimination of pollution $x$, while $D_i x(T)$ is a terminal cost. We consider a cooperative mode of the game, i.e., the players initially agree to use the optimal control maximizing the total profit assuming $t_0 = 0$, i.e.,

$$\sum_{i=1}^{3} K_i(x_0, T, u) \to \max_u . \tag{3.3}$$

**Proposition 3.1.** *In the cooperative mode of a differential game defined by an objective function of the grand coalition (3.3) subject to (3.1), the optimal cooperative control is given by*

$$u_i^*(t) = b_i - (T - t) h_{123} - D_{123}, \quad i \in \{1, 2, 3\}, \tag{3.4}$$

*the corresponding optimal state trajectory is*

$$x^*(t) = (b_{123} - 3D_{123} - 3h_{123} T)t + \frac{3}{2} t^2 h_{123} + x_0, \tag{3.5}$$

*where* $h_{123} = h_1 + h_2 + h_3$, $D_{123} = D_1 + D_2 + D_3$, $b_{123} = b_1 + b_2 + b_3$, $D_{ij} = D_i + D_j$.

*Proof.* The maximization problem (3.3) can be solved in the open-loop strategies [39, 79]. The Hamiltonian function goes as

$$H(x, u, \psi) = \sum_{i=1}^{3} \left[ (b_i - \frac{1}{2} u_i) u_i - h_i x(t) \right] + \psi(u_1 + u_2 + u_3).$$

The optimal control has to maximize the Hamiltonian function by complying with necessary condition that

$$\frac{\partial H}{\partial u_i} = b_i - u_i + \psi = 0,$$

$$u_i^* = b_i + \psi, i = 1, 2, 3.$$

Since the second derivative of $H$ at $u = u_i^*$ is negative, we confirm that the optimal control $u_i^*$ does maximize the Hamiltonian function.

The conditions for an adjoint equation are

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H}{\partial x} = h_{123}, \tag{3.6}$$

$$\psi(T) = \frac{\partial H}{\partial x}\Big|_{t=T} = -D_{123}. \tag{3.7}$$

Solving (3.6) and (3.7) together, we obtain

$$\psi(t) = (t - T)h_{123} - D_{123}.$$

Then the optimal control and optimal state trajectory are acquired accordingly as presented in (3.4) and (3.5). □

Let us introduce additional restrictions on the parameters of the model, which guarantee that the result to solve some other auxiliary optimization problem is admissible and belongs to the compact set $[0, b_i]$:

$$\begin{aligned} &D_i \in [0, \min(b_1 - D_{123}, b_2 - D_{123}, b_3 - D_{123})], \\ &h_i \in [0, \frac{\min(b_1, b_2, b_3) - D_{123}}{T} - h_{123}], \quad i \in \{1, 2, 3\}. \end{aligned} \tag{3.8}$$

Since calculation of the optimal control and optimal state trajectory is of the same character, for the following two cases: terminal cost is known/unknown, we generalize the problem and focus on the performance of a symbolized player, say, player $i$.

### 3.1.1 Terminal cost is known

The situation, when the terminal cost is known to the player is investigated in this section. Contrary to the joint optimization problem (3.3), in this section, we consider a noncooperative game when players are aimed at maximizing function (3.2) subject to (3.1). We find the Nash equilibrium, i.e., characterize the players' equilibrium strategies and the corresponding equilibrium state trajectory. When we say that terminal costs are known to the players, we mean that they include linear costs on terminal stock reduction into their objective functions. Therefore, any player $i$ maximizes function (3.2).

It is well-known that for a linear-quadratic differential game given by (3.1), (3.2), the Nash equilibrium exists and it is unique [6]. Similar to the calculations in

Proposition 3.1, we define the Nash equilibrium using the Pontryagin maximum principle [79].

**Proposition 3.2.** *In a differential game defined by objective function (3.2) subject to (3.1), the Nash equilibrium strategy profile is given by*

$$u_i^{NE}(t) = b_i - (T-t)h_i - D_i, \quad i \in \{1,2,3\}. \tag{3.9}$$

*The equilibrium state trajectory is*

$$x^{NE}(t) = (b_{123} - D_{123} - h_{123}T)t + \frac{1}{2}t^2 h_{123} + x_0. \tag{3.10}$$

*The equilibrium payoff for player 1 is*

$$\begin{aligned} K_1^{NE} &= K_1(x_0, T, u_1^{NE}) \\ &= -x_0(Th_1 + D_1) + D_1 T(D_{123} - b_{123} + Th_{123}) + \frac{T(b_1^2 - D_1^2)}{2} + \\ &+ \frac{T^2(h_1 D_{23} - h_1 b_{123} - D_1 h_{123})}{2} + \frac{T^3 h_1 (h_{123} + h_{23})}{6}. \end{aligned} \tag{3.11}$$

*For player $i = 2,3$, payoff in the Nash equilibrium can be obtained by a cyclic permutation of the indices in (3.11).*

*Proof.* To find the Nash equilibrium, we define the Hamiltonian function for player $i$, that is

$$H_i(x,u,\psi) = (b_i - \frac{1}{2}u_i)u_i - h_i x(t) + \psi(u_1 + u_2 + u_3), \quad i = 1,2,3.$$

The Nash equilibrium strategy $u_i^{NE} = b_i + \psi, i = 1,2,3$ maximizing the Hamiltonian function due to the second derivative of $H_i$ at $u = u_i^{NE}$ is equal to a negative value.

The conditions for adjoint equation are

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H_i}{\partial x} = h_i. \tag{3.12}$$

$$\psi(T) = \frac{\partial H_i}{\partial x}\Big|_{t=T} = -D_i. \tag{3.13}$$

Solving (3.12) and (3.13) together, we obtain

$$\psi(t) = (t-T)h_i - D_i, \quad i = 1,2,3.$$

Then the Nash equilibrium strategy and equilibrium state trajectory are obtained and given in (3.9) and (3.10). The player's payoff in the Nash equilibrium (3.11) is calculated by substituting the obtained control value into (3.2) and integrating the expressions. □

### 3.1.2 Terminal cost is unknown

We start with interpreting what we mean by "terminal cost is unknown". Suppose the player does not know whether at the terminal time she will be assigned a terminal penalty dependent on $x(T)$ or not. So, we assume that the player ignores the terminal costs in her payoff function, i.e., we set the terminal cost parameter $D_i = 0$ in payoff function (3.2). Then the so-called *simulated* optimization problem for any player $i$ is

$$\bar{K}_i(x_0, T, u_i) = \int_{t_0}^{T} \left( \big(b_i - \frac{1}{2}u_i(t)\big)u_i(t) - h_i x \right) dt \to \max_{u_i}, \quad i = 1, 2, 3. \tag{3.14}$$

We consider a *simulated* differential game in which players solve problem (3.14) subject to state dynamics (3.1). However, we can derive the actual payoff to the player that is, her payoff calculated by formula (3.2) by substituting the Nash equilibrium strategies when players solve the simulated optimization problem. It is interesting to estimate how ignorance of the terminal payoff in the optimization problem affects the actual players' payoffs. We attain it by substituting the Nash equilibrium of the simulated differential game into payoff function (3.14), and estimate the importance of information about a terminal cost. We call this payoff to the player as her "actual payoff".

**Proposition 3.3.** *In a simulated differential game defined by objective function (3.14) subject to (3.1), the so-called simulated Nash equilibrium strategy is given by*

$$u_i^{*NE}(t) = b_i - (T - t)h_i, \quad i \in \{1, 2, 3\}. \tag{3.15}$$

*The simulated equilibrium state trajectory obtained as a solution of equation (3.1) substituting strategies (3.15) is*

$$x^{*NE}(t) = (b_{123} - h_{123}T)t + \frac{1}{2}t^2 h_{123} + x_0. \tag{3.16}$$

*The actual payoff according to (3.2) for player 1 is*

$$\begin{aligned} K_1^{*NE} &= K_1(x_0, T, u_1^{NE}) \\ &= -x_0(D_1 + Th_1) - D_1 T(b_{123} - Th_{123}) + \frac{Tb_1^2}{2} - \\ &- \frac{T^2(D_1 h_{123} + h_1 b_{123})}{2} + \frac{T^3 h_1(h_{123} + h_{23})}{6}. \end{aligned} \tag{3.17}$$

*For player $i = 2, 3$, the actual payoff can be obtained by a cyclic permutation of the indices in (3.17).*

*Proof.* In the optimization problem defined in (3.14), the Hamiltonian function is

$$H_i(x, u, \psi) = (b_i - \frac{1}{2}u_i)u_i - h_i x(t) + \psi(u_1 + u_2 + u_3), \quad i = 1, 2, 3.$$

The Nash equilibrium strategy $u_i^{*NE} = b_i + \psi, i = 1, 2, 3$, maximizing the Hamiltonian function due to the second derivative of $H_i$ is negative at $u = u_i^{*NE}$.

