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Review of the Doctoral Thesis of Andrej V. Rodin:

„Axiomatic Architecture of Scientific Theories“

This is an excellent analysis of the axiomatic method in theory building which is based on the historical development of axiomatization in mathematics from Euclid, Hilbert, and Bourbaki up to modern category theory and univalent foundations. The author aims at an architecture of theories which he calls „constructive axiomatic method“, but which must not be confused with mathematical constructivism and intuitionism in the strict mathematical sense. The thesis is written in a clear and didacticly understandable style. It starts with an introduction which makes precise the pertinence of the author's research and the aim of the present study. The following brief overview of the content is helpful for the reader. Chapters 1-3 could be read as a systematic and historical introduction in axiomatic mathematical methods from Euclid to univalent mathematics. The last chapter 4 is reserved to the authors concept of a „constructive axiomatic method“.

Chapter 1 („From Euclid to Hilbert“) explains the differences between Euclid's and Hilbert's axiomatic method. Euclid distinguished postulates as rules of constructions (with compass and ruler) from axioms of propositions. Hilbert's formalization became important for the application of formal logic in mathematics in the 20th century. The author also underlines the revendication of the genetic method besides axiomatic methods since Euclid which will become important for his own notion of a „constructive axiomatic method“.

Chapter 2 is dedicated to the axiomatic method at work in mathematical and scientific practice. After a short introduction of set theory, the structuralistic approach of Bourbaki is described. From a didactic point of view, the section on „Bourbaki and mathematical education“ is remarkable. The author compares the movement of „New Math“ in the USA as well as in the former UDSSR and explains the bias of this approach in the practice of schools. The following section concerns axiomatic approaches in science and in the philosophy of science. There are are only two short subsections on physics and biology.

In the subsection 2.3.1, I miss a discussion of Hilbert's important contribution to a unified field theory of gravitation and electromagnetism which was considered by him as an explicit application of his axiomatic method. Actually, there were only few applications of the axiomatic method in the history of biology. But, a glance on modern mathematical applications in the life sciences would be helpful. Subsection 2.3. on „Computer Science and Engineering“ consists of only two pages. The author shortly mentioned modern artifical intelligence (AI) and knowledge-based systems and the bias of an axiomatic method in

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computer science. We will come back to this important point, when in chapter 4 the author's approach of a constructive axiomatic method is to be discussed.

Chapter 3 considers two novel axiomatic approaches. A highlight of this dissertation is section 3.1 on category-theoretic foundations of mathematics and topos theory. In this domain, the author is an expert who already delivered a remarkable book on this topic. Especially, he analyzes the works of William Lawvere and his axiomatic treatment of topos theory on the basis of general category theory. The second section 3.2 covers the Homotopy type theory (HoTT) and the related project of building new foundations of mathematics which was initiated by the Russian mathematician and Fields medal winner Vladimir Voevodsky. This approach is important for the author's suggestions on a constructive axiomatic method.

In chapter 4, the author's crucial concept of a „constructive axiomatic method" is explained. Again, I should unerline that the author's notion of „constructive" can be misleading for experts in mathematical foundations. In section 4.2, the author defines his notion and illustrates its philosophical, epistemic, and historical background (e.g., the philosopher Cassirer). The author says:

„Hilbert and Bernays 1934 remarks, in which the authors point to limits of their standard „existential" axiomatic method and discuss the possibility of a more general axiomatic approach, which combines this standard method with the more traditional Euclid-style „genetic" method (1.3). Hilbert and Bernays call this hypothetical general method *constructive*. We borrow this name from Hilbert and Bernays (understanding the risks of terminological confusion) along with the idea."

This is the hard core of the whole dissertation: The author suggests to continue a more general axiomatic approach in the meaning of Hilbert/Bernays 1934 overcoming the strict Hilbert-style axiomatic formalism. This generalized method dates back to the more Euclid-style *generic* method. Thus, in order to avoid misunderstanding, the better notion of the author's thesis should be *„generic axiomatic method"*. This is, of course, a valuable and remarkable contributation to running discussions on the general foundations of science, in order to cover more disciplines in science by axiomatic methods.

Nevertheless, it must be underlined why constructivism in the strict mathematical sense is important to understand Homotopy Type Theory and its relation to computer science and software engineering which the author also mentioned. At first, let us consider the essential steps to Voevodsky's Homotopy typ theory which the author also briefly sketched. Type theories are bridges between logic, mathematics, and computer science. Haskell Curry related proofs in formal Hilbert-style systems to computational methods of combinatory logic. In 1969, William A. Howard observed that Gentzen's proof system of natural deduction can be directly interpreted in its intuitionistic version as a typed variant of the mode of computation known as lambda calculus.

In addition to the type formers of the Curry-Howard interpretation, Per Martin-Löf extended the basic intuitionistic type theory (containing Heyting's arithmetic of higher types HA^{ω} and Gödel's system T of primitive recursive functions of higher type) with primitive identity types, well founded tree types, universe hierarchies and general notions of inductive and inductive–

recursive definitions. His extension increases the proof-theoretic strength of type theory and its application to programming as well as to formalization of mathematics.

