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REVIEW

of the Habilitation Thesis of **Fedor Vladimirovich Petrov** on the topic "Problems of Continuous and Polynomial Combinatorics", submitted for the degree Doctor of Science in Physics and Mathematics in the specialization 01.01.01 – Real, Complex and Functional Analysis.

Brief overview of the work

The thesis consists of two parts, the first dealing with continuous graphs and hypergraphs and the second focusing on combinatorial applications of the polynomial method and of group rings. Both topics have been very active in recent years, and the thesis contains a number of significant, novel contributions. The techniques in the first part combine combinatorial tools with ideas from functional analysis, measure theory and ergodic theory, whereas the ones in the second part include methods from enumerative combinatorics, additive combinatorics, group rings and algebraic geometry. This is an impressive work, based on a significant number of publications by the author himself and by the author and his collaborators.

Scientific value of the results

The first part of the work, described in Chapter 1, deals with continuous combinatorics. The basic object of study is a measurable m-homogeneous hypergraph, in which the vertex space V is a probability space with a measure μ. The hypergraph can then be represented as a measurable function from the m-th symmetric power of the space (V, μ) to $\{0, 1\}$ or to $[0, 1]$. The case $m = 2$ and the function which is identically $p \in (0, 1)$ corresponds to the Erdős-Renyi model of binomial random graphs and the countable random graph corresponding to this measure is, with probability 1, isomorphic to the universal Rado graph. This is a universal graph for the class of

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all finite graphs. Getting a similar universal graph for other classes of graphs, like the family of all triangle-free or K_d-free graphs, is more complicated. The first result described in Section 1.1 here proves the existence of such a graph by building what the author (with Vershik) calls a topological universal graph.

Section 1.2 deals with the correction of continuous graphs and hypergraphs. This is the continuous analog of results that have been studied in the investigation of property testing of graphs in the finite case, where the objective is to identify properties of graphs for which if random constant size induced subgraphs of the graph satisfy the property with high probability, then the graph can be modified without changing much of its representation so that it will globally satisfy it. One of the best known early examples of this type is the triangle-removal lemma of Ruzsa and Szemerédi, which assets that if a graph on n vertices contains only $o(n^3)$ triangles, then one can remove from it $o(n^2)$ edges and get a triangle-free graph. This has a natural measurable version , and Section 1.2 deals with the tools required in the investigation of this type of questions, developed in a paper of the author and Zatitskiy. Another related result proved in a sole paper of the author, is that the property of being a (finite) metric space does not yield an efficient correction procedure of this type.

The remaining sections of the first chapter deal with metrics and virtual continuity and contain several intriguing results and applications.

The second part of the thesis, presented in Chapter 2, is focused on the development and application of the polynomial method in Combinatorics, and on the study of related methods using group rings.

The Combinatorial Nullstellensatz is a method for proving that a multivariate polynomial does not vanish on a box provided its degree and the dimensions of the box satisfy certain constraints and provided an appropriate coefficient of the polynomial can be shown to be nonzero. This has been used to establish a large number of applications in Combinatorics, Graph theory, Additive Number Theory and Geometry. The first result here, described in Section 2.1, is a simple proof of the theorem in its original form using the Euler-Jacobi formula, as well as an interesting generalization for multisets. The arguments used in this proof provide a simple proof of the Dyson conjecture (which has been proved earlier by Gunson, by Wilson and by Good.) Identities of this type are closely related to Selberg's integral formula, as explained in Section 2.2 which includes also a description of several additional results extending the Aomoto formula and Forrester conjecture. Additional enumerative results are discussed in Section 2.3. These give as a special case the hook length formula (which gives the dimensions of irreducible representations of the symmetric group) and other identities obtained by applying the polynomial method.

In Section 2.4 the author describes applications in Additive combinatorics. These appear in two of his solo papers and in another one, jointly with Volkov. The relevance of the Combinatorial Nullstellensatz to variants of the Cauchy-Davenport Theorem has been noted before, but the results here provide many additional and interesting novel statements. These include results about expansion of polynomial mappings over finite fields and sharp estimates for several sum and product problems.

Section 2.5 contains applications in Graph Theory, including an interesting strengthening of the cycle and triangle problem which asserts that any graph on $3n$ vertices whose set of edges is the union of a Hamilton cycle and n vertex disjoint triangles is 3-colorable. Section 2.6 describes a symmetrization technique and Section 2.7 uses the group ring approach to derive several extensions and variants of the recent results about the cap set problem (about the maximum size of a subset of Z_3^n with no three-term arithmetic progression). The relevance of group rings to questions of this type was noted already by Olson, who used it to find a precise formula for the Davenport constant of any p-group, but the author showed that the technique is powerful in the study of intriguing questions for nonabelian groups as well.

Overall Assessment

The thesis submitted by Fedor Vladimirovich Petrov on the topic "Problems of Continuous and Polynomial Combinatorics" contains a significant number of novel techniques and results. These results are described in 15 papers of the author, including several joint papers and ones in which he is the only author. 14 of the papers already appeared in leading journals in the relevant areas. There is no doubt that the work meets the requirements and that Fedor Vladimirovich Petrov fully deserves the award of the degree of doctor of science in Physics and Mathematics in the specialty 01.01.01 – Real, Complex and Functional Analysis. I strongly support the award of this degree.

Sincerely,

N. Alon

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