The conditions for an adjoint equation are

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H_i}{\partial x} = h_i. \tag{3.18}$$

$$\psi(T) = \frac{\partial H_i}{\partial x}\Big|_{t=T} = 0. \tag{3.19}$$

Solving (3.18) and (3.19) together, we obtain

$$\psi(t) = (t - T)h_i, \quad i = 1, 2, 3.$$

Then the simulated Nash equilibrium strategy and corresponding state trajectory are obtained and given in (3.15) and (3.16). The player's actual payoff (3.17) is calculated by substituting the obtained control value into (3.2) and integrating the expression. □

### 3.1.3 Evaluation of value of information

Before defining the value of information, it's also important to compare the (simulated) Nash equilibrium control and trajectory for the previous two cases: known or unknown terminal costs. We assume that

$$\begin{aligned} &b_1 = 300, b_2 = 305, b_3 = 303, \\ &T = 20, h_1 = 2, h_2 = 3, h_3 = 4, \\ &D_1 = 3, D_2 = 6, D_3 = 5, x_0 = 5. \end{aligned}$$

We notice that these parameters have to satisfy the conditions in (3.8). As shown in the left figure of Fig. 3.1, for each player, the Nash equilibrium strategy when the terminal cost is unknown in a simulated game is above another one obtained when the terminal cost is known, which confirms the regular expectation that the players would increase their pollution, i.e., improve the production, when they misunderstand that the cost for pollution is lower due to the lack of sufficient information. Consequently, the pollution stock is higher in this case as depicted in Fig.3.1.

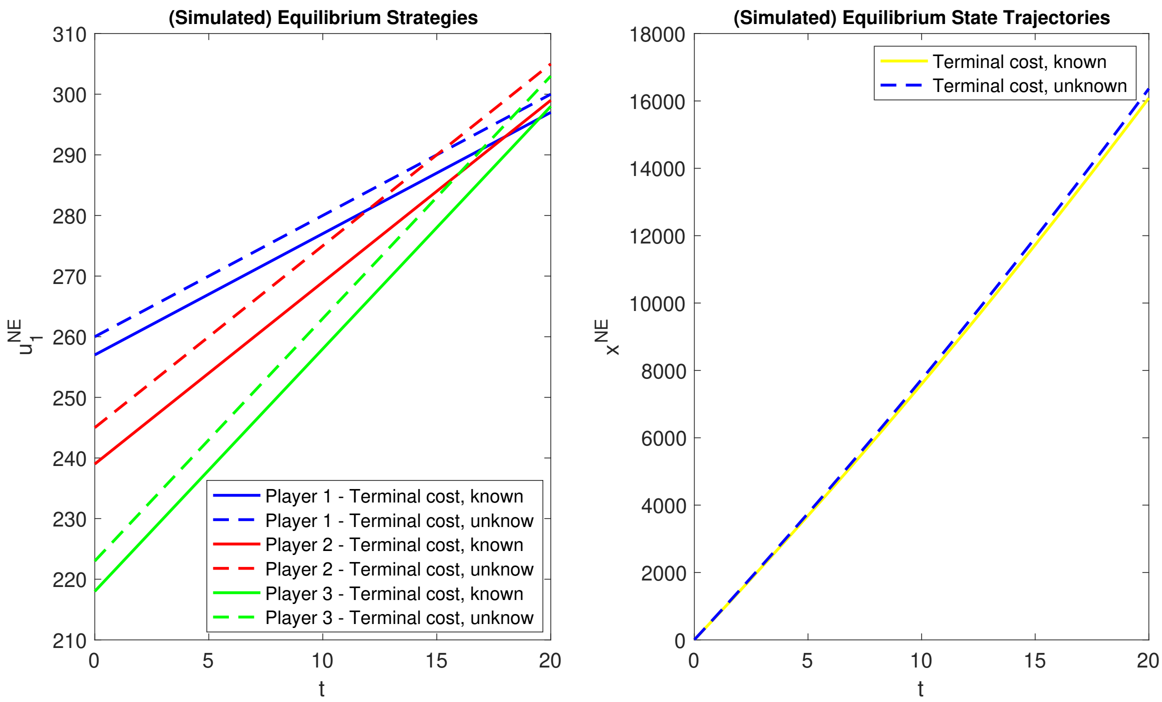


Figure 3.1: (Simulated) Nash equilibirum strategies and state trajectories when the terminal cost is known (unknown). The left figure represents (simulated) Nash equilibirum strategies for player 1.

Now in order to evaluate the *value of information* initially proposed in [80] for a problem with incomplete information about the terminal cost, we formulate a definition of the normalized value of information (NVI) calculated on the basis of the payoffs obtained in Proposition (3.2) and (3.3) as follows.

**Definition 3.1.** *For differential games given by (3.2) and (3.14), the normalized value of information about the terminal cost is given by:*

$$NVI_i = \left| \frac{K_i^{NE} - K_i^{*NE}}{K_i^{NE}} \right| \times 100\%, \quad i = 1, 2, 3. \tag{3.20}$$

Solving (3.20) with the numerical example given above, we obtain that the $NVI_1 = 0.96\%$, $NVI_2 = 1.73\%$, $NVI_3 = 4.34\%$ which is quite small. But if we say that the payoff is counted in billions, then the information may cost relatively much more.

The described method of calculating the NVI in the form given in (3.20) can be extended for a cooperative case of the game:

$$NVI_i = \frac{\xi_i - \xi_i^*}{\xi_i} \times 100\%,$$

where $\xi_i, i \in N$, is a component of the cooperative solution (e.g., the Shapley value, $\tau-$value, etc.) of the game calculated under conditions of complete information, and $\xi_i^*, i \in N$, is a component of the cooperative solution calculated with a lack of information.

## 3.2 Value of information for uncertainty about possible adjustment of upper boundary of control

In the classical theory of optimal control [72], the problems are commonly considered without changing a structure or information that a decision maker possesses. The work [44] embodies the dynamic update of information. The authors in [10] explore a differential game with regime switching and spillovers. While in our case, we are analyzing a model of industrial production, in which at some point the upper limit of an admissible level of pollution caused by production can be changed. Under this condition, we are studying the maintaining optimal controls and optimal trajectories. We simplify the model above and consider a process of pollution control with a unique player. We assume that pollution is proportional to the volume of production, and the rate of emissions in the atmosphere is most likely restrained by the government or other parties. The player aims to select an optimal control to maximize her profit. The model can be generalized for the cases of a noncooperative or cooperative differential game.

The dynamics of the pollution stock is defined by

$$\dot{x} = u(t), \quad x(0) = x_0, \quad u \in [0, b], \quad t \in [0, T]. \tag{3.21}$$

The player solves the maximization problem:

$$J(x_0, T, u) = \int_0^T \Big( \big(b - \frac{1}{2}u(t)\big)u(t) - hx(t) \Big) dt \to \max_u . \tag{3.22}$$

**Proposition 3.4.** *In the optimization problem defined by objective function (3.22) subject to (3.21), the optimal control is given by*

$$u^*(t) = \begin{cases} 0, & t \in [0, T - \frac{b}{h}], \\ b - h(T - t), & t \in [T - \frac{b}{h}, T]. \end{cases} \tag{3.23}$$

*The optimal state trajectory is*

$$x^*(t) = \begin{cases} x_0, & t \in [0, T - \frac{b}{h}], \\ x_0 + \frac{(b - Th + ht)^2}{2h}, & t \in [T - \frac{b}{h}, T]. \end{cases} \tag{3.24}$$

*Proof.* In the optimization problem (3.22) subject to (3.21), the Hamiltonian function takes the form:

$$H(x, u, \psi) = (b - \frac{u}{2})u - hx + \psi u. \tag{3.25}$$

Solving the equation:

$$\frac{\partial H}{\partial u} = b - u + \psi = 0,$$

we obtain the optimal control

$$u^* = b + \psi. \tag{3.26}$$

Since the second derivative of $H$ at $u = u_i^*$ is negative, we confirm that we get the maximal Hamiltonian function at optimal control $u^*$.