In computer programs, data types are used to reduce software bugs. Historically, they are rooted in logical type theory which was introduced to avoid logical paradoxes. Since Church's typed λ-calculus, type theory has been extended to represent more and more abstract mathematical objects. An example is Martin-Löf's intuitionistic type theory. The foundational program of univalent mathematics is inspired by the idea of a proof-checking software to guarantee correctness, trust, and security in mathematics even for abstract structures and categories. Homotopy Type Theory (HoTT) is based on the type-theoretical Calculus of Constructions (CoC) and the extended version of the Calculus of inductive Constructions (CiC) which is also the basis of the proof assistant Coq.

Behind all that, there is the fundamental epistemic question how far even abstract mathematical objects such as sets, structures, and categories can be reduced to the digital world of computers (represented by data types of computer languages). I made this point clear in my book on "The Digital and the Real World" [2]. In a final step, homotopy type theory (HoTT) tried to develop a universal ("univalent") foundation of mathematics as well as computer languages with respect to proof assistants for advanced mathematical proofs.

In modern mathematics, algebraic topology uses tools of abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence. Homotopy theory is an outgrowth of algebraic topology and homological algebra with relationships to higher category theory which can be considered as fundamental concepts of mathematics. Type theory is a branch of mathematical logic and theoretical computer science. Homotopy type theory (HoTT) interprets types as objects of abstract homotopy theory. Therefore, HoTT tried to develop a universal („univalent") foundation of mathematics as well as computer language with respect to the proof assistant Coq.

In accordance with HoTT, Vladimir Voevodsky suggested a formalization of modern mathematics which became known as "univalent foundation." It is a version of Martin-Löf type theory extended by a new axiom (UA) considering a universal property of identity and equivalence of mathematical structures. Equivalence ($\simeq$) refers to higher-dimensional objects from set theory up to category theory (e.g., equal elements, isomorphic sets, equivalent groupoids): The univalent axiom UA expresses that „everything" is preserved by equivalence. In particular, equivalent types (e.g., isomorphic structures) are identical.

My point is that this approach is constructive in the strict meaning of mathematical intuitionism. Its semantics is constructive in the strict meaning of the Brouwer-Heyting Kolmogorov (BHK) interpretation. With respect to applications in computer science, we must consider another constructive type-theoretical approach which is only shortly mentioned by the author: In analogy to Martin-Löf intuitionistic type theory, another higher-order typed lambda calculus was introduced by Thierry Coquand. In general, Coquand is important as co-author of the main manifesto of HoTT in the book "Homotopy Type Theory" published by the Institute of Advanced Study (Princeton).

Coquand's calculus of constructions (CoC) became the type-theoretic fundament of the proof assistant Coq. CoC is a type theory which can serve as typed programming language as well as constructive foundation of mathematics. With inductive types, the calculus of inductive constructions (CiC) removes impredicativity. It extends the Curry-Howard isomorphism to proofs in the full intuitionistic predicate calculus.

HoTT allows mathematical proofs to be translated into a computer programming language for computer proof assistants (e.g., Coq, CiC) even for advanced mathematical categories with "isomorphism as equality" (UA). An essential goal of HoTT is the transformation of type checking in highly advanced type theories into proof checking in higher categories for advanced mathematical proofs. It should be mentioned that HoTT is consistent with respect to a model in the category of Kan complexes. A constructive justification of the Univalence Axiom (UA) was recently suggested by Thierry Coquand. The Cubical Type Theory allows a constructive interpretation of the univalence axiom [1]. Therefore, again, the constructive semantics of HoTT can be made precise in the strict mathematical meaning of constructivism which should be supplemented in a publication of the doctorial thesis.

Let me close this review with some remarks on the impact of these approaches of axiomatic and constructive methods (in the strict mathematical sense) in science, technology, and society which the author is also interested in: In the age of digitalization, correctness, security, and trust in algorithms turn out to be general problems of computer technology with deep societal impact. Therefore, proof theory has practical consequences in applied computer science [3]. In this case, instead of formal theories and proofs, we consider formal models and computer programs of processes in industry and society. Formal derivations of formulas correspond to, e.g., steps in industrial production. In order to avoid additional costs of mistakes and biases, we should test and prove that computer programs are correct and sound before applying them. Verification of computer programs is a crucial demand of software engineering. In an ideal case, computer programs could be verified such as formal systems by mathematical proofs in mathematical logic.

Coming back, again, to the meaning of construction: Obviously, computer science, logic, and mathematics are connected in the constructions of intuitionistic type theory. In this sense, mathematical constructivism seems to open new avenues of research combining foundations of logic and mathematics with highly topical challenges of IT- and AI-technology. On the other side, it is remarkable that mathematical constructivism is deeply rooted in the philosophy of Kant, Brouwer, Markov, Bishop, Lorenzen et al. which we discussed elsewhere [2].

But, the Kuhnian belief in a sequence of paradigm shifts as progress of science is a simplified view. All these methods have advantages and disadvantages, depending on their constraints of application. Historical approaches are not necessarily overcome by alternative methods. As philosophers of science and technology, we carefully must determine the constraints of methods, their limitation and potentiality. In economic words, each procedure has a methodological price. There is no free lunch even in science and technology. This more relaxed, perhaps vague, and extended meaning of a "constructive" axiomatic method is defended by the author in his dissertation. He explains us how the generic method in Euclidian tradition may be helpful to axiomatize a wide variety of disciplines in modern science. In the end, we learn from Andrej V. Rodin's thesis how old methodological ideas may be integrated and transformed ("aufgehoben") into new ones in a Hegelian sense.

I strongly recommend the acceptance of the doctorial thesis of Andrey V. Rodin.

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