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ОТЗЫВ

члена диссертационного совета на диссертацию
Старчака Михаила Романовича на тему:
"Алгоритмы квазиэлиминации кванторов и вопросы выразимости
в арифметиках с делимостью",
представленную на соискание ученой степени
кандидата физико-математических наук
по научной специальности
1.1.5 Математическая логика, алгебра,
теория чисел и дискретная математика.

The story started in 1900, when David Hilbert, the recognized leader of 19 century mathematicians, was asked to select a few important challenges for the 20 century mathematics. Hilbert selected his famous 23 problems. In his introduction to these problems, he stated his belief that, in contrast to natural sciences, where there may be statements about which we will never know whether they are true or not, in mathematics there is no Ignoranbimus – we will never know, that each statements will eventually proven to be either true or false. Since – as Hilbert himself mentioned – a proof is a sequence of symbols, with precise checkable rules describing when a sequence of symbols constitutes a proof, we can thus, in principle, have what we now call an algorithm for deciding whether a given mathematical statement is true or not, simply by generating all possible sequences of symbols and checking whether a generated symbol is a proof of the given statement or of its negation. This algorithm may require an unrealistic time, but it is an algorithm.

This overoptimistic viewpoint was shattered in 1931 by Goedel, who proved that some statements cannot be proven or disproven, and that, in effect, no general deciding algorithm is possible for mathematics (even for arithmetic – i.e., for the first order theory of natural numbers with addition and multiplication). The pendulum swung to pessimism, when most mathematicians believed that general deciding algorithms are not possible.

The situation became more optimistic again in the 1940s when Tarski presented his famous algorithm for the first order theory of real numbers with addition and multiplication.

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This algorithm was probably the first to use quantifier elimination idea, when we reduce a formula with a quantifier in front to an equivalent formula that does not have a quantifier in front – and thus, has fewer quantifiers in general. We continue this process until we get to a decidable quantifier-free formula.

Since then, the situation somewhat oscillates between optimistic and pessimistic, with the ultimate goal to find the boundary between decidable and undecidable classes of statements. Let us just mention two results: the famous Matiyasevich theorem showing already the existential theory of integers is undecidable – and which answers (in the negative way) tenth Hilbert's problem, and – closer to the topic of the dissertation – Bel'tuykov-Lipschitz theorem that the existential theory of natural numbers with 1, addition, and divisibility is decidable.

At first glance, Bel'tuykov-Lipschitz theorem may sound like a pure mathematical exercise, but the corresponding theorem actually has a natural physical interpretation. For example, if we have objects with masses proportional to some fixed mass (e.g., composed of atoms or molecules of the same type), then, by using scales, we can compare their masses, the sum corresponds to placing two objects on the same plate, and divisibility means that by placing several copies of object x on one plate, we can get the same weight as object y. This theorem has not (yet) been used in weight measurement, but it has been used in similar applications, e.g., in temporal logic, where we measure and compare time intervals.

Of course, in this research area, there are many open questions, and this is where the dissertation comes in. What the dissertation starts with – and what I consider the main result of this dissertation – is the development of a new method of proving decidability, a method that the author calls quasi-quantifier elimination. The idea is that instead of completely eliminating the in-front quantifier, we reduce the original formula to a simpler one, where the quantifier is still present, but the corresponding expressions are now much simpler – which makes the resulting equivalent formula easier to decide.

The author shows that this technique can lead to a new proof of the Bel'tuykov-Lipschitz theorem, a proof that leads to a more natural algorithm, and, what is more important, it helps to prove several new related decidability and undecidability results – for example, decidability of the existential theory of natural numbers with 1, addition, and co-primeness, and the impossibility to define non-co-primeness (i.e., the existence of a nontrivial common divisor) in the positive existential fragment of this theory.

Another important result of the author – also naturally interpretable in terms of weight measurements – is the decidability of an existential theory of real numbers with 1, addition, subtraction, equality, inequality, divisibility, and the "floor" function returning the largest integer smaller than or equal to the given real number. Not only is this result interesting, it is a solution to a problem that remained open since 1999, for 20+ years! This shows how skilled is the author and how good are his methods.

There are many more interesting results. I think these techniques are results open the door for new developments. For example, quantifier elimination has been studied a lot in model theory, and this study helps understand why for some theories this method works and for some it does not. It would be great to come up with something similar for the candidate's quasi-quantifier elimination technique.

Summarizing: the dissertation by Starchak Mikhail Romanovich is very good, it definitely satisfies all the requirements for such dissertations, and its author definitely deserves the degree of Candidate of Physico-Mathematical Sciences in speciality 1.1.5 “Mathematical logic, algebra, number theory, and discrete mathematics”.

Sincerely yours,

(V. Kreinovich)

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