The adjoint equation is

$$\frac{\partial \psi}{\partial t} = -\frac{\partial H}{\partial x} = h. \tag{3.27}$$

The terminal state $x(T)$ is free and we find the solution of the differential equation (3.27) with the terminal condition $\psi(T) = 0$, therefore,

$$\psi(t) = h(t - T).$$

Then substituting $\psi$ into (3.26), and since the sign of control has to be nonnegative, as $b - h(T - t) < 0$, we expect to set $u = 0$. Considering the solution of the optimization problem with Kuhn-Tucker condition [48], we confirm that the optimal control given by (3.23) does maximize the Hamiltonian function (3.25). Finally, the optimal control $u^*$ and the optimal state trajectory are such as those given in Proposition 3.4. □

### 3.2.1 Change of upper boundary of control

Suppose that at time moment $\tau_s \in [0, T]$, the upper boundary of a control variable is changed to $\bar{b}$, i.e. for $t \in [\tau_s, T]$, the control should satisfy the constraint $u(t) \in [0, \bar{b}]$, and in case of complete information, the player is aware of that. We also assume that moment $\tau_s$ is given and known to the player.

In the case when the upper boundary of control can be changed, generally, there are two scenarios: (i) changed upper boundary $\bar{b} < b$, (ii) changed upper boundary $\bar{b} \geq b$. Moreover, when the player can't perceive the relevant information about the changed upper boundary, these two scenarios will make no differences when describing what we are going to discuss in the following sections, because the player complies with exactly the same objective function defined in (3.22).

### 3.2.2 Upper boundary of control $\bar{b} < b$

**The change of upper boundary of control is unknown**

In this section, the player does not know about the change of upper boundary and we assume that the changed upper boundary of control $\bar{b}$ is lower than the original one $b$, i.e., $\bar{b} < b$ in this case. Supposedly, the so-called simulated optimal control under this situation is identical to the one in (3.23), because the player will still behave to maximize the objective function (3.22) due to the ignorance of information about the change of upper boundary. Meanwhile, suppose the optimal control will reach $\bar{b}$ at time $\theta$, i.e.,

$$b - h(T - \theta) = \bar{b},$$
$$\theta = T + \frac{\bar{b} - b}{h} < T.$$

In addition, we suppose $\theta \geq \tau_s$, which means that the change of upper boundary of control happens before the optimal control reaches a new limit. Thus, in this case, this constraint on $\tau_s$ is trivial because the player does not know about the change of upper boundary.

Consequently, exceeding the changed upper boundary $\bar{b}$ after time $\theta$, the player has to be punished. Here we define a penalty estimated by the integral where the so-called simulated optimal control surpasses $\bar{b}$ over time, i.e., $P\int_\theta^T (b - h(T-t) - \bar{b})dt$, where $P > 0$ is a penalty coefficient. Thus we can write the objective function as follows:

$$J^L(x_0, T, u) = \int_0^T \Big( \big(b - \frac{1}{2}u\big)u - dx \Big) dt - P\int_\theta^T \big(b - h(T-t) - \bar{b}\big)dt. \quad (3.28)$$

It is noticeable that the so-called simulated optimal control and trajectory are obtained through the objective function (3.22) considering the player has no information about the change of upper boundary of control, however, the calculation of a player's payoff is solved by substituting the optimal control and trajectory in (3.28).

**Proposition 3.5.** *In the case when the change of upper boundary of control is unknown, when $\bar{b} < b$, for the optimization problem defined by objective function (3.22) subject to (3.21), the simulated optimal control is given by*

$$u^{*LN}(t) = \begin{cases} 0, & t \in [0, T - \frac{b}{h}], \\ b - h(T - t), & t \in [T - \frac{b}{h}, T]. \end{cases}$$

*The simulated optimal state trajectory is*

$$x^{*LN}(t) = \begin{cases} x_0, & t \in [0, T - \frac{b}{h}], \\ x_0 + \frac{(b-Th+ht)^2}{2h}, & t \in [T - \frac{b}{h}, T]. \end{cases}$$

*The actual payoff according to (3.28) for the player is*

$$J^{*LN} = -Thx_0 + \frac{b^3}{6h} - \frac{P(b-\bar{b})^2}{2h}.$$

*Proof.* See Proposition 3.4. □

**The change of upper boundary of control is known**

In this section, just like in the previous case, we also suppose $\theta \geq \tau_s$ and in contrast to that case, the prominent difference is that the control won't exceed $\bar{b}$ if it proceeds with adequate information about the change of upper boundary of control, i.e.,

$$0 \leq u \leq \bar{b} < b. \tag{3.29}$$

Therefore, there will be no extra cost in this case.

**Proposition 3.6.** *In the case when the change of upper boundary of control is known, when $\bar{b} < b$, for the optimization problem defined by objective function (3.22) subject to (3.29), the optimal control is given by*

$$u^{*LY}(t) = \begin{cases} 0, & t \in [0, T - \frac{b}{h}], \\ b - h(T-t), & t \in [T - \frac{b}{h}, T + \frac{\bar{b}-b}{h}], \\ \bar{b}, & t \in [T + \frac{\bar{b}-b}{h}, T]. \end{cases} \tag{3.30}$$

*The optimal state trajectory is*

$$x^{*LY}(t) = \begin{cases} x_0, & t \in [0, T - \frac{b}{h}], \\ x_0 + \frac{(b-Th+ht)^2}{2h}, & t \in [T - \frac{b}{h}, T + \frac{\bar{b}-b}{h}], \\ x_0 + \frac{\bar{b}^2}{2h} - \bar{b}(T-t) + \frac{\bar{b}(b-\bar{b})}{h}, & t \in [T + \frac{\bar{b}-b}{h}, T]. \end{cases}$$

*The maximal payoff to the player is*

$$J^{*LY} = -Thx_0 + \frac{\bar{b}(b-\bar{b})^2}{2h} + \frac{\bar{b}^2(3b-2\bar{b})}{6h}.$$

*Proof.* Due to the constraint $u \leq \bar{b}$ as indicated in (3.29), we construct the Lagrange function:

$$L(u, \lambda) = (b - \frac{u}{2})u - hx + \psi u + \lambda(\bar{b} - u).$$

By applying Kuhn-Tucker condition, we have the following conditions needed to be satisfied,

$$\frac{\partial L}{\partial u} = b - u + \psi - \lambda \leq 0, \tag{3.31}$$

$$\frac{\partial L}{\partial \lambda} = \bar{b} - u \geq 0, \tag{3.32}$$

$$\lambda, u \geq 0, \tag{3.33}$$

$$\lambda \frac{\partial L}{\partial \lambda} = \lambda(\bar{b} - u) = 0, \tag{3.34}$$

$$u \frac{\partial L}{\partial u} = u(b - u + \psi - \lambda) = 0. \tag{3.35}$$

Taking (3.34) as a starting point, if $\lambda = 0$, we have $u = 0$ or $u = b + \psi$. If $u = \bar{b}$, we have $\lambda = b - \bar{b} + \psi = 0$. Combined with $\psi$ in Proposition 3.4, the optimal control (3.30) can be obtained, and finally the optimal state trajectory and maximal payoff will be also obtained. □

### 3.2.3 Upper boundary of control $\bar{b} \geq b$

Since the upper boundary of control changed to $\bar{b} > b$, the admissible control will not be larger than $\bar{b}$ no matter the information about it is known or unknown. Therefore, the specification of constraints of $\tau_s$ is not necessary as long as $\tau_s$ belongs to $[0, T]$.

#### The change of upper boundary of control is unknown

When the change of upper boundary of control is unknown, it means that the player will solve the optimization problem defined in (3.22) subject to

$$0 \leq u \leq b \leq \bar{b}. \tag{3.36}$$

**Proposition 3.7.** *In the case when the change of upper boundary of control is unknown, and $\bar{b} \geq b$, for the optimization problem defined by objective function (3.22) subject to (3.36), the optimal control is given by*

$$u^{*HN}(t) = \begin{cases} 0, & t \in [0, T - \frac{b}{h}], \\ b - h(T - t), & t \in [T - \frac{b}{h}, T]. \end{cases}$$

*The optimal state trajectory is*

$$x^{*HN}(t) = \begin{cases} x_0, & t \in [0, T - \frac{b}{h}], \\ x_0 + \frac{(b - Th + ht)^2}{2h}, & t \in [T - \frac{b}{h}, T]. \end{cases}$$

*The maximal payoff according to (3.22) for the player is*

$$J^{*HN} = -Thx_0 + \frac{b^3}{6h}.$$

*Proof.* See Proposition 3.4. □

**The change of upper boundary of control is known**

Similar to the optimization problem described in Proposition 3.7, when the change of upper boundary of control is known, the objective function and constraints remain the same.

**Proposition 3.8.** *In the case when the change of upper boundary of control is known, more specifically, when $\bar{b} \geq b$, for the optimization problem defined by objective function (3.22) subject to (3.36), the optimal control is given by*

$$u^{*HY}(t) = \begin{cases} 0, & t \in [0, T - \frac{b}{h}], \\ b - h(T-t), & t \in [T - \frac{b}{h}, T]. \end{cases} \tag{3.37}$$

*The optimal state trajectory is*

$$x^{*HY}(t) = \begin{cases} x_0, & t \in [0, T - \frac{b}{h}], \\ x_0 + \frac{(b - Th + ht)^2}{2h}, & t \in [T - \frac{b}{h}, T]. \end{cases}$$

*The maximal payoff to the player is*

$$J^{*HY} = -Thx_0 + \frac{b^3}{6h}.$$

*Proof.* Repeating the same procedure as described in Proposition 3.6, with Kuhn-Tucker conditions, we get the identical conditions (3.31)-(3.35) needed to be satisfied.

If $\lambda = 0$, we have $u = 0$ or $u = b + \psi$. If $\lambda \neq 0$, we have $u = \bar{b}$ and $\lambda = b - \bar{b} + \psi \geq 0$. However, since $b - \bar{b} \leq 0$ in our case, and $\psi < 0$ as shown in Proposition 3.4, therefore, $\lambda < 0$ which doesn't meet the condition. In this way, we only have $u = 0$ and $u = b + \psi$, two options as described in (3.37). As for the optimal state trajectory and maximal payoff, see Proposition 3.7. □

### 3.2.4 Evaluation of value of information

Suppose $T = 25, b = 20, h = 1, x_0 = 10, P = 8$ for both cases. Let $\bar{b} = 5$ when $\bar{b} < b$ and $\bar{b} = 25$ when $\bar{b} \geq b$. As shown in Fig. 3.2, the information about the change of upper boundary of control plays an important role only in the case when $\bar{b} < b$ is known and influences the (simulated) optimal control and (simulated) optimal state trajectories. Especially, when $\bar{b} \geq b$, the (simulated) optimal control and (simulated) optimal state trajectories do not change, and the payoffs under these two cases are equal, which shows that the information brings no value to the player.

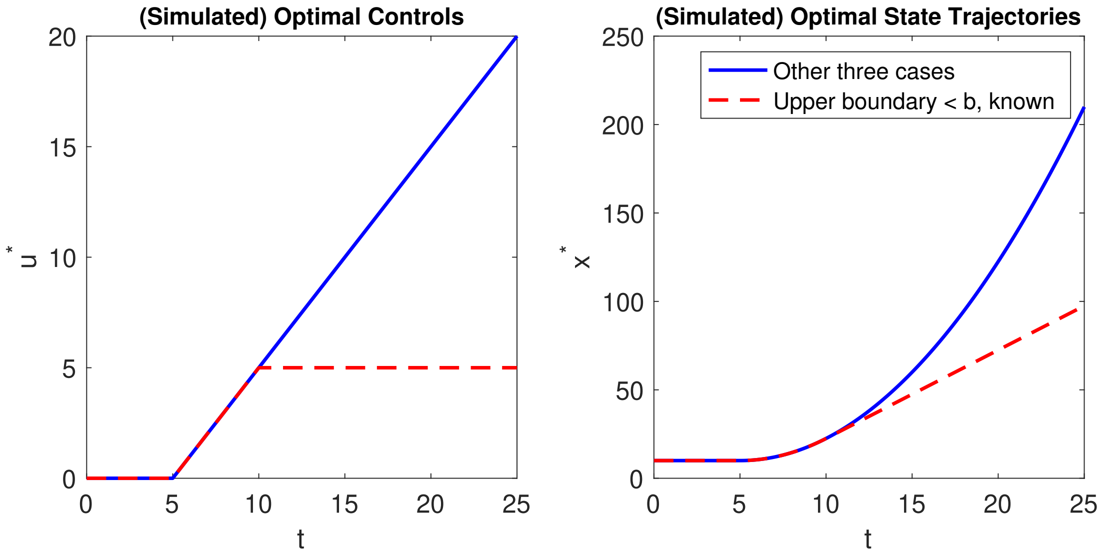


Figure 3.2: (Simulated) optimal controls and state trajectories under the situation when the upper boundary of control is changed. The red (dashed) line denotes the case when $\bar{b} < b$ is known (see Proposition 3.6), and the blue (solid) line represents other three cases considered in Propositions 3.5, 3.7 and 3.8.

For evaluating the *value of information*, we again consider the normalized value of information.

**Definition 3.2.** *In the optimization problem (3.22) and (3.28) when the change of upper boundary of control is known or unknown, the normalized value of information about the terminal cost is given by:*

$$NVI = \left| \frac{J^{MY} - J^{MN}}{J^{MY}} \right| \times 100\%, \quad M \in \{L, H\}.$$

For the numerical example above, when $\bar{b} < b$, we have $NVI = 21.6\%$ and as expected, this value is an increasing function of parameter $P$, which indicates that the information about the change of upper boundary of control is quite crucial in this case. Moreover, the information about $\bar{b} \geq b$ is useless, because the analysis shows that the player does not improve her payoff, and the NVI is equal to zero.

## 3.3 Value of information about initial pollution stock

In this section, we consider a differential game with two firms $z \in \{i, j\}$ working on resource extraction with disturbance stock $p_t$ at time $t \in [0, T]$. The amount of firm's extraction $\gamma^z p_t$, the environmental disturbance magnitude $e_t^z = \epsilon^z \gamma^z p_t$ and the abatement $a_t^z = \alpha^z \tau_t^z$ are separately denoted, where $\gamma^z, \epsilon^z, \alpha^z$ are positive constants and the reclamation effort $\tau_t^z$ is a control variable of player $z$.

The system dynamics are given by

$$\dot{p}_t = (\epsilon^i \gamma^i + \epsilon^j \gamma^j - \delta) p_t - \alpha^i \tau_t^i - \alpha^j \tau_t^j, \ p_t > 0, \ p_{t=0} = p_0, \tag{3.38}$$

where $\delta > 0$ is a natural recovery rate of the environment, and a growth rate of the environmental pollution stock without abatement is $\epsilon^i \gamma^i + \epsilon^j \gamma^j - \delta > 0$.

The goal of the firms is to take optimal reclamation effort $\tau_t^z$ to minimize their cost devoted to recovery procedures. Assume that the players behave cooperatively aiming at minimizing the sum of their costs. The cooperative objective functional is

$$K(p_0, \tau_t^i, \tau_t^j) = \int_0^T \left[ \frac{(\tau_t^i)^2}{2} + \frac{(\tau_t^j)^2}{2} \right] dt + \phi p_T^2 \to \min, \tag{3.39}$$

while the noncooperative objective functional of player $z$ is

$$J_z(p_0, \tau_t^z) = \int_0^T \frac{(\tau_t^z)^2}{2} dt + \frac{1}{2} \phi p_T^2 \to \min, \tag{3.40}$$

where $p_0$ stands for the initial disturbance stock, the reclamation cost is denoted as $\ell(\tau_t^z) = \frac{(\tau_t^z)^2}{2}$, the abandonment reclamation fee for each firm at the terminal time is $f(p_T) = \phi \frac{p_T^2}{2}$, where $\phi$ is a positive constant.

If we assume the players to be symmetric, i.e., having equal coefficients $\epsilon^i = \epsilon^j = \epsilon, \alpha^i = \alpha^j = \alpha, \gamma^i = \gamma^j = \gamma$, we can rewrite (3.38) as

$$\dot{p}_t = (2\epsilon\gamma - \delta) p_t - \alpha(\tau_t^i + \tau_t^j) \tag{3.41}$$

with the updated constraint $2\epsilon\gamma - \delta > 0$.

This game is presented in [62], where the closed-loop solutions are considered. In our setting, we assume that the players do not have accurate information about the initial stock change and cannot observe $p_t$ at any moment of time. Under such conditions, it is not possible to use closed-loop control, so we focus on the open-loop equilibrium and estimate the value of information about initial state of the system.

### 3.3.1 Cooperative case

**Proposition 3.9.** *In the cooperative differential game defined by joint objective function (3.39) subject to (3.41), the optimal cooperative control for the firms is*

$$(\tau_t^i)^* = (\tau_t^j)^* = -\psi\alpha = 2\alpha\phi e^{(2\epsilon\gamma-\delta)(T-t)} p_T^*, \tag{3.42}$$

*where* $p_T^* = \frac{(2\epsilon\gamma-\delta)p_0 e^{(2\epsilon\gamma-\delta)T}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}$.

*The corresponding optimal state trajectory is*

$$p_t^* = p_0 e^{(2\epsilon\gamma-\delta)t} - \frac{2\alpha^2\phi e^{(2\epsilon\gamma-\delta)(T+t)} p_T^*}{2\epsilon\gamma-\delta} + \frac{2\alpha^2\phi e^{(2\epsilon\gamma-\delta)(T-t)} p_T^*}{2\epsilon\gamma-\delta}. \tag{3.43}$$

*The total payoff of a coalition of two firms is*

$$K(p_0, (\tau_t^i)^*, (\tau_t^j)^*) = p_0\phi e^{(2\epsilon\gamma-\delta)T} p_T^*.$$

*Proof.* Following [78], to find an open-loop solution of the cooperative problem to minimize (3.39), we define the Hamiltonian function as

$$H = -\frac{(\tau_t^i)^2 + (\tau_t^j)^2}{2} + \psi\big[(2\epsilon\gamma-\delta)p_t - \alpha(\tau_t^i + \tau_t^j)\big],$$

where $\psi$ is the adjoint variable which can be obtained through a canonical system with the transversality condition $\psi(T) = -\frac{d}{dp_t}\phi p_T^2 \mid_{t=T} = -2\phi p_T$, we get $\psi(t) = -2\phi e^{(2\epsilon\gamma-\delta)(T-t)} p_T$. According to the first order extremality condition and the sign of the second derivative of $H$ on control variable, the optimal control for each firm is given in (3.42). Correspondingly, the optimal trajectory is obtained as (3.43).

The total payoff of a coalition of two firms is

$$\begin{aligned} K(p_0, (\tau_t^i)^*, (\tau_t^j)^*) &= \sum_{z\in i,j} \int_0^T \frac{((\tau_t^z)^*)^2}{2} dt + \phi(p_T^*)^2 \\ &= \int_0^T \left[\frac{2\alpha\phi(2\epsilon\gamma-\delta)p_0 e^{(2\epsilon\gamma-\delta)(2T-t)}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}\right]^2 dt + \phi(p_T^*)^2 \\ &= \frac{p_0^2\phi(2\epsilon\gamma-\delta)e^{2(2\epsilon\gamma-\delta)T}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)} = p_0\phi e^{(2\epsilon\gamma-\delta)T} p_T^*, \end{aligned}$$

where $P_T^*$ is found substituting $t = T$ into the expression of $p_t^*$ given above. We can notice that using control $(\tau_t^z)^*$, $z = i, j$, the players will not completely clean up the environment by time T, since $p_T^* > 0$. ☐

With respect to the uncertainty of the initial pollution stock, there are two possible cases which could occur: (i) the value of initial stock can be overestimated compared with the actual one; (ii) the initial stock can be underestimated. Actually, there is the third case in which the estimation coincides with the actual value, but this case is trivial and we leave it without further discussion.

Suppose $\hat{p}_0$ represents the estimated initial stock. If we substitute $\hat{p}_0$ into the expressions of the optimal strategies and optimal state trajectory, we obtain the following:

$$(\hat{\tau}_t^i)^* = (\hat{\tau}_t^j)^* = -\psi\alpha = \frac{2\alpha\phi(2\epsilon\gamma-\delta)e^{(2\epsilon\gamma-\delta)(2T-t)}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}, \tag{3.44}$$

$$\hat{p}_t^* = p_0 e^{(2\epsilon\gamma-\delta)t} - [e^{(2\epsilon\gamma-\delta)t} - \frac{1}{e^{(2\epsilon\gamma-\delta)t}}]\frac{2\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}. \tag{3.45}$$

**Overestimation of the initial stock**

When the estimated initial stock $\hat{p}_0$ is above the actual one (i.e. $\hat{p}_0 > p_0$), there are two possible outcomes, which depend on how much $\hat{p}_0$ differs from $p_0$. If the difference is not very large, then we get $\hat{p}_T^* \geq 0$, as in the original problem. Otherwise, it is possible that firms will clean up and reclaim the environment by the time $\hat{t} < T$, and we have $\hat{p}_T^* < 0$.

First, consider the case when $\hat{p}_T^* \geq 0$.

According to (3.45), we have

$$p_0 e^{(2\epsilon\gamma-\delta)T} - [e^{(2\epsilon\gamma-\delta)T} - \frac{1}{e^{(2\epsilon\gamma-\delta)T}}]\frac{2\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)} \geq 0.$$

This inequality holds if

$$\frac{p_0}{\hat{p}_0} \geq \frac{2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}. \tag{3.46}$$

Writing $\hat{p}_0$ in terms of $p_0$ as $\hat{p}_0 = rp_0$, we rewrite inequality (3.46) in the form:

$$1 < r \leq 1 + \frac{2\epsilon\gamma-\delta}{2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}. \tag{3.47}$$

In the case when inequality (3.47) is satisfied, the players using controls (3.44) will have the total payoff in the following form:

$$K(p_0,(\hat{\tau}_t^i)^*,(\hat{\tau}_t^j)^*) = \sum_{z\in i,j}\int_0^T \frac{((\hat{\tau}_t^z)^*)^2}{2}dt + \phi(\hat{p}_T^*)^2$$
$$= \int_0^T \left[\frac{2\alpha\phi(2\epsilon\gamma-\delta)\hat{p}_0 e^{(2\epsilon\gamma-\delta)(2T-t)}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}\right]^2 dt + \phi(\hat{p}_T^*)^2$$
$$= \frac{2\alpha^2\phi^2\hat{p}_0^2(2\epsilon\gamma-\delta)e^{4(2\epsilon\gamma-\delta)T}(1-e^{-2(2\epsilon\gamma-\delta)T})}{(2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1))^2} + \phi(\hat{p}_T^*)^2.$$

When condition (3.47) is not satisfied, we have $\hat{p}_T^* < 0$. This means that firms will clean up and reclaim the environment by the time $\hat{t} < T$. To find the moment $\hat{t}$ such that $\hat{p}_{\hat{t}} = 0$, one can solve the equation to pinpoint $e^{-2(2\epsilon\gamma-\delta)\hat{t}}$ instead of explicit $\hat{t}$:

$$p_0 e^{(2\epsilon\gamma-\delta)\hat{t}} - [e^{(2\epsilon\gamma-\delta)\hat{t}} - \frac{1}{e^{(2\epsilon\gamma-\delta)\hat{t}}}]\frac{2\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)} = 0.$$

Then

$$e^{-2(2\epsilon\gamma-\delta)\hat{t}} = 1 - \frac{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}{2r\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}}.$$

The resulting expression for the controls in this scenario thus reads as

$$(\hat{\tau}_t^i)^{**} = (\hat{\tau}_t^j)^{**} = \begin{cases} \frac{2\alpha\phi(2\epsilon\gamma-\delta)e^{(2\epsilon\gamma-\delta)(2T-t)}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}, & t\in[t_0,\hat{t}], \\ 0, \quad t\in[\hat{t},T]. \end{cases}$$

The corresponding trajectory goes like this:

$$\hat{p}_t^{**} = \begin{cases} p_0 e^{(2\epsilon\gamma-\delta)t} - [e^{(2\epsilon\gamma-\delta)t} - \frac{1}{e^{(2\epsilon\gamma-\delta)t}}]\frac{2\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}\hat{p}_0}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}, & t\in[t_0,\hat{t}], \\ 0, \quad t\in[\hat{t},T]. \end{cases}$$

So, if $r > 1 + \frac{2\epsilon\gamma-\delta}{2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}$, the current total payoff is

$$K(p_0,(\tau_t^i)^{**},(\tau_t^j)^{**}) = \sum_{z\in i,j}\int_0^T \frac{((\hat{\tau}_t^z)^{**})^2}{2}dt + \phi(\hat{p}_T^{**})^2$$
$$= \int_0^{\hat{t}} \left[\frac{2\alpha\phi(2\epsilon\gamma-\delta)\hat{p}_0 e^{(2\epsilon\gamma-\delta)(2T-t)}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}\right]^2 dt$$
$$= \frac{2\alpha^2\phi^2\hat{p}_0^2(2\epsilon\gamma-\delta)e^{4(2\epsilon\gamma-\delta)T}(1-e^{-2(2\epsilon\gamma-\delta)\hat{t}})}{(2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1))^2}$$
$$= \frac{p_0\hat{p}_0\phi(2\epsilon\gamma-\delta)e^{2(2\epsilon\gamma-\delta)T}}{2\epsilon\gamma-\delta+2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}.$$

**Underestimation of the initial stock**

In the case when the players underestimate the initial stock, i.e., $\hat{p}_0 < p_0$ $(0 < r < 1)$, again we get $\hat{p}^*_T \geq 0$. The players use strategies (3.44) and the total payoff is equal to $K(p_0, (\hat{\tau}^i_t)^*, (\hat{\tau}^j_t)^*)$.

### 3.3.2 Noncooperative case

**Proposition 3.10.** *In the differential game defined in (3.40) subject to (3.41), the Nash equilibrium strategy for firm z is*

$$(\tau^z_t)^{NC} = -\psi\alpha = \alpha\phi e^{(2\epsilon\gamma-\delta)(T-t)}p^+_T, \tag{3.48}$$

*where* $p^+_T = \frac{(2\epsilon\gamma-\delta)p_0 e^{(2\epsilon\gamma-\delta)T}}{2\epsilon\gamma-\delta+\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}$.

*The equilibrium trajectory is*

$$p^{NC}_t = p_0 e^{(2\epsilon\gamma-\delta)t} - \frac{\alpha^2\phi e^{(2\epsilon\gamma-\delta)(T+t)}p^+_T}{2\epsilon\gamma-\delta} + \frac{\alpha^2\phi e^{(2\epsilon\gamma-\delta)(T-t)}p^+_T}{2\epsilon\gamma-\delta}. \tag{3.49}$$

*The equilibrium payoff to firm z is*

$$J_z(p_0, (\tau^z_t)^{NC}) = \frac{1}{2}\phi(p^+_T)^2[\frac{\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}{2(2\epsilon\gamma-\delta)} + 1]. \tag{3.50}$$

*Proof.* Switching to (3.40), we define a new Hamiltonian function to find the firm's Nash equilibrium strategy, that is

$$H_z = -\frac{(\tau^z_t)^2}{2} + \psi[(2\epsilon\gamma-\delta)p_t - \alpha(\tau^i_t + \tau^j_t)].$$

As the same solution is described in a cooperative case, the Nash equilibrium strategy for each firm is given by (3.48), and the equilibrium trajectory is defined by (3.49). The equilibrium payoff to firm $z$ is given by (3.50). Then the sum of two firms' noncooperative payoffs is

$$J = \sum_{z\in i,j} J_z(p_0, (\tau^z_t)^{NC}) = \phi(p^+_T)^2[\frac{\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}{2(2\epsilon\gamma-\delta)} + 1].$$

□

**Two cases concerning estimation of the initial stock**

In a noncooperative case, the analysis of classification of the initial stock estimation is identical to the previous one including overestimation, equality and underestimation. Assuming that the estimation of the initial stock $\tilde{p}_0$ makes no difference to both firms, then the corresponding Nash equilibrium strategy and equilibrium trajectory are

$$(\tilde{\tau}_t^z)^{NC} = -\psi\alpha = \frac{\alpha\phi(2\epsilon\gamma-\delta)e^{(2\epsilon\gamma-\delta)(2T-t)}\tilde{p}_0}{2\epsilon\gamma-\delta+\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)},$$

$$\tilde{p}_t^{NC} = p_0e^{(2\epsilon\gamma-\delta)t} - [e^{(2\epsilon\gamma-\delta)t} - \frac{1}{e^{(2\epsilon\gamma-\delta)t}}]\frac{\alpha^2\phi e^{2(2\epsilon\gamma-\delta)T}\tilde{p}_0}{2\epsilon\gamma-\delta+\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T}-1)}.$$

Due to the similar calculations on overestimation and underestimation as described above, the result in this case repeats the one given in Proposition 3.12.

### 3.3.3 Normalized value of information about initial stock

We investigate the ideas of research in [21, 94], and define the problem of determining the value of information in a continuous-time differential game.

**Definition 3.3.** *The normalized value of information in a cooperative game is defined as*

$$\mathcal{V}^C = \left|\frac{K(p_0,(\tau_t^i)^*,(\tau_t^j)^*) - K(p_0,(\hat{\tau}_t^i)^*,(\hat{\tau}_t^j)^*)}{K(p_0,(\tau_t^i)^*,(\tau_t^j)^*)}\right|,$$

*where $K(p_0,(\tau_t^i)^*,(\tau_t^j)^*)$ is the summarized players' payoff obtained with known information about the values of parameters, while $K(p_0,(\hat{\tau}_t^i)^*,(\hat{\tau}_t^j)^*)$ is the summarized players' payoff obtained for an imprecise estimation of the values.*

In contrast to the cooperative case, for a noncooperative game, when using unreliable information, the costs of the players could be less than with known information. In this case, we assume that the value of information about the initial condition is equal to zero.

**Definition 3.4.** *The normalized value of information in a noncooperative game for player $z \in \{i,j\}$ is defined as*

$$\mathcal{V}_z^{NC} = \begin{cases} \left|\frac{J_z(p_0,(\tau_t^z)^{NC}) - J_z(p_0,(\tilde{\tau}_t^z)^{NC})}{J_z(p_0,(\tau_t^z)^{NC})}\right|, & J_z(p_0,(\tau_t^z)^{NC}) < J_z(p_0,(\tilde{\tau}_t^z)^{NC}), \\ 0, & J_z(p_0,(\tau_t^z)^{NC}) \geq J_z(p_0,(\tilde{\tau}_t^z)^{NC}). \end{cases}$$

*where $J_z(p_0,(\tau_t^z)^{NC})$ is the costs of player $z$ obtained in the case of exact information about the values of parameters, while $J_z(p_0,(\tilde{\tau}_t^z)^{NC})$ is the costs of player $z$ obtained in the case of uncertain estimation of the values.*

Clearly, the larger the values of $\mathcal{V}^C$ and $\mathcal{V}^{NC}$ are, the more important the information about initial disturbance stock in cooperative and noncooperative cases is. Furthermore, grounding on the results obtained in Section 3.3.1, we can formulate the following propositions.

**Proposition 3.11.** *In the cooperative game (3.39) subject to (3.41), the value of information about the initial disturbance stock* $p_0$ *is given by:*

$$\mathcal{V}^C = \begin{cases} \frac{(1-r)^2}{\theta}, & 0 < r \leq 1+\theta, \\ r-1, & r > 1+\theta. \end{cases}$$

*Here* $r = \frac{\hat{p}_0}{p_0}$, $\hat{p}_0$ *is an estimation of the initial disturbance stock and*

$$\theta = \frac{2\epsilon\gamma - \delta}{2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T} - 1)}.$$

*Proof.* See Section 3.3.1 and Definition 3.3.

Note that when $0 < r < 1$ or $1 < r \leq 1+\theta$, for the value of information we use the formula:

$$\mathcal{V}^C = \left| \frac{K(p_0, (\tau_t^i)^*, (\tau_t^j)^*) - K(p_0, (\hat{\tau}_t^i)^*, (\hat{\tau}_t^j)^*)}{K(p_0, (\tau_t^i)^*, (\tau_t^j)^*)} \right|.$$

If $r = 1$, then $\mathcal{V}^C = 0$.

And if $r > 1+\theta$, NVI is as follows:

$$\mathcal{V}^C = \left| \frac{K(p_0, (\tau_t^i)^*, (\tau_t^j)^*) - K(p_0, (\hat{\tau}_t^i)^{**}, (\hat{\tau}_t^j)^{**})}{K(p_0, (\tau_t^i)^*, (\tau_t^j)^*)} \right|.$$

□

**Proposition 3.12.** *In the noncooperative game (3.40) subject to (3.41), the value of information about the initial disturbance stock* $p_0$ *is given by:*

$$\mathcal{V}_z^{NC} = \begin{cases} \frac{(1-r)(r(1+\theta)-3\theta-1)}{\theta(1+4\theta)}, & 0 < r \leq 1 \text{ or } \frac{3\theta+1}{\theta+1} < r \leq 1+2\theta, \\ 0, & 1 < r \leq \frac{3\theta+1}{\theta+1}, \\ \frac{2\theta+1}{4\theta+1} r - 1, & r > 1+2\theta. \end{cases}$$

*Here* $r = \frac{\tilde{p}_0}{p_0}$, $\tilde{p}_0$ *is an estimation of the initial disturbance stock and*

$$\theta = \frac{2\epsilon\gamma - \delta}{2\alpha^2\phi(e^{2(2\epsilon\gamma-\delta)T} - 1)}.$$

*Proof.* See Section 3.3.2 and Definition 3.4. □

### 3.3.4 Analysis of theoretical results and numerical examples

**Comparison of cooperative and noncooperative cases**

In the considered game, the players are symmetric which means we can conclude that in a cooperative game, each player will receive half of the total payoff. Then the value of information for the total payoff will coincide with the value of information for each player separately. With such knowledge, a way of comparing the value of information about the initial disturbance stock in a cooperative and noncooperative game is quite explicit.

The result of comparison between $\mathcal{V}^C$ and $\mathcal{V}^{NC}$ is presented in Tables 3.1, 3.2. In Table 3.1, it is assumed that $0 < \theta \leq 1$. The results for $\theta > 1$ are given in Table 3.2.

Table 3.1: Comparison of NVI for $0 < \theta \leq 1$

| $0 < r < 1$ | $1 < r \leq \frac{1+3\theta}{\theta+1}$ | $\frac{1+3\theta}{\theta+1} < r < 1+2\theta$ | $r \geq 1+2\theta$ |
|---|---|---|---|
| $\mathcal{V}^C > \mathcal{V}^{NC}$ | $\mathcal{V}^C > \mathcal{V}^{NC}$ | $\mathcal{V}^C < \mathcal{V}^{NC}$ | $\mathcal{V}^C > \mathcal{V}^{NC}$ |

Table 3.2: Comparison of NVI for $\theta > 1$

| $0 < r < 1$ | $1 < r \leq \frac{1+3\theta}{\theta+1}$ | $\frac{1+3\theta}{\theta+1} < r < 1+\theta$ | $r \geq 1+\theta$ |
|---|---|---|---|
| $\mathcal{V}^C > \mathcal{V}^{NC}$ | $\mathcal{V}^C > \mathcal{V}^{NC}$ | $\mathcal{V}^C < \mathcal{V}^{NC}$ | $\mathcal{V}^C > \mathcal{V}^{NC}$ |

It can be noticed that in most cases the value of information in a cooperative game is higher than that of in a noncooperative one, which indicates that the impact of estimation of the initial stock on the players' payoffs in a cooperative game is much more significant. Therefore, if the players choose cooperative behavior, they should pay more attention to the accuracy of information about the initial condition.

**Comparison of overestimation and underestimation cases**

In order to figure out whether overestimation and underestimation at relatively the same level will have an identical impact on the final payoff in each case, first of all, a detailed analysis of a cooperative case is given. The value $\overline{\mathcal{V}}^C$ represents the NVI in the case of overestimation with the shift $\beta p_0$ away from $p_0$, and $\underline{\mathcal{V}}^C$ is the NVI for the case of underestimation with the same shift. There are various cases depending on parameters values:

1. Assume $0 \leq \beta < 1$, $\theta < 1$.

   In the case of overestimation, for $\hat{p}_0 = p_0 + \beta p_0$ we have

$$\overline{\mathcal{V}}^C = \begin{cases} \frac{\beta^2}{\theta}, & 0 \leq \beta \leq \theta, \\ \beta, & \theta < \beta < 1. \end{cases}$$

   In the case of underestmation, if $\hat{p}_0 = p_0 - \beta p_0$, then

$$\underline{\mathcal{V}}^C = \frac{\beta^2}{\theta}, \text{ for all } 0 \leq \beta < 1.$$

   We can find out that

$$\overline{\mathcal{V}}^C = \underline{\mathcal{V}}^C, \text{ if } 0 \leq \beta \leq \theta,$$
$$\overline{\mathcal{V}}^C < \underline{\mathcal{V}}^C, \text{ if } \theta < \beta < 1.$$

2. Assume $0 \leq \beta < 1$, $\theta \geq 1$. In this case, $\overline{\mathcal{V}}^C = \underline{\mathcal{V}}^C$ for all $0 \leq \beta < 1$.

   These two cases show that the influence generated by an underestimation rate always outperforms or is equal to it in overestimation of the counterpart rate.

3. For $\beta \geq 1$, it does not make sense to investigate more information about uncertain parameter under this circumstance. But it can still be noted, that the loss in a case of overestimation outweighs the largest lost in a case of underestimation only if $\beta > \frac{1}{\theta}$.

Similar analyses can be made for a noncooperative case and a terminal cost coefficient case.

**Numerical example**

Based on Propositions 3.11, 3.12, a numerical example demonstrates how to analyze the influence of various cases of estimation. The parameter values are assigned randomly, let they be $\gamma = 1.8,\ \epsilon = 0.1,\ \alpha = 1,\ \phi = 0.06,\ p_0 = 100,\ \delta = 0.1, r \in (0, 2]$ for both cases. Specifically, in our example, we compare four different cases with terminal times $T = 10, 12, 14$ and $16$.

Fig. 3.3 and Fig. 3.4 show that a mode of game (cooperative and noncooperative) does demonstrate a similar form of the curve, but with a different magnitude. Obviously, in these two figures, if we compare the NVI under overestimation, i.e., $r > 1$, and underestimation, i.e., $r \leq 1$, the underestimation would bring larger costs to

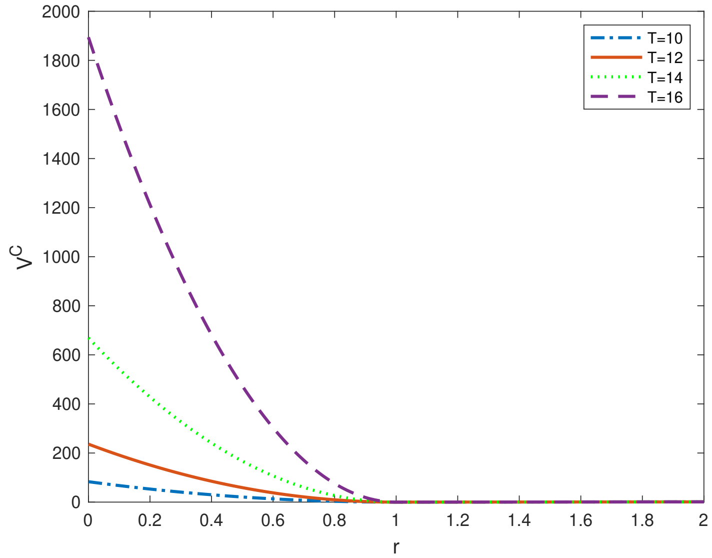


Figure 3.3: Normalized value of information for different estimated initial pollution stocks under various time intervals (X-axis denotes the estimation rate, Y-axis denotes the value of $\mathcal{V}^C$)

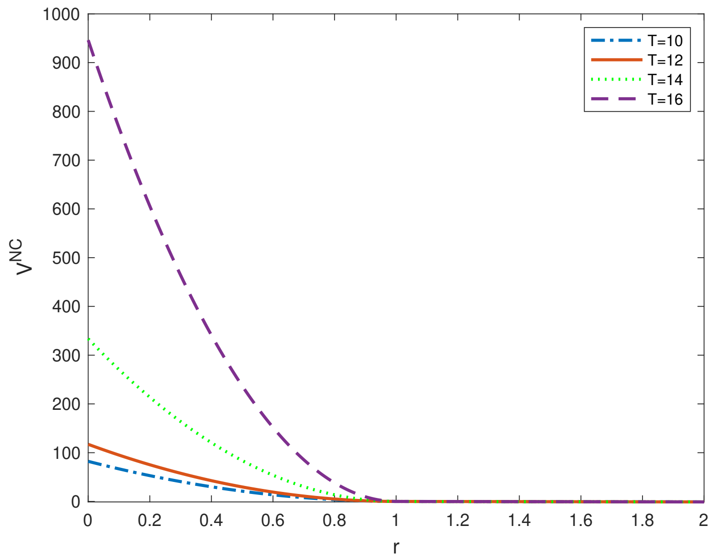


Figure 3.4: Normalized value of information for different estimated initial pollution stocks under various time intervals (X-axis denotes an estimation rate, Y-axis denotes a value of $\mathcal{V}^{NC}$)

the players if we observe them on the whole, which is also proved in our theoretical analysis. With the fixed initial time is, the larger the terminal time, the sharper the increased costs would be generated by underestimation. On the contrary, the linear-like curve of the NVI under overestimation is much more stable and gentle. Having combined what we have obtained in an analytical analysis, we may believe that a decision maker would consider to add more weight to the overestimation of the initial stock after obtaining an observed range of initial volume, e.g. with the range of initial stock $[A \pm B]$, in our case, we turn to $A + B$. Moreover, because of the larger influence on a cooperative case, it requires extra efforts for a decision maker to be sure of a high reliability of the estimated initial stock.

Additionally, we think it's meaningful to study the implication of the change of terminal cost coefficient $\phi$. This parameter determines the penalty that a firm should pay to the regulator if he/she fails to reclaim the environment to the terminal time. Fig. 3.5 demonstrates four plots with $\phi = 0.01$, $\phi = 0.03$, $\phi = 0.05$ and $\phi = 0.1$. The design of these four values for $\phi$ is out of consideration of the full scale testing from upper to lower interval. The overall trends of the NVI at different terminal time in cooperative and noncooperative cases are similar, we only present the result with $T = 10$.

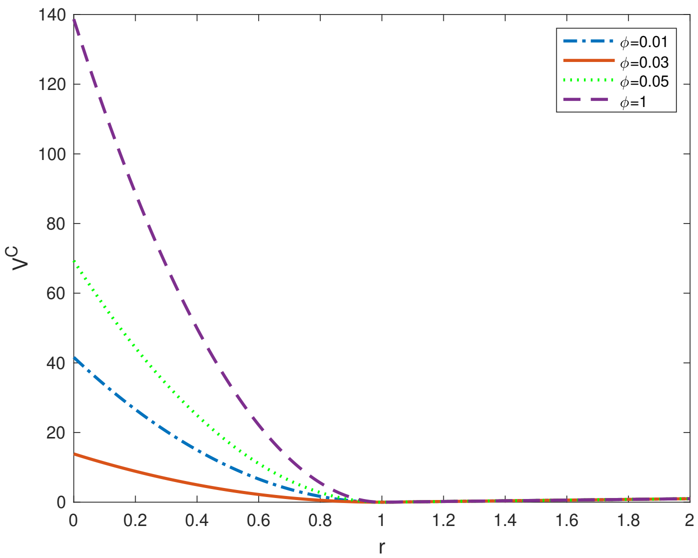


Figure 3.5: Normalized value of information for different estimated initial pollution stocks under different coefficients $\phi$ when $T = 10$ (X-axis denotes an estimation rate, Y-axis denotes a value of $\mathcal{V}^C$)

From Fig. 3.5, we can learn that in the case of underestimation, the NVI greatly increases with the growth of $\phi$. Moreover, the overall plot here is in accordance with the performance of the NVI at different terminal time. Hence, we think it's more acceptable to make overestimation of the initial pollution stock when a decision maker can't guarantee that the estimation he obtains is accurate enough to use it in his solution.

## 3.4 Conclusion to Chapter 3

In this chapter, we explore three different scenarios in which information available to the players is not accurate. Starting with the uncertainty of a terminal cost and the unawareness of the upper boundary of control to the estimation of the initial pollution stock, we use the normalized value of information (NVI).

In the terminal costs and upper boundary of control scenarios, we define the NVI to separately capture the value of knowledge embedded in terminal costs and upper boundary of control. Due to the complexity of the theoretical results, we present numerical examples in which we calculate the value of information. The numerical examples in both scenarios have demonstrated that the absence of knowledge mentioned at the beginning does reduce the players' payoffs with different levels of pollution reduction, except the latter case when the change of upper boundary of control $\bar{b} \geq b$. The lack of such information in the latter case has no impact on the players' payoffs.

In the scenario with unknown initial pollution stock, we study how the estimated initial stock could influence the performance of two players involved in a cooperative and non-cooperative differential game. Two cases including overestimation and underestimation are being explained and the NVI is applied to quantify the value of information about the initial stock parameter. We conclude that the uncertainty about the initial stock influences more a cooperative case a noncooperative case, which reminds players to be more careful with information accuracy in the cooperative mode of play. For both cooperative and noncooperative cases, a common phenomenon that the overestimation of initial stock carries less weight to the final payoff of the players is being observed. The results of the numerical example demonstrate that the players could greatly reduce their costs if they incline towards overestimation when they receive the evaluation report of the estimated initial stock from the experts, especially in a cooperative case.

# Conclusions

This thesis is devoted to the stability analysis in pollution control problems with one or many decision makers. The research can be essentially divided into two parts: examining internal factors such as coalition structure and objective functions on stability of cooperation (Chapter 1 and 2), and stability analysis of external factor influence such as uncertainty of the terminal costs, possible adjustment of upper boundary of the control variables as well as estimation of initial pollution stock (Chapter 3). More specifically, a static pollution control problem with four players is examined in Chapter 1 in which two stability concepts are defined and three mechanisms are provided to make some particular coalition structures stable, while Chapter 2 deals with a dynamic model of pollution control and introduces the trade-off mechanism inspired by the contract design typically used in supply chain modeling. In Chapter 2, an asymmetric three-player differential game where the consistency of stable coalition structure in dynamics is also investigated. The stability analysis given in Chapter 3 is focused on investigating how the uncertain information could affect the stability of decisions. In Chapter 3, a new characteristic called normalized value of information is suggested to quantify the value of information of uncertainty with respect to terminal cost, upper boundary of control, and initial pollution stock.

The main results of the work are as follows:

1. We examine the game between polluting countries which are different in their attitudes to environment protection policies. Some countries take the pollution reduction costs in their optimization problems, while other countries do not. We examine different scenarios of cooperation including cooperative, noncooperative, and partially cooperative scenarios when any coalition structures can be formed. We examine the stability of all scenarios and obtain that there are no stable scenarios but a unique individually stable scenario. To sustain stability of desirable scenarios we provide three mechanisms to make them stable

including (i) implementation of transfers based on any cooperative solution, (ii) design of the system of transition costs in case of their deviations, and (iii) restrictions on coalition formation and design of the set of feasible scenarios. We provide numerical examples of four-player games and introduce how the proposed mechanisms can be realized. We highlight that the number of computations increases exponentially with an increase of the number of players. Therefore, theoretically, the results can be applied to the games with any number of players, but practically, it is difficult to make computations for the proposed mechanisms.

2. We introduce a trade-off mechanism involving a trade of pollution disposal between vulnerable and invulnerable players to reduce the actual pollution stock and provide a new solution of a pollution multi-player control problem. The comparison between a trade-off mechanism scenario, noncooperative, and cooperative scenarios specifies the advantages and limitations of the trade-off mechanism. Basing on a numerical example, we conclude that we can find the parameters of the trade-off mechanism to outperform a noncooperative scenario. It is also clear that this mechanism is worse than a fully cooperative scenario in terms of improving both players' profits. A differential game of pollution control with developing and developed countries is also investigated. We examine stability of different cooperative scenarios when players can partially cooperate. To examine all possible coalition structures, we propose three types of scenarios:(i) cooperative, (ii) noncooperative, and (iii) partially cooperative, in which the coalition's profit is dependent on the outside players' behavior (in particular, it depends on if they form coalitions or not). The general conditions of Nash and individual stability of coalition structures or scenarios are determined. Two numerical examples demonstrate the procedure of finding a stable scenario in a three-player game. We also introduce the procedure of making a particular scenario stable (if possible) by defining a special transfer scheme.

3. We consider three models of the pollution emission control with uncertain information and introduce a normalized value of information proposed to represent the value of information in a numerical way. In the analysis of uncertainty in terminal cost and upper boundary of control, we examine when information about the existence of such a component, i.e., terminal cost and changed up-

per boundary is available or is not available. We also examine the influence of such information on the final players' payoffs. The results show that when the information is unavailable, the players' payoffs will be reduced except the case when the changed upper boundary of control is above the original, such information is meaningless for players. Moreover, we accomplished the study of how the estimated initial stock could influence the performance of two players in terms of the rehabilitation process in cooperative and noncooperative differential games. Through the rigorous analytical analysis by comparing normalized value of information under various terminal times in both cooperative and noncooperative cases, we find out that the overestimation of initial stock impacts the final payoff in a trivial way and the uncertainty about the initial stock brings more disturbance in a cooperative case. It is expected that the decision maker would incline toward overestimation, especially, in a cooperative case.

We conclude that all the tasks formulated in this thesis are achieved, and the objectives are fully accomplished.

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